Two questions on completely invariant domains

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1. Makienko Conjecture.
Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational function, and J its Julia set. Suppose that D is a component of $\overline{\mathbb{C}} \setminus J$ such that $\partial D = J$. Then D is completely invariant for the second iterate of f, that is $f^{-2}(D) = D$.

There are several restatements of this conjecture. Suppose that D is invariant, then $f : D \to D$ is a ramified covering; let m be the degree of this covering, then D is completely invariant if and only if $m = d := \deg f$. So any counterexample to this conjecture must involve a region D, and a ramified covering $f : D \to D$ such that f is m-to-1 in D, while f is d-to-1 on ∂D, with $d > m$. It is not even known whether such thing is possible for continuous functions, even with $m = 1$ and $d = 2$.
One can show that for any counterexample, J must be an indecomposable continuum [3].

2. How many completely invariant components can the Fatou set of an entire function have?

A rational function can have at most two completely invariant components. For long time, it was believed that I. N. Baker proved in 1970 that a transcendental entire function can have at most one. However, a mistake in Baker's proof was found in 2017, and a counterexample *to his proof* was found.

Baker was proving a more general statement: Let D_1 and D_2 be disjoint simply connected domains. Then there is no transcendental entire function such that both $f^{-1}(D_j)$, $j = 1, 2$ are connected. Surprisingly this turned out to be wrong: even for a simple entire function like $e^z + z$ one can construct infinitely many disjoint simply connected domains whose preimages are connected [4]

It is known that for meromorphic functions of class S (with finitely many singular values) the number of completely invariant domains is at most 2, [2] and for entire functions of class S this number is at most 1, [4].


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