Stabilizability by static output feedback

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Consider a linear system

$$\begin{aligned} \dot{x} &= Ax + Bu & (1) \\ y &= Cx & (2) \end{aligned}$$

Here $A \in \mathrm{Mat}(n \times n), B \in \mathrm{Mat}(n \times p), C \in \mathrm{Mat}(m \times n)$ are real matrices, x, u, y are functions of a real variable t with values in $\mathbf{R}^n, \mathbf{R}^p$ and $\mathbf{R}^m$, respectively. The functions x, u, y are called the state, input and output, respectively.

A static output feedback is an equation

$$u = Ky \tag{3}$$

where $K \in \mathrm{Mat}(p \times m)$.

Controlling the system by a static output feedback means that equations (1),(2) and (3) are combined. Then y and u can be eliminated and we obtain

$$\dot{x} = (A + BKC)x, \tag{4}$$

which is called the closed loop system. The eigenvalues of this closed loop system are the roots of the characteristic polynomial

$$\phi_K(z) = \det(zI - A - BKC).$$

The *pole placement problem* is: *for given* A, B, C *and given set* $\{z_j\}_{j=1}^n \subset \mathbf{C}$, *symmetric with respect to the real line, to find real* K *so that the polynomial* ϕ_K *has roots* z_j.

A less ambitious (and more important for engineering applications) is the *stabilizability problem*: *for given* A, B, C, *to find* K *so that all roots of* ϕ_K *lie in the left half-plane.*

So the pole placement is equivalent to solving a system of equations with respect to K

$$\phi_K(z_j) = 0, \quad 1 \leq j \leq n. \tag{5}$$

If this system is underdetermined, that is $n < mp$, it always has solutions for generic A, B, C. This result of A. Wang is non-trivial because we are looking for *real* solutions; existence of complex solutions is much easier.

Of course, this implies that a generic system of dimensions m, n, p with $mp > n$ is stabilizable.

From now on we assume that $n = mp$. In this case, the following is known. If m and p are both even, there is an open set of systems (A, B, C) for which the pole placement is unsolvable for some choice of $\{z_j\}$. This is the result of [2].

If $m = p = 2$ then there is an open set of systems (A, B, C) which are not stabilizable [1].

The cases $m = 1$ and $p = 1$ are special; in these cases the system of equations (5) is linear, and pole placement is possible for a generic system.

Question. For which m and p the generic system with $n = mp$ is stabilizable?

We recall a representation of ϕ_K which permits to give a geometric interpretation to our question. Consider the rational matrix $C(zI - A)^{-1}B$ which is called the transfer function of the system (1), (2). There exists a factorization:

$$C(zI - A)^{-1}B = D^{-1}(z)N(z), \quad \det D(z) = \det(zI - A),$$

where D and N are polynomial matrices of sizes $m \times m$ and $m \times p$, respectively. Using this factorization and the well-known property

$$\det(I - AB) = \det(I - BA)$$

for any rectangular matrices for which both sides are defined, we obtain

$$\begin{aligned}
\phi_K(z) &= \det(zI - A - BKC) = \det(zI - A)\det(I - (zI - A)^{-1}BKC) \\
&= \det(zI - A)\det(I - C(zI - A)^{-1}BK) \\
&= \det(zI - A)\det(I - D^{-1}(z)N(z)K) = \det(D(z) - N(z)K).
\end{aligned}$$

So the condition $\phi_K(z) = 0$ can be written as

$$\begin{vmatrix} D(z) & N(z) \\ K & I \end{vmatrix} \tag{6}$$

which reveals that the pole placement equations (5) is a Schubert problem: each equation (5) says that the p-space spanned by the rows of $[K, I]$ intersects n given m-spaces spanned by the rows of $[D(z_j), N(z_k)]$.

When we consider generic solvability, it is useful to compactify both the set of the systems considered and the set of admissible feedbacks. Instead of $[D, N]$ we consider an arbitrary $m \times (m+p)$ polynomial matrix $V(z)$ with the following property:

The $m \times m$ minors do not have a common factor, and their maximal degree is n.

We denote the set of such matrices V by $Q(m, m+p)$. (Such matrices correspond to the so-called "autoregressive systems" [3]. In other context Q is known as a "quantum Grassmannian". This is just the set of rational maps from the projective line to the Grassmannian $G(m, m+p)$ of degree $n = mp$. This degree is the same as the degree of the curve in the projective space obtained by the Plücker embedding of the Grassmannian).

Instead of $[K, I]$ we consider an arbitrary element L of the Grassmannian $G(p, m+p)$, and the pole placement problem becomes

$$\begin{vmatrix} V(z) \\ L \end{vmatrix} = c\phi(z).$$

where ϕ is a given polynomial and $c \neq 0$. The pole placement problem is generically solvable if the pole placement map

$$L \mapsto \begin{vmatrix} V(z) \\ K \end{vmatrix}, \quad G(p, m+p) \to \mathbf{P}^n \tag{7}$$

is surjective for every $V \in Q(m, m+p)$.

Now we restate in the similar way the stabilizability problem.

A system V as above is called *degenerate* if there is $L \in G(p, m+p)$ such that the determinant in (7) is identically equal to 0. Anderson and Byrnes [1] proved the following:

For given m and p (and $n = mp$), the generic system is stabilizable if and only if for every non-degenerate $V \in Q(m, m+p)$ the equation (7) with $\phi(z) = z^{mp}$ has a real solution.

They also gave the following counterexample for $m = p = 2$:

$$V(z) = \begin{pmatrix} z^2 & 1 & z & 0 \\ z+1 & z^2 & 1 & z \end{pmatrix}.$$

It is easy to see that $\deg V = 4$, and that V is non-degenerate. Equation

$$\det \begin{pmatrix} V(z) \\ L \end{pmatrix} = cz^4$$

can be explicitly solved with respect to L and the conclusion is that it has no real solutions.

I am aware of no regular procedure which would permit to find such examples. The question is for which m and p they exist.

The theorem of Anderson and Byrnes is easy to explain. Consider the map (7) as a map

$$Q(m, m+p) \times G(p, m+p) \to \mathbf{P}^{mp}. \tag{8}$$

where $\mathbf{P}^{mp}$ is the set of non-zero polynomials of degree mp modulo proportionality. The map is well defined on the non-degenerate subset of $Q(m, m+p)$ which is open and dense. We have the following $SL(2)$ action on the polynomials P of degree at most k:

$$P(z) \mapsto (cz+d)^k P\left(\frac{az+b}{cz+d}\right), \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1.$$

This action naturally extends to $Q(m, m+p)$. It is easy to see that for every L our map (8) splits these two actions. So if for every V there is L such that $\phi_L(z) = z^{mp}$, then for every V there is L such that $\phi_L(z) = (z+1)^{mp}$, so every V is stabilizable.

Now suppose that every V is stabilizable. This means that for every V there exists L such that ϕ_L has all zeros in the left half-plane. Then, by $SL(2, \mathbf{R})$ action we conclude that for every V there is L such that all zeros of ϕ_L belong to a given circle centered on the real line. By passing to the limit (all our manifolds are compact!) we can move all zeros of ϕ_L to the point 0.

References

[1] C. Byrnes and B. Anderson, Output feedback and generic stabilizability, SIAM J. Control and Optimization, 22, 3 (1984) 362–380.

[2] A. Eremenko and A. Gabrielov, Pole placement by static output feedback for generic linear systems, SIAM J. Control and Optimization, 41, 1 (2002) 303–312.

[3] A. Wang, Pole placement by static output feedback, J. Math. Syst. Estim. Control, 2 (1992) 205-218.