On metrics of curvature 1 with four conic singularities on tori and on the sphere

Alexandre Eremenko[*] and Andrei Gabrielov[†]

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Abstract

We discuss conformal metrics of curvature 1 on tori and on the sphere, with four conic singularities whose angles are multiples of $\pi/2$. Besides some general results we study in detail the family of such symmetric metrics on the sphere, with angles $(\pi/2, 3\pi/2, \pi/2, 3\pi/2)$.

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1 Introduction

The general problem studied here concerns conformal Riemannian metrics of constant positive curvature with conic singularities on a compact Riemann surface S. For the history of the problem and its relevance to mathematics and physics we refer to [30, 24, 28, 23, 15, 11, 12, 25]. The goal is to classify such metrics up to isometry, in particular *to determine for a given Riemann surface, how many such metrics exist with prescribed singularities and prescribed angles at each singularity.*

Analytically the problem consists in the study of the Liouville equation

$$\Delta u + e^{2u} = 0 \tag{1}$$

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on the punctured Riemann surface, with prescribed singularities at the punctures. If $\rho(z)|dz|$ is the length element of the metric in a local conformal coordinate, then $\rho = e^u$, and equation (1) expresses the fact that the curvature equals 1. The behavior at a singularity $a \in S$ is $\rho(z) \sim c|z|^{\alpha-1}$, where $c > 0$, z is the local coordinate which is 0 at a, and $2\alpha > 0$ is the angle at the singularity.

Here and in what follows we measure the angles at the singularities in *half-turns*, for example "integer angle" means an integer multiple of π radians.

An important special case is that of *symmetric* metrics, that is metrics which are invariant with respect to an anti-conformal involution which leaves all singularities fixed. In the case of the sphere, such metrics are in one-to-one correspondence with spherical polygons [12]. So the problem in this case consists in classification and enumerating spherical polygons of prescribed conformal type with prescribed angles at the corners.

Every surface of constant curvature 1 is locally isometric to a piece of the standard sphere. This isometry is analytic in conformal coordinates, so it admits an analytic continuation along every path not passing through the singularities, and we obtain a multi-valued analytic function

$$f : S \backslash \{a_0, \ldots, a_{n-1}\} \to \overline{\mathbf{C}},$$

where a_j are the singularities, and $\overline{\mathbf{C}}$ is the Riemann sphere. This function f is called the *developing map*, and the length element of the metric is expressed in the form

$$\rho(z)|dz| = \frac{2|f'||dz|}{1+|f|^2}.$$

Monodromy of the developing map consists of linear-fractional transformations (rotations of the sphere), so the Schwarzian derivative

$$\mathbf{S}[f] := \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

is single valued on $S\backslash\{a_0, \ldots, a_{n-1}\}$. The behavior of f near the conic singularities a_j implies that $\mathbf{S}[f]$ has double poles at these points, with principal parts related to the angles (see below). In the case of the sphere, $\mathbf{S}[f]$ is a rational function R, and we obtain a Schwarz differential equation $\mathbf{S}[f] = R$. General solution of this Schwarz equation is a ratio of two linearly independent solutions of the linear differential equation with regular singular points

$$w'' + \frac{1}{2}Rw = 0. \tag{2}$$

In the case of the sphere, R in this equation is a rational function with poles of order 2 at the singularities, and the coefficients at $(z-a_k)^{-2}$ are determined by the angles. The exponent differences at the singular points are the angles. After a normalization $n-3$ free parameters remain which are called *accessory parameters*. In terms of this equation, our general problem is equivalent to the following:

For given singular points and the exponent differences, how many values of accessory parameters exist, so that the monodromy group of (2) is conjugate to a subgroup of $PSU(2)$.

In the case of three singularities, there is no accessory parameter, the equation (2) is hypergeometric, and the problem was completely solved for this case in [19, 7, 15]. The case of four singularities corresponds to Heun's equation which has one accessory parameter. The assumption that the angles are half-integer leads to the special case of the Heun equation, the Darboux–Treibich–Verdier equation [32], which is easier to study because its general solution is meromorphic, and can be expressed in terms of Abelian integrals.

For $n \geq 4$, the existing results show that the general problem described above is very difficult, see, for example, the recent study of metrics with one singularity of integer angle on a torus [23, 4, 5], or classification of such symmetric metrics with four angles, at least one of them integer, on the sphere [11, 12, 13].

In this paper we study new special cases. We restrict ourselves to the cases of four singularities with integer angles on tori and half-integer angles on the sphere.

In the first part of this paper we establish a relation between the metrics on the sphere and the metrics on tori, and restate the results obtained in [10, 11, 12, 13]. In the second part we study the simplest family which is not covered by the existing results, metrics on "rectangular" tori with angles $(1, 3, 1, 3)$. By the results of the first part, this is equivalent to the study of symmetric metrics on the sphere with angles $(1/2, 3/2, 1/2, 3/2)$.

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2 Metrics on the sphere and on tori

We consider a metric of curvature 1 on a torus

$$\mathbf{T} = \mathbf{C}/\Lambda, \quad \Lambda = \{m\omega_1 + n\omega_3 : m, n \in \mathbf{Z}\}, \quad \tau = \omega_3/\omega_1, \quad \operatorname{Im}\tau > 0,$$

with four conic singularities at half-periods, $\omega_j/2,\ 0 \leq j \leq 3$, where $\omega_0 = 0,\ \omega_2 = \omega_1 + \omega_3$, and the metric has an integer angle at each singularity. Let the angles be $m_j \in \mathbf{N}$. Then the developing map F must satisfy the Schwarz equation

$$\mathbf{S}[F] = -2\sum_{j=0}^{3} n_j(n_j+1)\wp(z-\omega_j/2) + \lambda.$$

with $n_j = (m_j - 1)/2$, as is explained below. The general solution is a ratio of two linearly independent solutions of the equation

$$w'' - \left(\sum_{j=0}^{3} n_j(n_j+1)\wp(z-\omega_j/2) + \lambda\right) w = 0. \tag{3}$$

The exponents at the singularities of (3) are determined from the characteristic equation

$$\rho(\rho-1) - n_j(n_j+1) = 0,$$

whose solutions are $\rho = 1/2 \pm (n_j + 1/2)$, so the difference of the exponents is $m_j = 2n_j + 1$, and these are our prescribed angles of the metric on $\mathbf{T}$.

