A conjecture about meromorphic functions in the plane

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Let $f : \mathbf{C} \to \overline{\mathbf{C}}$ be a non-constant meromorphic function. How many disjoint simply connected regions $G_k \subset \overline{\mathbf{C}}$ can exist, such that each preimage $f^{-1}(G_k)$ is connected ?

If f has a limit $f(\infty)$, that is f is rational, the number of such regions can be at most 2. Indeed, extend f to $\overline{\mathbf{C}}$, and let $D_k = f^{-1}(G_k) \subset \overline{\mathbf{C}}$ be connected. Let f_k be the restrictions of f on D_k. Then f_k are ramified coverings of degree $d = \deg f$. If some G_k is the sphere, then evidently $k = 1$. Otherwise, by the Riemann–Hurwitz relation,

$$\chi(D_k) = d - I_k,$$

where I_k is the number of critical points in D_k. If D_k is the sphere then $k = 1$. Otherwise $\chi(D_k) \leq 1$, so $I_k \geq d - 1$, and we obtain that $k \leq 2$ because the total number of critical points is $2d - 2$.

The same is true if we assume (instead of the existence of a limit $f(\infty)$) that there is a finite set $A \subset \overline{\mathbf{C}}$ such that the restriction

$$f : \mathbf{C} \backslash f^{-1}(A) \to \overline{\mathbf{C}} \backslash A$$

is an unramified covering [2, 3].

I conjecture that $k \leq 2$ is true for all meromorphic functions.

In the case $f : \mathbf{C} \to \mathbf{C}$ and $G_k \subset \mathbf{C}$, $k \leq 1$ holds [1].

References

[1] I. N. Baker, Completely invariant domains for entire functions, in the book: H. Shankar, ed., Math Essays dedicated to A. J. Macintyre, Ohio Univ. press, Athens, OH, 1970, 33–35.

[2] I. N. Baker, J. Kotus and Y. Lü, Iterates of meromorphic functions, Ergodic Theory and Dynamical Systems, 11 (1991) 241–248.

[3] W. Bergweiler and A. Eremenko, Meromorphic functions with two completely invariant domains, in the book: Ph. Rippon and G. Stallard, eds, Transcendental Dynamics and Complex Analysis, p. 74-89 (LMS Lect. Notes Ser., 348) Cambridge UP, 2008.