Possible shapes of spherical quadrilaterals

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A quadrilateral is a bordered surface homeomorphic to the closed disc, with 4 marked boundary points called vertices, and equipped with a Riemannian metric of constant curvature $K \in \{0, 1, -1\}$, such that the sides (the arcs between the vertices) are geodesic. So the metric has the so-called conic singularities at the vertices.

Let $\pi\alpha_j > 0$ be the interior angles at the vertices.

When $K = 0$, we know that the angles must satisfy $\sum \alpha_j = 2$, and there exists a unique up to scaling quadrilateral with prescribed angles and prescribed conformal modulus. Existence and uniqueness are immediate consequences of the Schwarz–Christoffel formula.

Similar result holds for $K = -1$. The necessary and sufficient condition is $\sum \alpha_j < 2$ and for every prescribed angles satisfying this condition and every prescribed conformal modulus there exists a unique hyperbolic quadrilateral.

The question is what happens when $K = 1$. Gauss–Bonnet theorem implies that

$$\sum \alpha_j > 2.$$

For which α_j satisfying this condition a spherical quadrilateral exists? Luo and Tian [10] proved that under the additional restriction $0 < \alpha_j \leq 1$, one has existence and uniqueness of the spherical quadrilateral with prescribed angles and conformal modulus.

However, when larger angles are allowed, there are additional conditions on the angles, and in general there is no uniqueness.

For spherical triangles the question has been completely solved using hypergeometric functions [1, 7]. A complete answer is also known for n-gons when all α_j are integers [11, 2, 3].

Other partial results concern the case when at least one α_j is an integer [4, 5, 6].

Spherical quadrilaterals whose sides are not necessarily geodesic but each side has constant geodesic curvature where studied by Ihlenburg [8, 9] in great detail. However a complete classification up to isometry was not obtained. See also [12].

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