Monodromy of the developing map cannot contain a parabolic transformation, so there should be no logarithms in the local solutions of (3) near the singularities, and equation (3) must have trivial monodromy at $\omega_j/2$. In other words, all singularities are *apparent*.

Then the monodromy coming from the fundamental group of the torus must be conjugate to an Abelian subgroup of $PSU(2)$. There are only two kinds of such groups [20]: the Klein Viergroup $(\mathbf{Z}/2\mathbf{Z})^2$ and a cyclic group (finite or infinite). By a post-composition of F with an element of $PSU(2)$, which does not change the metric, we can achieve that F satisfies one of the following:

$$F(z+\omega_1) = -F(z), \quad F(z+\omega_3) = 1/F(z), \tag{4}$$

or

$$F(z+\omega_j) = e^{2\pi i\alpha_j}F(z), \quad j = 1, 3. \tag{5}$$

We call two metrics *equivalent* if their developing maps are related by a linear-fractional transformation. Thus in the case (5) each metric is a member of a one-parametric family of equivalent metrics with developing maps kF, $k \neq 0$.

Theorem 1. *In case (4) the metric is even, that is invariant under $z \mapsto -z$, while in case (5) each equivalence class contains an even metric.*

This generalizes results from [4], sections 2.1 and 5.1.

Proof. It is sufficient to show that

$$F(-z) = TF(z), \tag{6}$$

with $T \in PSU(2)$.

That F satisfies (6) with *some* $T \in PSL(2, \mathbf{C})$ is clear because the Schwarzian $\mathbf{S}[F]$ is even. It is also clear from (6) that

$$T^2 = \mathrm{id}, \tag{7}$$

so T is either an elliptic transformation with multiplier -1 or the identity. If $T = \mathrm{id}$, the proof is finished, so assume that the multiplier of T is -1.

Suppose now that F is a meromorphic function, satisfying (6), and

$$F(z + \omega) = SF(z) \tag{8}$$

where S is a linear-fractional transformation. We claim that in this case the equation

$$STS = T \tag{9}$$

must hold. Indeed, (8) implies

$$F(z - \omega) = S^{-1}F(z) \tag{10}$$

and also

$$F(-z + \omega) = SF(-z). \tag{11}$$

Using (6), (11) and (10) we obtain

$$STF(z) = SF(-z) = F(-z + \omega) = F(-(z - \omega)) = TF(z - \omega) = TS^{-1}F(z),$$

which implies (9).

Suppose that (4) holds. Both equations in (4) are of the form (8) with $S^{-1} = S$, thus $ST = TS$. Using the first equation (4) in which $S(z) = -z$, we conclude that $T(z) = cz$ or $T(z) = c/z$, $c \neq 0$. In the first case (7) implies $c^2 = 1$, so $T \in PSU(2)$. In the second case, we use the second equation in (4) which implies that T commutes with $S(z) = 1/z$, and conclude again that $c^2 = 1$ and $T \in PSU(2)$.

Suppose now that (5) holds, and $S(z) = kz,\ k = e^{2\pi i\alpha},\ k \neq 1$. Then (9) gives

$$kT(kz) \equiv T(z). \tag{12}$$

Substituting

$$T(z) = \frac{az+b}{cz+d}$$

to (12), after simple computation we obtain

$$bd = 0 \quad \text{and} \quad ac = 0,$$

and

$$ad = 0 \quad \text{or} \quad k = -1.$$

If $b = 0$, then $a \neq 0$, $d \neq 0$, so $k = -1$ and $T(z) = (a/d)z$. Using (7) we conclude that $T(z) = -z$.

If $d = 0$, then $c \neq 0$, $b \neq 0$, so $a = 0$ and $T(z) = c/z$, and after multiplication of F by a constant we obtain $T \in PSU(2)$.

Finally, if we have (5) with both multiplies $k_j = 1$, then F is an elliptic function, with $F(-z) = TF(z)$, where T is an elliptic transformation satisfying (6). Let $V \in PSL(2, \mathbf{C})$ be such that $VTV^{-1} \in PSU(2)$ then the metric with the developing map VF is even.

This completes the proof of Theorem 1.

Every even metric on the torus is a pull-back of a metric from the sphere via the $\wp$ function. Thus we obtain a metric on the sphere, which has 4 singularities at the critical values of $\wp$ that is at ∞, e_1, e_2, e_3, and the angles at these singularities are $\alpha_j = m_j/2$. We call f the developing map on the sphere, so that $F = f \circ \wp$. Now we consider four cases.

Case 1. *All m_j are odd.*

Then the monodromy of f is generated by four non-trivial involutions.

Proposition 1. *In Case 1, F satisfies (5).*

This generalizes a result [4, Thm. 0.3].

Proof. Proving this by contradiction, assume that we have (4). Then the monodromy group of F is the Klein Viergroup $V = (\mathbf{Z}/2\mathbf{Z})^2$. Let σ_j be conformal involutions of $\mathbf{C}$ with fixed points $\omega_j/2$,

$$\sigma_j(z) = \omega_j - z, \quad 0 \leq j \leq 3.$$

Their images in the monodromy group are non-trivial elements of V, because all m_j are odd. As V contains only three non-trivial elements, two of the distinct involutions σ and σ' have the same image, so the image of $\sigma\sigma'$ in the monodromy group must be the identity. But $\sigma\sigma'$ is a translation by some ω_j, so we obtain a contradiction with (4), which proves the Proposition.

