Math 530, Spring 2024, midterm exam

NAME:

1. Let ε be an n-th root of unity, $\varepsilon^n = 1$ and $\varepsilon \neq 1$. Prove these formulas:

$$1 + \varepsilon + \varepsilon^2 + \ldots + \varepsilon^{n-1} = 0. \tag{1}$$

$$1 + 2\varepsilon + 3\varepsilon^2 + \ldots + n\varepsilon^{n-1} = \frac{n}{\varepsilon - 1}. \tag{2}$$

Solution.
(1) By geometric progression formula,

$$1 + \varepsilon + \varepsilon^2 + \ldots + \varepsilon^{n-1} = \frac{1 - \varepsilon^n}{1 - \varepsilon} = 0,$$

since the numerator is 0 while the detominator is not.

(2) Multiply both sides on $\varepsilon - 1$:

$$(\varepsilon + 2\varepsilon^2 + 3\varepsilon^3 + \ldots + n\varepsilon^n) - (1 + 2\varepsilon + 3\varepsilon^2 + \ldots + n\varepsilon^{n-1})$$

$$= (-1 - \varepsilon - \varepsilon^2 - \ldots - \varepsilon^{n-1}) + n\varepsilon^n = 0 + n = n,$$

where we used (1) and $\varepsilon^n = 1$.

2. For which real a, b is the function

$$u(x, y) = x^3 + ax^2y + bxy^2 + y^3$$

harmonic? For these a, b find an analytic function $f(z),\ z = x + iy$, whose real part is u.

Solution.

$$u_{xx} + u_{yy} = (6 + 2b)x + (2a + 6)y \equiv 0,$$

so $a = b = -3$.

Now for $u(x, y) = x^3 - 3x^2y - 3xy^2 + y^3$, solve the Cauchy–Riemann equations for v. We obtain the general solution

$$v(x, y) = x^3 + 3x^2y - 3xy^2 - y^3 + C.$$

Since only one solution is required, set $C = 0$. Now it is easy to guess that $f(z) = (1 + i)z^3$.

Solution in the form

$$f(z) = u(x, y) + iv(x, y),$$

where v is also acceptable.

3. Find all solutions of the equation

$$\sin z = i,$$

and make a picture of them.

Solution.
Let $w = e^{iz}$. Then our equation is

$$\frac{w - w^{-1}}{2i} = i, \quad \text{or} \quad w^2 + 2w - 1 = 0.$$

By quadratic formula we obtain two real solutions, one positive another negative:

$$w_{1,w} = -1 \pm \sqrt{2}.$$

Solving $e^{iz} = w_1 = -1 + \sqrt{2}$ we obtain the first series

$$z_{1,n} = -i\mathrm{Log}\,(\sqrt{2} - 1) + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots.$$

Similarly

$$z_{2,n} = -i\mathrm{Log}\,(\sqrt{2} + 1) + \pi + 2\pi n.$$

To make a good picture, notice that $(\sqrt{2}-1)(\sqrt{2}+1) = 1$, therefore $\mathrm{Log}\,(\sqrt{2}-1) = -\mathrm{Log}\,(\sqrt{2}+1) < 0$. I will describe the pictire with words: it consists of two horizontal rows of dots parallel to the x-axis, one above the x-axis and another below, on equal distance from the x-axis. The upper row contains a point on the imaginary axis, with positive imaginary part, and the lower row contains a point whose real part is π, and both rows are arithmetic progressions with increment 2π.

4. Let a, b be complex numbers, and $|a| < r < |b|$. Evaluate the integral

$$\int_{|z|=r} \frac{dz}{(z-a)(z-b)} dz.$$

Solution.
The partial fraction decomposition is

$$\frac{1}{a-b}\left(\frac{1}{z-a} - \frac{1}{z-b}\right).$$

By assumption, the pole at a is inside the circle $|z| = r$, and the other pole is outside. So by the residue theorem the integral is

$$\frac{2\pi i}{a-b}.$$

5. Find the radii of convergence of these series:

a) $\sum_{n=1}^{\infty} \frac{n^n}{n!} z^n$,

b) $\sum_{n=0}^{\infty} n! e^{-n^2} z^n$,

c) $\sum_{n=0}^{\infty} e^{-\sqrt{n}} z^n$.

Solution.

a) Use the ratio test:

$$\frac{(n+1)^{n+1}|z|^{n+1}}{(n+1)!} \frac{n!}{n^n|z|^n} = \left(1 + \frac{1}{n}\right)^n |z| \to e|z|,$$

so the radius of convergence is $1/e$.

b) Again the ratio test:

$$\frac{(n+1)! e^{-(n+1)^2}|z|^{n+1}}{n! e^{-n^2}|z|^n} = (n+1)e^{-2n-1}|z| \to 0,$$

so the radius of convergence is ∞.

c) Use the root test:

$$\left(e^{-\sqrt{n}}\right)^{1/n} = e^{-1/\sqrt{n}} \to 1,$$

so the radius of convergence is 1.