Goldberg's constant and its relatives

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Let f be a holomorphic function in the unit disc having exactly one simple zero z_0 and two 1-points (counting multiplicity) z_1, z_2. Let ρ be the radius of the smallest hyperbolic disc containing z_0, z_1, z_2. Goldberg [4] proved that $\rho \geq \mu > 0$, where μ is an absolute constant. What is the best (largest value) of μ for which this is true? This optimal value is called the Goldberg constant A_2.

There is a conjecture that the extremal function must have one double 1-point. Assuming this, the extremal function was found in [1].

There are several similar problems.

1. Let f be a function holomorphic in the unit disc, $f(0) = 0$, $f'(0) \neq 0$, and no other zeros, and such that the equation $f(z) = 1$ has two solutions, z_1, z_2 counting multiplicity.

a) What is the minimal value of $\max\{|z_1|, |z_2|\}$?

b) What is the maximal value of $|f'(0)|$?

2. Same questions with the additional assumption that f is real.

3. Let f be a real holomorphic function in the unit disc, having one simple zero at 0 and two simple 1-points at $\pm ia$. What is the minimal value of a ? This minimal value is called the Belgian Chocolate Constant because the prize of $1kg$ of fine Belgian chocolate is offered for it [2, p. 149f].

4. All these problems can be also asked for rational functions of given degree. I conjecture that the extremal function for Problems 1,2 and for the Goldberg constant is a Belyi function with 1-point of multiplicity 2.

All these problems are relevant for control theory [2, 3].

References

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