In the remaining cases, some m_j are even, thus the metric on the sphere has at least one integer angle. All these cases were studied in detail and completely classified in [27, 10, 11, 12, 13]. We state the results.

Case 2. *Exactly one of the m_j is even.*

Let $m_0 = 2\alpha_0$ be the even angle. Then it follows from [13, Proposition 4.2] that there at most α_0 distinct metrics, and for generic torus there are exactly α_0 of them. The accessory parameters λ are found from the algebraic equation that expresses the absence of logarithms in the solution of the Heun equation.

Case 3. *Exactly two of the m_j are even.*

Let $m_j = 2\alpha_j$, $0 \leq j \leq 3$ and assume that

$$\alpha_3 = r + 1/2, \quad \alpha_0 = s + 1/2, \quad \alpha_1 = p, \quad \alpha_2 = q.$$

Then according to [11, Thm. 4.1] there are two subcases:

Subcase 3a. $p + q + r + s$ is even, $|r - s| \leq p + q$, and the number of classes of metrics is at most $(p + q - |r - s|)/2$ with equality for generic tori,

Subcase 3b. $p + q + r + s$ is odd, $r + s + 3 \leq p + q$, and the number of classes of metrics is at most $(p + q - r - s - 1)/2$ with equality for generic tori.

If three of the m_j are even then all four must be even, because the product of local monodromies is the identity, and we obtain the last case:

Case 4. *All m_j are even.*

In this case, f is a rational function with four critical points of orders $m_j/2$. Such functions were classified in [27, 10].

So in all cases 2,3,4, we have a complete classification of metrics in question.

In the next sections we consider a special family which belongs to Case 1.

3 Spherical rectangles. Statement of results

Here we begin to study spherical quadrilaterals whose angles are odd multiples of 1/2. We call them *spherical rectangles*.

Metrics on the sphere corresponding to spherical rectangles are symmetric with respect to a circle, and without loss of generality we may assume that this circle is the real line. Singularities lie on the real line. Consider the ramified covering of degree 2 from a torus to the sphere, ramified over these singularities. It is a composition of an element of $PSL(2, \mathbf{R})$ with the Weierstrass $\wp$ function whose invariants $e_j = \wp(\omega_j/2)$ are real. Such Weierstrass functions are defined on "rectangular" tori, which means that $\omega_3/\omega_1 = iK,\ K > 0$. The pull back of the metric on the torus has singularities with integer angles at half-periods $\omega_j/2$.

In each quadrilateral we mark one corner. Two quadrilaterals are considered equal if there is an orientation preserving isometry sending one to another and sending the marked corner to the marked corner.

We usually choose the upper half-lane H as a conformal model of a spherical polygon. Then the corners a_j become points on the real line, and the metric has the length element $\rho(z)dz$, $\rho = e^u$ where u satisfies (1). Then the spherical polygon is a pair $Q = (H, \rho)$, where ρ has conic singularities at a_j and the sides (a_j, a_{j+1}) are geodesic with respect to the metric ρ.

Developing map of a quadrilateral satisfies the Schwarz equation associated with a Heun equation, as explained in the Introduction (see also [11]).

Our first result is

Theorem 2. *Let $f : Q \to \overline{\mathbf{C}}$ be the developing map of a spherical quadrilateral whose angles are odd multiples of $1/2$. Then there are two opposite sides whose f-images are contained in the same circle.*

Thus there are two types of such spherical quadrilaterals: in the first type one of the two sides whose images belong to the same circle ends at the marked corner, and in the second type such a side begins at the marked corner.

To state our main result we define the conformal modulus. Let us map conformally our quadrilateral onto the rectangle $(0, 1, 1 + iK, iK)$, so that the marked corner is mapped to 0. Then $K > 0$ is called the modulus.

Theorem 3. *There are two continuous families of spherical quadrilaterals with angles $(3/2, 1/2, 3/2, 1/2)$, and angle $3/2$ at the marked corner. One family consists of quadrilaterals of the first type, and the modulus in this family varies from 0 to $K_{\text{crit}} < 1$. The second family consists of quadrilaterals of the second type, and the modulus varies from $1/K_{\text{crit}}$ to ∞. Each family contains exactly one quadrilateral of the given modulus in the described range.*

So there is a "prohibited interval" $[K_{\text{crit}}, 1/K_{\text{crit}}]$ for the moduli of spherical quadrilaterals with angles $(3/2, 1/2, 3/2, 1/2)$.

Theorem 3 is proved in the next section.

Proof of Theorem 2. Let us enumerate the great circles C_j which contain the images of sides according to positive orientation of the boundary, so that j is a residue modulo 4. We may assume that C_1 and C_2 are real and imaginary axes, then C_3 must cross the real axis perpendicularly at the points $r, -1/r$, and C_4 must cross the imaginary axis perpendicularly at the points $it, -i/t$. Without loss of generality $r \geq 0$ and $t \geq 0$.

If the 4 circles are in general position (that is there are no triple intersections), then both r and t are finite, positive, and there is a crossing point A of C_3 and C_4. The centers of C_3 and C_4 are at $c_3 = (r - 1/r)/2$ and $c_4 = i(t - 1/t)/2$. Their radii are $r_3 = (r + 1/r)/2$ and $r_4 = (t + 1/t)/2$. Then we obtain from the right triangle $(c_2, 0, c_4)$ that the square of the distance between the centers is

$$(r - 1/r)^2/4 + (t - 1/t)^2/4.$$

On the other hand the angle between the radii of C_3 and C_4 at A is right, so the same distance is

$$(r + 1/r)^2/4 + (t + 1/t)^2/4,$$

a contradiction. Therefore the images of two opposite sides must lie on the same circle.

