## 5. Cauchy inequalities

Consider a polynomial

$$f(z)=\sum_{n=0}^{d}a_nz^n,$$

and let

$$M(r,f)=\max_{|z|=r}|f(z)|.$$

We are going to show that

$$|a_n|\leq M(r,f)/r^n, \tag{1}$$

for every $n$ and every $r>0$.

Let $\epsilon=\exp(2\pi i/(d+1)$ be the primitive root of unity. Then we have by the geometric progression formula

$$\sum_{j=0}^{d}\epsilon^{kj}=\frac{\epsilon^{k(d+1)}-1}{\epsilon^k-1}=0, \tag{2}$$

for all integers $k$ such that $|k|\leq d$.

Let us fix an integer $n\in[0,d]$ and set

$$g(z)=\frac{f(z)}{z^n}=\frac{a_0}{z^n}+\ldots+a_n+\ldots+a_dz^{d-n}.$$

Then

$$\frac{1}{d+1}\sum_{j=0}^{d}g(\epsilon^jz)=a_n,$$

because according to (2) everything else in the sum cancels. So

$$|a_n|\leq\max_{|z|=r}|g(z)|=M(r,f)/r^n,$$

so we proved (1).

Now suppose that we have a convergent series

$$f(z)=\sum_{n=0}^{\infty}a_nz^n,$$

whose radius of convergence is $R>0$. Then we can truncate the series, use the inequality (1) for partial sums and then pass to the limit.

Thus the Cauchy inequalities (1) hold for the coefficients of every convergent power series.