Problems of potential theory arising in the value distribution theory of meromorphic functions

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1. Let μ be a positive measure in the plane, satisfying

$$\mu(\{z : |z| \leq r\}) \leq cr^{\alpha}, r \geq 0,$$

where $0 < \alpha < 1/2$ and $c > 0$ are some constants. Is it possible that the subharmonic function

$$u(z) = \int \log \left| 1 - \frac{z}{\zeta} \right| d\mu_{\zeta}$$

is locally constant on an open set D which intersects every circle $|z| = r$?

Positive or negative solution of this problem will imply a solution of the old question of W. H. J. Fuchs: is it true that $\delta(0, f'/f) = 0$ for every entire function f of order less than $1/2$?

2. Let u_k be a sequence of subharmonic functions in the unit square $\{x+iy : |x| < 1, |y| < 1\}$, and suppose that $u_k(x, y) \to x$ uniformly. Consider the level set $D_k = \{z : u_k(z) < 0\}$. This set has a "large connected component", D_k^*, the one that contains $-1/2$. Is it possible that $D_k \backslash D_k^*$ intersects every horizontal segment $[-1 + it, 1 + it]$ for $-1 < t < 1$?

Positive or negative solution will imply a solution of an old problem of Edrei and Fuchs: can en entire function with large sum of deficiencies have infinite number of deficient values?

3. Let u be a subharmonic function in the ring $1 < |z| < 2$. Let $D_k, 1 \leq k < \infty$, be disjoint open sets, each of them is a union of components of the level set $\{z : u(z) < 0\}$. Suppose that for every $r \in (1, 2)$ and every k we have

$$\int_{\theta : re^{i\theta} \in D_k} u(re^{i\theta}) d\theta \leq -\delta_k.$$

What can be said about the rate of decrease of the sequence δ_k?

Some known results. Eremenko constructed an example with $\delta_k > c^k$ with some $c \in (0, 1)$ thus disproving a conjecture of Arakelyan. In the opposite direction, Lewis and Wu proved that

$$\sum_k \delta_k^\alpha < \infty$$

with some universal constant $\alpha < 1/3$. It is desirable to close the gap between these results.

References

[1] A. Eremenko, A counterexample to the Arakelyan conjecture. Bull. Amer. Math. Soc. 1992, vol. 27, N 1, p. 159-164.

[2] J. Lewis and J-M. Wu, On conjectures of Arakelyan and Littlewood, J. d'Analyse, 50 (1988).