The non-generic cases are easy and we leave them to the reader. This proves Theorem 2.

Let our three circles be the real line, the line ℓ_α through the origin making angle α with positive ray, and the unit circle. The f-images of the two opposite sides lie on the unit circle and the images of the other two sides on $\mathbf{R}$ and ℓ_α, respectively. We have one continuous parameter $\alpha \in (0, \pi)$, and the modulus of the quadrilateral depends on the angles and on α.

The group generated by reflections in the f-images of the sides has a subgroup of index 2 consisting of conformal maps; this subgroup is the monodromy group of our developing map $H \to \overline{\mathbf{C}}$. The monodromy group is generated by 3 elliptic involutions with fixed points $(0, \infty)$, $(1, -1)$ and

$(e^{i\alpha}, -e^{i\alpha})$, so it consists of the maps of the form

$$z \mapsto e^{2\pi i m\alpha} z \quad \text{and} \quad z \mapsto e^{2\pi i m\alpha}/z, \tag{13}$$

where m is any integer. When α is rational, this group is finite, it is a dihedral group, and the developing map is algebraic. For this case, there is a theorem of Klein (see [2]) which says that the Heun equation is a pullback of a hypergeometric equation with finite monodromy group. This means that the developing map factors as $h \circ b$, where h is a solution of the hypergeometric equation (in our case h^{-1} is a conformal map of the triangle with angles $(1/2, 1/2, 2\alpha)$ onto the upper half-plane, and b is a Belyi rational function which can be computed algebraically).

Our monodromy group (13) belongs to the class of groups considered by Van Vleck which consist of transformations of the form

$$z \mapsto cz, \quad z \mapsto d/z. \tag{14}$$

Van Vleck [31] proved that Fuchsian differential equations with such projective monodromy groups can be characterized by the property that they have two linearly independent solutions whose product is a polynomial. This remarkable property was first discovered by Hermite [18] for the case of Lamé equation, and generalized by Darboux [6] to equation (3). It permits to solve the equation in a closed form. If P is this polynomial, then two linearly independent solutions can be represented in the form

$$w_1 = \sqrt{P} \exp\left(C \int \prod (z - a_j)^{\alpha_j - 1} \frac{dz}{P(z)} \right), \tag{15}$$

$$w_2 = \sqrt{P} \exp\left(-C \int \prod (z - a_j)^{\alpha_j - 1} \frac{dz}{P(z)} \right). \tag{16}$$

Hermite found the method of determining P in terms of a_j, α_j and the accessory parameter. Equation on the accessory parameter comes from the condition that the monodromy group has the special form (13). In section 4 we perform all computations explicitly in a simple case.

To pull back our metric on a torus, we model our spherical quadrilateral on a flat rectangle, instead of the half-plane. Consider the rectangle $(0, \omega_1/2, \omega_2/2, \omega_3/2)$, where ω_1 and ω_3/i are positive and $\omega_2 = \omega_1 + \omega_3$. Now the developing map f is defined on this rectangle and maps it to the standard sphere. Without loss of generality one can assume that $\omega_1 = 2$ and $\omega_3 = 2iK$, but sometimes we will use ω_1 and ω_3. Then the developing map extends by reflection to the whole plane. We assume without loss of generality that the vertical sides of the rectangle are mapped by f into the unit circle, and the side $(0, \omega_1/2)$ to the real line. Then the developing map is real on the real line and satisfies

$$f(z + \omega_1) = f(z), \quad f(z + \omega_3) = e^{2\pi i \alpha} f(z), \tag{17}$$

which is a special case of (5). To this one can add

$$f(-z) = 1/f(z) \tag{18}$$

and similar properties for conformal involutions with fixed points at $\omega_j/2$. If the angles α_j of our spherical rectangle are $n_j + 1/2$, then the map f is ramified at the corners and is $m_j = (2n_j + 1)$-to-1 near $\omega_j/2$, that is f' has a zero of order $m_j - 1$ at $\omega_j/2$ and no other zeros.

In (17), α and the parameter $\tau = \omega_3/\omega_1 = iK$ are connected by some relation which also depends on the m_j.

Thus we have the problem:

For given periods and given integers m_j, how many functions f exist (up to proportionality) that satisfy (17) and have prescribed critical point pattern: critical points of order $m_j - 1$ at ω_j and no other critical points.

The Schwarzian derivative of f is doubly periodic with periods $2\omega_1, 2\omega_3$, and has double poles at ω_j, where we must have apparent singularities and the exponent difference must be m_j. So our developing map is the ratio of two solutions of the Darboux–Treibich–Verdier equation (3). Thus we have an equivalent problem:

For given periods and n_j, how many real values of λ exist, so that a ratio $f = w_1/w_2$ of some linearly independent solutions of (3) has properties (17) with some real α.

(Parameter λ must be real because the Schwarzian of f is real on the real line, and $\wp(z - \omega_j)$ are real on the real line for the lattices with real ω_1 and imaginary ω_3.)

4 Rectangles with angles $(3/2, 1/2, 3/2, 1/2)$

In this section we prove Theorem 3. We consider the simplest case of a spherical rectangle whose angles are odd multiples of $1/2$ (measured in half-turns, so that $1/2$ means $\pi/2$ radians). We take the angles $(3/2, 1/2, 3/2, 1/2)$.

