Some constants coming from the work of Littlewood

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1. Let

$$\phi(n)=\sup_{\deg p=n}\int_{|z|<1}\frac{|p'|}{1+|p|^2}dm,$$

where the sup is taken over all polynomials of degree n and dm stands for the area element. Littlewood [11] gives several equivalent definitions of $\phi(n)$. He observed that $\phi(n)\leq\pi\sqrt{n}$. Let

$$\alpha=\limsup_{n\to\infty}\frac{\log\phi(n)}{\log n}.$$

Littlewood's conjecture that $\alpha<1/2$ was proved by Lewis and Wu [10], who showed that in fact $\alpha<2^{-1}-2^{-264}$. These authors used an approach from [7, 8] where a weaker result was obtained. Early lower estimates of α are due to Paley and Hayman, but only in [5] it was proved that $\alpha>0$. The rigorous numerical estimate $\alpha>10^{-5}$ is due to Baker and Stallard [1]. Using a computer, Kraetzer [9] showed that in fact the method proposed in [5] gives $\alpha>0.242$. This confirms an earlier computation of Carleson and Jones [3].

2. Let E be a regular compact subset of the plane and G the Green function of $\overline{\mathbf{C}}\backslash K$ with the pole at infinity. For every $\epsilon>0$, let $l(\epsilon)$ be the length of the level curve $\{z:G(z)=\epsilon\}$. Put

$$\beta_E=\limsup_{\epsilon\to 0}\frac{\log l(\epsilon)}{-\log\epsilon},$$

and

$$\beta=\sup\beta_E$$

over all *connected* compact sets. It is known that $.17 < \beta < .49$. The upper bound is due to Clunie and Pommerenke [4] and the lower bound to Pommerenke [12]. The problem of estimating β comes from Littlewood's work on univalent functions. He proved that $\beta > 0$. Computer experiments of Carleson and Jones [3] and Kraetzer [9] indicate that $\beta > .242$, which seems to confirm the conjecture Carleson and Jones that $\beta = 1/4$.

One can ask the same question about $\sup \beta_E$ over all regular compact sets, not necessarily connected. I conjecture that the sup is attained on connected sets. Paper [6] gives some evidence of this.

3. All lower estimates in [1, 3, 5, 9] are all based on iteration theory, more precisely on "thermodynamic formalism". Let $p_c(z) = z^2 + c$ be a hyperbolic polynomial ("hyperbolic" means that the trajectory of 0 under the iterates of p_c tends to an attracting cycle, possibly to infinity). Denote the n-th iterate of p_c by p_c^n. The number

$$P_c = \lim_{n\to\infty} \frac{1}{n} \log \sum_{z:p_c^n(z)=1} \|(p_c^n)'|^{-1}$$

is called the *pressure, corresponding to* $|(p_c^n)'|^{-1}$. The existence of the limit and positivity of P_c for $c \neq 0$ was shown by Ruelle [13] who also proved that $c \mapsto P_c$ is a real analytic function on every component of the set of parameters c where p_c is hyperbolic. It can be shown [5] that $\alpha \geq P_c/\log 2$ for all such c. Similarly, it is shown in [3] that $\beta \geq P_c/\log 2$ for all c such that the trajectory of 0 tends to a finite attracting cycle.

4. I do not know of any evidence in favor or against the Carleson and Jones conjecture, except the computer experiments mentioned above. But there are several questions which seem to be easier:

A. Is there any connection between α and β? Is it true that $\alpha = \beta$?

B. How to obtain better estimates of α, β and $\sup_c P_c$, even with the help of a computer?

C. Are the polynomials P_c^n extremal or nearly extremal for α?

D. For which c is the pressure P_c close to its supremum?

Remark added on March 17 2015 Beliaev and Smirnov [2] claim that they proved $\alpha = \beta$. However their argument is based on an unpublished result of Binder and Jones, which is still not available (March 2015).

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