Composite periodic entire functions

A. Eremenko

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This question goes back to A. and C. Renyi [1].

Let f, g be entire functions such that $f(g)$ is periodic. What can one say about g?

Here are some possibilities:

a) g is periodic,

b) $g(z+T) = g(z) + K$ and f is K-periodic.

c) g is a polynomial of degree 2. For example, $g(z) = z^2$ and $f(z) = \cos\sqrt{z}$.

d) $g = P \circ h$, where P is a quadratic polynomial, and $h(z+T) = h(z)+K$, such that $f \circ P$ has period K.

It is conjectured that a), b), c), d) essentially exhaust all possibilities. A. and K. Renyi proved this under the additional assumption that either f or g is a polynomial.

In the 1980s, E. Gleizer proved this under the additional condition that f and g are real entire functions. This proof was seen and checked by several people, including myself, but unfoirtunately it is lost. Gleizer has not published it and does not remember it, no manuscript survived, and those people who checked the proof are either dead or do not remember it.

In 1988, Gaida [2] published an announcement of a complete proof of the conjeture, but no proof was ever published.

References

[1] A. Renyi and C. Renyi, Some remarks on periodic entire functions, J. Analyse Math. 14 (1965), 303–310.

[2] Yu. Gaida, Monodromy groups of mappings of a Riemann surface and compositions of meromorphic functions, Dokl. Akad. Nauk Ukrain. SSR Ser. A 1988, no. 10, 6–9, 84.