Lemma 2. *There are two families of spherical rectangles with angles* $(1/2, 3/2, 1/2, 3/2)$*, each depending on parameter* $\alpha \in (0, \pi)$*. Rectangles of these families are isometric to subsets of the sphere whose oriented boundaries are shown in Fig. 1.*

Proof. The area of a rectangle with angles as in Lemma 2 is $1/2$ of the area of the sphere; this follows from the Gauss–Bonnet formula. We can normalize the developing map so that the image of the boundary is contained in the union of the real line, the line ℓ_α as in the previous section, and the unit circle.

The image of any side cannot cover more than $1/2$ of the great circle to which this image belongs, otherwise this image would contain either a half-plane or the exterior or the interior of the unit disk. Since the area of the whole rectangle is $1/2$ of the area of the sphere, it would follow that the rectangle is isometric to a hemisphere, which is of course impossible.

From Theorem 1 we know that some pair of opposite sides is mapped by the developing map into the same circle. Without loss of generality we assume that this circle is the unit circle. Let s, s' be these two sides. They are oriented in the usual way, so that the rectangle is on the left of each side. The angle at the beginning of s is the same as that at the beginning of s', and it is either $1/2$ or $3/2$. So we have (at least) two types of such rectangles.

Consider a rectangle of the first type. Let the sides be (d, s, d', s') in the order of positive orientation. Without loss of generality, the image of d is on the real line, oriented in the increasing direction. This image must be either $(-1, 1)$ or its complement. Without loss of generality, let it be the complement.

Then the image of s must be a clockwise arc of the unit circle from -1 to some point $e^{i\alpha}$ in the upper half-plane. Tracing the rest of the boundary we arrive at a picture in the top of Fig. 1.

Rectangles of type 2 are considered in the similar way. This proves the lemma.

An alternative way to understand spherical quadrilaterals is through their *nets* [8, 12, 13]. The developing map f maps the sides of a quadrilateral Q to three transversal great circles on the sphere. The *net* Γ of Q is the preimage in Q of these three circles. For each circle C, its preimage in Q is called C-*net*, denoted Γ_C. An *arc* of the net Γ_C is a connected component of $\Gamma_C \setminus \partial Q$. The net Γ defines a triangulation of Q, each face of it being mapped by f one-to-one onto one of the triangles into which the three circles partition the sphere.

Lemma 3. *Let C be the circle to which two opposite sides s and s' of a quadrilateral Q with angles $1/2$, $3/2$, $1/2$, $3/2$ are mapped. Then there is an arc γ of Γ_C with the ends at the two corners of Q having angles $3/2$.*

Proof. Since the area of Q is $1/2$ of the area of the sphere, and each corner having angle $3/2$ has three faces of Γ adjacent to it, with the total area greater than $1/4$ of the area of the sphere, there should be a face F of Γ adjacent to both of these corners. Thus these corners are connected by one of the sides of F which must be an arc of Γ_C because C is the only circle to which the images of both of these corners belong.

Corollary. *Each quadrilateral Q with angles $(1/2, 3/2, 1/2, 3/2)$ is a union of two isometric halfs of hemispheres, each of them bounded by the arcs of two great circles intersecting at a right angle.*

Remark. The statement of Lemma 3 is a special case of the following statement, which will be proved elsewhere:

Let Q be a spherical n-gon such that all angles of Q are not integer and all sides of Q are mapped to three transversal great circles by the developing map. Then there is (in general, non-unique) triangulation of Q by $n-3$ disjoint arcs of its net, each arc having both ends at the corners of Q.

We recall that our quadrilateral has marked corner, and two quadrilaterals are considered equal if there is an isometry between them which preserves the orientation and sends the marked corner to the marked corner.

We choose the marked corner so that the angle at it is $3/2$, and the two families are distinguished by the property that the side ending at the marked corner in the first family has the image on the same circle as its opposite side, while in the second family the side that ends at the marked corner and its opposite side have images on the different circles.

Note that the choice of one of the two corners with the angle $3/2$ as marked is not important, as our families have isometries $z \mapsto e^{i\alpha}/z$ and

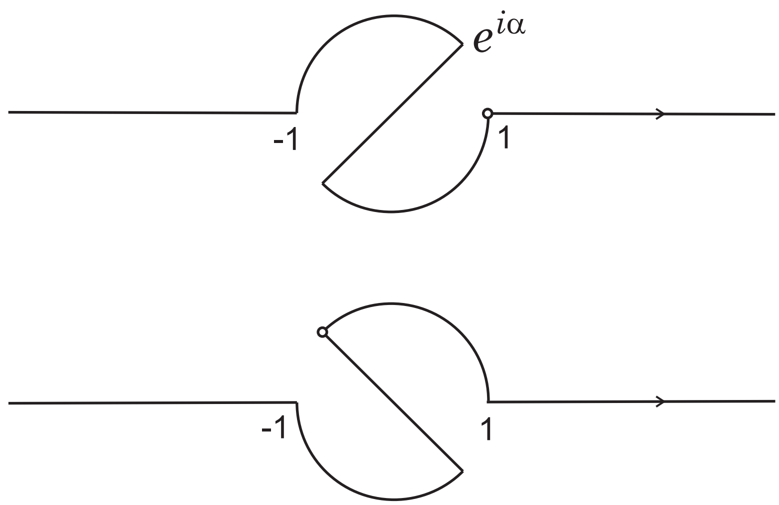

Figure 1: Two families of quadrilaterals

$z \mapsto e^{-i\alpha}/z$, respectively, exchanging the two corners.

Let us map our quadrilateral on a rectangle with two sides $[0,1]$ and $[0,Ki]$, where 0 is the marked corner. Then $K>0$ is the *modulus* of the quadrilateral.

