Reversion of Cartan's Second main theorem

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Let $f:\mathbf{C}\to\mathbf{P}^n$ be a holomorphic curve with a homogeneous representation $(f_0,\ldots,f_n)$ where we suppose that f_j are entire functions without zeros common to all of them. Let $\|f\|=\sqrt{|f_0|^2+\ldots+|f_n|^2}$, and $u=\log\|f\|$. The Cartan - Nevanlinna characteristic is

$$T(r,f)=\frac{1}{2\pi}\int_0^{2\pi}u(re^{i\theta})d\theta-u(0).$$

If E is a projective subspace (of any dimension), we denote by $d(f(z),E)$ the distance from $f(z)$ to E, using the Fubini–Study metric. For example, if E is a hyperplane defined by the equation

$$a_0w_0+\ldots+a_nw_n=0,$$

then

$$d(f(z),E)=\frac{|a_0f_0(z)+a_1f_1(z)+\ldots+a_nf_n(z)|}{\|f\|\|a\|}.$$

If E has codimension k then $E=E_1\cap\ldots\cap E_k$, where E_j are hyperplanes, and we have for the distance

$$\log d(f(z),E)=\max_j\{\log d(f(z),E_j)\}+O(1),$$

where the $O(1)$ term is estimated by a constant depending only on dimension, if we choose the E_j orthogonal to each other.

Let A be a complete flag, $A=(A^{(0)}\subset A^{(1)}\subset\ldots\subset\mathbf{P}^n)$. Then the distance from $f(z)$ to this flag is defined by

$$\log d(f(z),A)=\sum_{j=0}^{n-1}\log d(f(z),A^{(j)}).$$

Following Yamanoi, we define

$$m_p(r,f)=\max_{\{A_1,\ldots,A_p\}}\frac{1}{2\pi}\int_0^{2\pi}\max_{1\leq k\leq p}\log\frac{1}{d(f(re^{i\theta}),A_k)}d\theta.$$

Suppose now that f is linearly non-degenerate, denote by W the Wronskian determinant of $(f_0,\ldots,f_n)$ and let $n_1(r,f)$ be the number of zeros of W in the disk $|z|\leq r$, counting multiplicity. Then we define

$$N_1(r,f)=\int_0^r\frac{n_1(t,f)-n_1(0,f)}{t}dt+n_1(0,f)\log r.$$

Conjecture. For every linearly non-degenerate curve we have

$$m_{\phi(r)}(r,f)+N_1(r,f)=(n+1+o(1))T(r,f),$$

when r avoids some exceptional set of finite logarithmic measure. Here $\phi(r)=[\log T(r,f)]^N$, or perhaps some other function of much smaller growth than $T(r,f)$.

When $n=1$, this is a special case of Yamanoi's theorem [2, Thm. 1]. In [1], the special case is proved when $f_0,\ldots,f_n$ are linearly independent solutions of any linear differential equations of the form

$$w^{(m)}+a_{m-1}w^{(m-1)}+\ldots+a_0w=0$$

with polynomial coefficients.

The inequality

$$m_p(r,f)+N_1(r)\leq(n+1+o(1))T(r,f)$$

is equivalent to Cartan's second main theorem (see [1]).

References

[1] A. Eremenko, On the second main theorem of Cartan, Ann. Acad. Sci. Fenn., 39 (2014) 895-871; Correction: 40 (2014) 503–506. Corrected version: arXiv:1409.4850.

[2] K. Yamanoi, Zeros of higher derivatives of meromorphic functions in the complex plane. Proc. Lond. Math. Soc. (3) 106 (2013), no. 4, 703–780.