Zeros of sums of simple fractions

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Let $z_n \to \infty$ be a sequence in the complex plane. Consider a meromorphic function

$$f(z) = \sum_{n=1}^{\infty} \frac{a_k}{z - z_k}, \quad \sum_{k=1}^{\infty} |a_k| < \infty, \quad \sum_{k=1}^{\infty} a_k \neq 0.$$

Is it true that every such function has zeros?

This is known in the following cases:

a) f is of finite order [1],

b) $a_k > 0$, [2].

References

[1] A. Goldberg and I. Ostrovskii, Distribution of values of meromorphic functions, AMS, Providence RI, 2002.

[2] A. Eremenko, J. Langley, J. Rossi, On the zeros of meromorphic functions of the form $\sum_{k=1}^{\infty} \frac{a_k}{z-z_k}$, Journ. d'Analyse Math., 62 (1994), 271-286.