Now we return to the proof of Theorem 3. Our two families are shown in Fig. 1. The marked corner is circled. Consider the first family. Taking log we obtain the region in Fig. 2. Now according to Schwarz and Christoffel, logarithm of the developing map defined in the upper half-plane is

$$L(z) := \log f(z) = A \int_k^z \sqrt{\frac{(1+\zeta)(k-\zeta)}{(1-\zeta)(k+\zeta)}} \frac{d\zeta}{(\zeta-c)(\zeta-d)}.$$

This should be compared with Van Vleck's general formula (15). The residue at c equals:

$$\frac{A}{c-d}\sqrt{\frac{(1+c)(k-c)}{(1-c)(k+c)}} = \pm 1.$$

Then the residues at d are ∓ 1 by the Residue theorem. This gives the condition

$$h(c) = h(d) \quad \text{where} \quad h(x) = \frac{(1+x)(k-x)}{(1-x)(k+x)}. \tag{19}$$

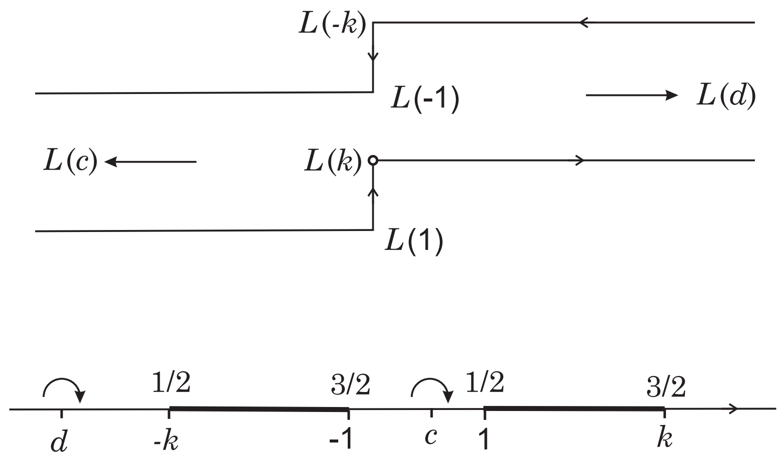

Figure 2: Conformal map, $L = \log f$.

This is the "Bethe-Ansatz equation" for this case. Thus $d = -k/c$ and

$$A = (c + k/c)\sqrt{\frac{(1-c)(k+c)}{(1+c)(k-c)}} > 0. \tag{20}$$

Therefore our integral is

$$L(z) = \int_k^z \frac{c + k/c}{\zeta + k/c}\sqrt{\frac{(1-c)(k+c)(1+\zeta)(k-\zeta)}{(1+c)(k-c)(1-\zeta)(k+\zeta)}}\frac{d\zeta}{\zeta - c} =: \int_k^z \frac{g(c,\zeta)d\zeta}{\zeta - c}, \tag{21}$$

where the branch of the square root in the upper half-plane is chosen so that the integrand is positive on (k, ∞).

Here c plays the role of the accessory parameter, and is simply related to it. The condition is that

$$\mathrm{Re}\int_{-1}^k \frac{g(c,\zeta)d\zeta}{\zeta - c} = \int_{-1}^1 \frac{g(c,\zeta)d\zeta}{\zeta - c} = 0,$$

because the part of the integral in (21) over $[1, k]$ is imaginary. From this condition $c \in (-1, 1)$ must be found. To regularize our integral on $(-1, 1)$. write it as

$$F(k,c) := \int_{-1}^1 (g(c,\zeta) - g(c,c))\frac{d\zeta}{\zeta - c} + g(c,c)\int_{-1}^1 \frac{d\zeta}{\zeta - c}.$$

Now $g(c,c)=1$ and
$$\int_{-1}^{1}\frac{dx}{x-c}=\log\frac{1-c}{1+c}.$$
The equation for c is $F(k,c)=0$.

We would like to know that $c \in (0,1)$ is uniquely determined by k. This would mean that there is at most one quadrilateral with prescribed modulus in each family.

Proposition 1. *There exists $K_{\text{crit}} \in (0,1)$ with the following property. For every $K \in (0, K_{\text{crit}})$ there exists unique $c \in (0,1)$ such that $F(k,c)=0$. Moreover, $c \to 0$ as $K \to 0$ and $c \to 1$ as $K \to K_{\text{crit}}$.*

Here K is a known (increasing) function of k [1] see also (22) below.

Proof of Proposition 1. To prove uniqueness of c it is sufficient to show that after multiplication of F by some function of k and c which is non-zero in the considered range, the result is strictly decreasing in c for each k. First we consider the function

$$\begin{aligned} F_1(k,c) &= \sqrt{\frac{(1+c)(k-c)}{(1-c)(k+c)}}F(k,c) \\ &= \int_{-1}^{1}\sqrt{\frac{(1+z)(k-z)}{(1-z)(k+z)}}\frac{(c+k/c)dz}{(z-c)(z+k/c)}. \end{aligned}$$

Then we do partial fraction decomposition

$$\frac{c+k/c}{(z-c)(z+k/c)}=\left(\frac{1}{z-c}-\frac{1}{z+k/c}\right).$$

The second partial fraction $-1/(z+k/c)$ is strictly decreasing in $c \in (0,1)$, for each $z \in (-1,1)$, so it remains to prove that

$$F_2(k,c)=\int_{-1}^{1}\sqrt{\frac{(1+z)(k-z)}{(1-z)(k+z)}}\frac{dz}{z-c}$$

is decreasing.

