A polynomial equation in infinitely many variables

Let N be a positive integer, (z_k) a sequence of complex numbers tending to zero, and

$$\sum_k z_k^n = 0 \quad \text{for every integer} \quad n > N,$$

where the series are absolutely convergent. Prove that all $z_k = 0$.

This can be generalized in the following way: Consider the system of equations with respect to complex numbers z_k and w_k:

$$\sum_k w_k z_k^n = a_n \quad \text{for every integer} \quad n > N.$$

If this system has a solution with *different* $z_k \to 0$, and such that the series converge absolutely, then such solution is unique.

In the case of finitely many variables, this is called the Sylvester–Ramanujan system, see for example,

Yuri I. Lyubich, The Sylvester–Ramanujan system of equations and the complex power moment problem, *Ramanujan J.* 8 (2004) 23–45.