Backward uniqueness for the heat equation

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Let D be a region in $\mathbf{R}^n$ with regular boundary for the Dirichlet problem. Consider two properties of D:

PI. *There exists a bounded temperature* $u(x,t) \not\equiv 0$

$$u_t = \Delta u, \quad (x,t) \in D \times [0,1],$$

with the property

$$u(x,1) \equiv 0, \quad x \in D.$$

PII. *There exists a harmonic function* $v(x) \not\equiv 0$ *in* D *with the property*

$$v(x) = O(\exp(-|x|^2)), \quad x \to \infty.$$

According to Gurarii and Matsaev [5], these properties are equivalent, but no proof was ever published.

Here is a summary of known results.

When $n = 1$, both properties hold if and only if D is a bounded interval. For PII this is evident, for PI this follows from the results of Tychonov [7].

When $n = 2$, and D is an angular sector $\{z : |\arg z| < \alpha\}$, the property PII holds if and only if $\alpha < 45°$. Recently, Escauriaza showed that for such sectors PI also holds. He actually proved that PII implies PI for cones. If v is a harmonic function in a cone satisfying PII, then

$$u(x,t) = (t-1)^{-n/2} \exp(-|x|^2/(4(t-1)))v(x/(t-1)), \quad (x,t) \in D \times (0,1)$$

is a temperature with the property PII. This formula is a special case of the *Appell transform* [4].

In the opposite direction, Li and Sverak [6] showed that PI cannot hold in cones of revolution (in any dimension) with opening angle greater than $2\arccos(1/\sqrt{3}) \approx 109.52^\circ$.

The equivalence of properties PI and PII would be useful because PII is much easier to verify for a given region. In dimension 2, Phragmén–Lindelöf theorems give quite general geometric conditions for PII. If PII holds on a region $D_0 \subset \mathbf{R}^2$, then it also holds for the wedge regions $D_0 \times \mathbf{R}^{n-1} \subset \mathbf{R}^n$. It is known that PII holds for cones of revolution in R^n if and only if the opening angle is less than 90°. For this and further results on the property PII see [1, 2, 3].

References

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