To do this we deform the contour of integration (see Fig. 3). We consider the simple closed curve γ on the Riemann surface of the integrand which

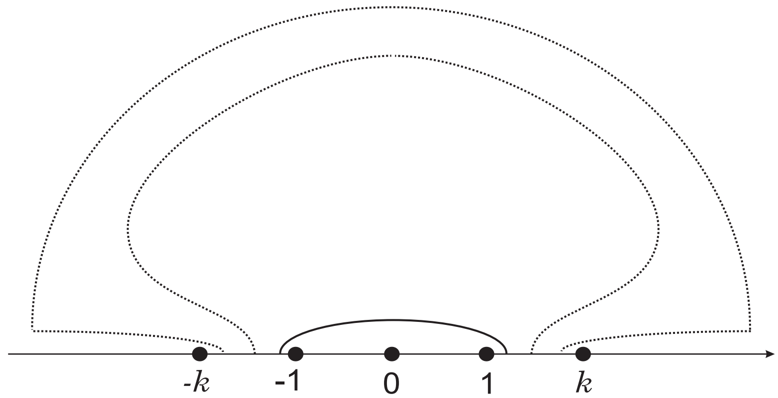

Figure 3: Deformation of the path of integration.

goes once around the segment $[-1,1]$ in the negative direction (clockwise). Then

$$F_2(k,c) = \frac{1}{2}\int_\gamma \sqrt{\frac{(1+z)(k-z)}{(1-z)(k+z)}}\frac{dz}{z-c}.$$

Then we deform the contour so that the part in the upper half-plane consists of the interval (k,R), $R>k$ oriented from R to k, the interval $(-R,-k)$ oriented from $-k$ to $-R$ and a large half-circle $|z| = R, \operatorname{Im} z > 0$. The integral over the half-circle is $\pm\pi i$ times residue at infinity, thus it is pure imaginary.

On the interval (k,R) the integral has the form

$$-\int_k^R \phi(k,x)\frac{dx}{x-c},$$

where $\phi > 0$ and it is decreasing as a function of c. On the interval $(-R,-k)$ the integral has similar form, and also decreasing as a function of c. This proves that for every k there exists at most one c such that $F(k,c) = 0$. Let this value be $c(k)$.

When $k \to 1+$, we have $c(k) \to 0+$ and $c(k)$ is increasing. So we have some interval $(0, k_{\text{crit}})$ on which c is increasing.

When k passes 1, c passes 0, and we obtain the second family.

When $k > k_{\text{crit}}$, the quadrilateral degenerates must have the shape as in Fig. 4. The conformal map is

$$L(z) = \log f(z) = A'\int_k^z \sqrt{\frac{(1+\zeta)(k-\zeta)}{(1-\zeta)(k+\zeta)}}\frac{d\zeta}{(\zeta-c)(\zeta+k/c)} =: A'\int_k^z G(\zeta,c)d\zeta.$$

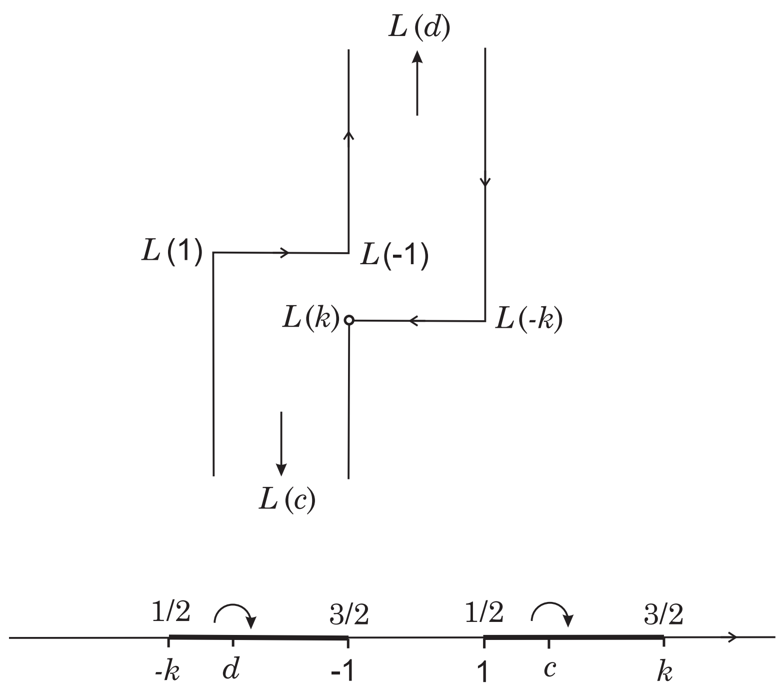

Figure 4: Conformal map $L = \log f$ for $k > k_{\text{crit}}$.

We want to keep $A' > 0$ but the residue at c now must be i, and the residue at d must be $-i$. That is

$$\frac{A'}{c-d}\sqrt{\frac{(1+c)(k-c)}{(1-c)(k+c)}} = \pm i.$$

Notice that the equation for d is exactly the same as before, thus $d = -k/c$, and that the expression under the root is now negative, and we obtain

$$A' = (c-d)\sqrt{\frac{(c-1)(k+c)}{(c+1)(k-c)}} > 0,$$

as desired. The condition now is that

$$\Re A \int_{-1}^{k} G(\zeta, c)d\zeta = 0 \ \leftrightarrow \int_{-1}^{1} G(\zeta, c)d\zeta = -\pi,$$

and this must define $c \in (1, k)$. The integral is now

$$\int_{-1}^{1} \frac{c^2+k}{cx+k}\sqrt{\frac{(c-1)(k+c)(1+x)(k-x)}{(c+1)(k-c)(1-x)(k+x)}}\frac{dx}{x-c}.$$

The only difference in comparison with the previous integral is that factor $(1-c)$ is replaced by $(c-1)$ under the square root, and the integral must be equal to $-\pi$. This integral does not require desingularization.

