Malmquist's theorem on algebroid solutions of first order differential equations

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A function $y(z)$ of a complex variable z is called an n-valued *algebroid* function if it satisfies an algebraic equation

$$a_n(z)y^n + \ldots + a_1(z)y + a_0(z) = 0,$$

where a_j are entire functions, $a_n \neq 0$. If y is not algebraic it is called transcendental.

In 1941 Johannes Malmquist published the following theorem [6]:

Theorem 1. *Let k be the field of all algebraic functions (the algebraic closure of $\mathbf{C}(z)$), and $F \in k[t_1, t_2]$ an irreducible polynomial. If the differential equation*

$$F\left(\frac{dy}{dz}, y\right) = 0 \tag{1}$$

has a transcendental n-valued algebroid solution y, then:

either there exists a polynomial $G \in k[t_1, t_2]$, with $\deg_{t_1} G = n$, and an algebroid function w solving

$$\frac{dw}{dz} = aw^2 + bw + c, \quad \textit{where} \quad a, b, c \in k, \tag{2}$$

such that $G(y, w) = 0$,

or there exists a polynomial $G_1 \in k[t_1, t_2, t_3]$ with $\deg_{t_1} G_1 = n$, $deg_{t_2} G_1 = 1$, and an algebroid function w solving

$$\left(\frac{dw}{dz}\right)^2 = aP(w), \quad \textit{where} \quad a \in k, \quad P \in \mathbf{C}[t], \quad \deg P = 3. \tag{3}$$

such that $G(y, y', w) = 0$.

For example, the equation

$$2y' + y + y^3 = 0$$

has a 2-valued algebroid solution $y = 1/\sqrt{e^z - 1}$; by the change of the variable $w = 1/y^2$ the equation is reduced to

$$w' = w + 1,$$

which of of type (2).

This result contains two special cases published by Malmquist earlier:

In 1913 he proved the special case for the equation (1) of first degree in y', that is of the form [4]:

$$y' = R(y), \quad R \in k(t).$$

In this case only the first possibility (2) can hold, and $w = y$.

In 1920 he proved the special case $n = 1$ for general equation (1), [5].

I know only two references on [6] in the literature: in [8], the authors after mentioning [5] write "see also [6]" with no other comments, and in [9] the authors mention [6] only to write that "The classical proofs of Malmquist are, however, incomprehensible for the modern reader."

On the other hand, [5] is well-understood: the first proof independent of Malmquist's paper was given in [2], and two other proofs in [8] and [1].

In the paper [3] a generalization of [4] is given (still with $n = 1$). Most of the literature where the name "Malmquist Theorem" occurs is concerned with the intersection of [4] and [5], that is the case when $n = 1$ and the equation is of the form (2). A survey of this literature is contained in [2].

The case $k = \mathbf{C}$ of Theorem 1 is also interesting. Then the differential equation is

$$F(y', y) = 0, \quad F \in \mathbf{C}[t_1, t_2], \tag{4}$$

and the inverse function of y is an Abelian integral. So the Theorem 1 says for this case that

If an Abelian integral has an algebroid inverse, then this inverse is either an algebraic function, or an algebraic function of the exponential or an algebraic function of an elliptic function.

Since equation (4) is autonomous, its general solution is of the form $y(z+c)$, where y is a particular solution. So if we have a transcendental algebroid solution, the general solution is also algebroid, and a theorem of Painlevé [7, Introduction, 8] applies.

An alternative proof can be obtained by considering periods of the function y satisfying (4). Unfortunately none of these two proofs of the special case apply to the general case.

References

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