A problem of B. Ya. Levin

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If all zeros of a polynomial P lie in the (closed) lower half-plane, then by the Gauss–Lucas theorem zeros of all derivatives $P^{(n)},\ n \geq 0$ also lie in the lower halfplane.

Consider the closure F of this set of polynomials, that is the set of entire functions which can be approximated (uniformly on compact subsets) by polynomials with zeros in lower half-plane. This class F is closed under differentiation.

Now suppose that we have an entire function with the property that all zeros of all derivatives belong to the lower half-plane. Does it follow that $f \in F$, or

$$f(z) = ce^a z, \quad \text{or} \quad f(z) = c(e^{ibz} - e^{id}), \tag{1}$$

where c, a are complex and b, d are real?

If the lower half-plane is replaced by the real line, we obtain the Laguerre–Pólya class LP instead of F. It is known [1] that for an entire function,i f the condition that all zeros of $ff'f''f'''$ are real implies that $f \in LP$ or has one of the forms (1).

But for the original problem any finite number of derivatives is not enough.

References

[1] S. Hellerstein, L-C Shen and J. Williamson, Reality of zeros of an entire function and its derivatives, TAMS, 275 (1983), 319–331.