Consider the rescaled degenerate configuration, Fig. 5. It is a quadrilateral but flat, not spherical. The map of the upper half-plane on this region is given by

$$\int \sqrt{\frac{(\zeta-k)(\zeta+1)}{(\zeta+k)(\zeta-1)}}\frac{d\zeta}{(\zeta+k)(\zeta-1)}$$

which is the same that we obtain if we put $c = 1$ in our integral. The condition would be now that

$$\int_{-1}^{1} \sqrt{\frac{(\zeta-k)(\zeta+1)}{(\zeta+k)(\zeta-1)}}\frac{d\zeta}{(\zeta+k)(\zeta-1)} = 0,$$

but this diverges at one end. Von Koppenfels [22, 5.3] integrates by parts to obtain his equation for critical k.

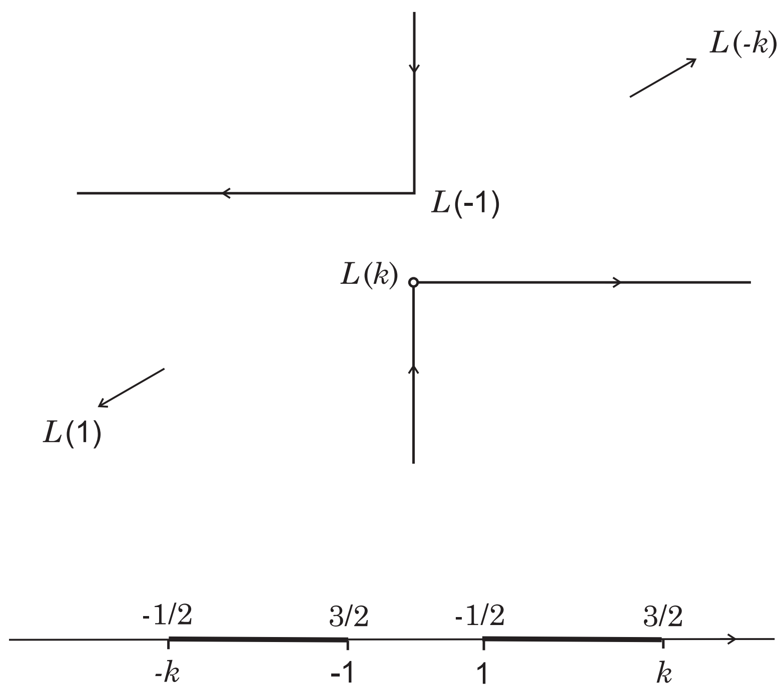

Figure 5: Limit of rescaled maps for $k = k_{\text{crit}}$.

The modulus of the degenerate region was computed by von Koppenfels [22, (5.3.30)] by solving the equation

$$K(\kappa') - 2E(\kappa') = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\kappa'^2 x^2)}} - 2\int_0^1 \sqrt{\frac{1-\kappa'^2 x^2}{1-x^2}}dx = 0. \quad (22)$$

The solution is approximately

$$\kappa'_{\text{crit}} = 0.9089085575\ldots. \quad (23)$$

The relation between our k and κ' is the following $k = (1+\kappa)/(1-\kappa)$, where $\kappa = \sqrt{1-\kappa'^2}$. This gives the critical value $k_{\text{crit}} = 2.4305$.

Thus as k varies from 1 to k_{crit}, c varies from 0 to 1, and K (the modulus varies from 0 to K_{crit}.

When $k \to 1+$, $c \to 0+$; after this point is passed, we obtain the second family, with modulus K' near infinity. This second family is symmetric to the first, so there is a correspondence $K' \mapsto 1/K$. As seen from the pictures, and using the domain extension principle we have $K_{\text{crit}} < 1$.

Thus for every $K > 0$ we have at most one quadrilateral with angles $(3/2, 1/2, 3/2, 1/2)$. It is of the first type when $K < K_{\text{crit}}$ and of the second type when $K > 1/K_{\text{crit}}$. This completes the proof of Theorem 3.

5 Remarkable constants

Paper [3] contains new proofs of some results from [23]. It is noticed in [3, Remark 2] that equation (22) and its solution κ'_{crit} in (23) is a famous constant discussed in Finch [14, 4.5]. To conform to Finch's notation, let

$$c := \kappa'_{\text{crit}},$$

and

$$\Lambda = \exp\left(\frac{-\pi K\left(\sqrt{1-c^2}\right)}{K(c)}\right) = 0.1076539192\ldots.$$

There was a constant known for some time as the "One-Ninth" constant in approximation theory, until Gonchar and Rakhmanov proved that it equals

Λ. Earlier the same constant Λ implicitly appeared in Euler's work on elastics. Halphen [17, p. 287] mentions c as the unique solution of the equation

$$\sum_{n=0}^{\infty}(2n+1)^2(-x)^{n(n+1)}, \quad 0 < x < 1,$$

and computes this solution to six digits.

But the most remarkable coincidence is the appearance of the related constants in [23], where there is a constant

$$b_1 = \frac{K(c)}{K\left(\sqrt{1-c^2}\right)}.$$

This constant b_1 appears in [23] in the study of metrics of curvature 1 with one conic singularity on real tori, but the real tori in [23] are *different* from the real tori studied here!

We recall that there are two types of real tori: in the first type, which we called rectangular in this paper, the parameter $\tau = \omega_3/\omega_1$ is pure imaginary. The second type, which is studied in [23] can be called "rhombic", and can be characterized by the property $\mathrm{Re}\,\tau = 1/2$.

The problem considered in [23] has no solutions for rectangular tori, while it has a unique solution for the rhombic tori with parameter $\tau = 1/2 + ib$ exactly when $b > b_1$ or $b < 1/(4b_1)$.

That the same constant arises in these two different settings suggest some hidden relation between the problem studied in [23] and the problem studied here in Section 4.

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Department of Mathematics, Purdue University, West Lafayette Indiana, 47907 USA