Let $f$ be a rational function of degree at least two. Consider Jordan analytic curves $C$ which are invariant under $f$. The simplest example of such a curve is a circle. If $f$ has an invariant circle $C$, one can conjugate $f$ by a linear-fractional transformation which sends $C$ to the extended real line $\mathbf{R} \cup \{\infty\}$. Rational functions which map the real line into itself are real rational functions $f(\overline{z}) = \overline{f(z)}$.

What are the other possibilities? If $f$ has a rotation domain (a Siegel disk or an Herman ring), then there is an analytic function $\phi$ in this domain which conjugates $f$ with an irrational rotation. Then the level lines $\{z : |\phi(z)| = c\}$ are analytic invariant curves.

**Question 1.** *Does there exist a rational function with an analytic invariant curve on which $f$ is topologically conjugate to an irrational rotation, and such that $C$ is neither a circle nor a level curve of a linearizer of a rotation domain?*

Such curves do not exist for polynomials or rational functions [1]. A Jordan curve $C$ is called a *degenerate Herman ring* if it is

a) contained in the Julia set,

b) is neither a circle nor a boundary component of a rotation domain, and

c) $f$ is conjugate to an irrational rotation on $C$.

There exist smooth degenerate Herman rings [6]. Question 1 asks whether there exist analytic degenerate Herman rings. Many non-trivial degenerate Herman rings (which are not smooth) are constructed in [4].

**Question 2.** (Bergweiler) *Is the number of degenerate Herman rings finite? Can it be estimated in terms of degree of $f$?*

Now we turn to invariant curves such that the restriction $f : C \to C$ is not one-to-one.

**Theorem [2].** *Let $C$ be an analytic invariant curve of a rational function $f$, and suppose that $f : C \to C$ is not a homeomorphism, and there is a repelling fixed point of $f$ in $C$. Assume in addition that $C \subset J(f)$ and $C$ contains no critical points or rational fixed points of $f$. Then either $f$ is a Latté function or $C$ is algebraic.*

Examples of the first possibility were constructed in [2], and of the second possibility in [5].

**Question 3.** Which conditions of this theorem can be removed?

# References


[1] Y. Azarina, Invariant analytic curves for entire functions, Siberian Math. J. 30 (1989) 349–353.

[2] A. Eremenko, Invariant curves and semiconjugacies of rational functions, Fund. Math., 219 (2012) 3, 263–270.

[3] P. Fatou, Sur les équations fonctionnelles, Troisième Mémoire, Bull. Soc. Math. France, 48 (1920) 208–314.

[4] Willie Rush Lim, A priori bounds and degeneration of Herman rings with bounded rotation number, arxiv:2302.07794.

[5] Peter Müller, Decompositions of rational functions over real and complex numbers and a question about invariant curves, Illinois J. Math., 59 (2015) 4, 825–838.

[6] Fei Yang, Rational maps with smooth degenerate Herman rings, arXiv:2207.06770.