A Theorem of Eakin and Sathaye and Green's Hyperplane Restriction Theorem

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Abstract

A Theorem of Eakin and Sathaye relates the number of generators of a certain power of an ideal with the existence of a distinguished reduction for that ideal. We prove how this result can be obtained as a special case of Green's General Hyperplane Restriction Theorem.

1   Introduction

The purpose of these notes is to show how the following Theorem 2.1, due to Eakin and Sathaye, can be viewed, after some standard reductions, as a corollary of Green's General Hyperplane Restriction Theorem.

THEOREM 2.1[EAKIN-SATHAYE] *Let (R, m) be a quasi-local ring with infinite residue field. Let I be an ideal of R. Let n and r be positive integers. If the number of minimal generators of I^i, denoted by $v(I^i)$, satisfies*

$$v(I^i) < \binom{i + r}{r},$$

then there are elements $h_1, \ldots, h_r$ in I such that $I^i = (h_1, \ldots, h_r)I^{i-1}$.

Before proving Theorem 2.1 we have to recall some general facts about Macaulay representation of integer numbers. This is needed for the understanding of Green's Hyperplane Restriction Theorem. For more details on those topics we refer the reader to [3] and [4].

1.1 Macaulay representation of integer numbers

Let d be a positive integer. Any positive integer c can then be uniquely expressed as

$$c = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_1}{1},$$

where the k_i's are non-negative and strictly increasing i.e $k_d > k_{d-1} > \cdots > k_1 \geq 0$. This way of writing c is called the d'th *Macaulay representation* of c, and the k_i's are called the d'th *Macaulay coefficients* of c. For instance, setting $c = 13$ and $d = 3$ we get $13 = \binom{5}{3} + \binom{3}{2} + \binom{0}{1}$.

REMARK 1.1 An important property of Macaulay representation is that the usual order on the integers corresponds to the lexicographical order on the arrays of Macaulay coefficients. In other words, given two positive integer $c_1 = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_1}{1}$ and $c_2 = \binom{h_d}{d} + \binom{h_{d-1}}{d-1} + \cdots + \binom{h_1}{1}$ we have $c_1 < c_2$ if and only if $(k_d, k_{d-1}, \ldots, k_1)$ is smaller lexicographically than $(h_d, h_{d-1}, \ldots, h_1)$.

DEFINITION 1.2 Let c and d be positive integers. We define $c_{<d>}$ to be

$$c_{<d>} = \binom{k_d - 1}{d} + \binom{k_{d-1} - 1}{d-1} + \cdots + \binom{k_1 - 1}{1}$$

where $k_d, \ldots, k_1$ are d'th Macaulay coefficients of c. We use the convention that $\binom{a}{b} = 0$ whenever $a < b$.

REMARK 1.3 It is easy to check that if $c_1 \leq c_2$ then $c_{1<d>} \leq c_{2<d>}$. This property, as we see in the following, allows us to iteratively apply Green's Theorem and prove Corollary 1.5.

1.2 Green's General Hyperplane Restriction Theorem

Let R be a standard graded algebra over an infinite field K. We can write R as $K[X_1, \ldots, X_n]/I$ where I is an homogeneous ideal. Given a generic linear form L we will denote by $R_L = K[X_1, \ldots, X_{n-1}]/I_L$ the restriction of R to the hyperplane given by L. Note that since L is generic we can write it as $L = l_1 X_1 + \cdots + l_n X_n$ where $l_n \neq 0$, therefore I_L is defined as

$$I_L = (P(X_1, \ldots, X_{n-1}, (L/l_n) - X_n)|P \in I).$$

We will denote by R_d the d'th graded component of R. Mark Green proved the following Theorem.

THEOREM 1.4 (GREEN'S GENERAL HYPERPLANE RESTRICTION THEOREM) *Let R be a standard graded algebra over an infinite field K, and let L be a generic linear form of R. Setting S to be R_L, we have*

$$\dim_k S_d \leq (\dim_K R_d)_{<d>}.$$

The General Hyperplane Restriction Theorem first appeared in [4], where it was proved with no assumption on the characteristic of the base field K.

A different, and more combinatorial, proof can be found in [3] where the characteristic zero assumption is a working hypothesis. A person interested in reading this last proof can observe that the arguments in [3] also work in positive characteristic with a few minor changes.

A direct corollary of Green's Theorem is the following

COROLLARY 1.5 *Let R be a standard graded algebra over an infinite field K, and let $L_1, \ldots, L_r$ be generic linear forms of R. Let $\binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_1}{1}$ be the Macaulay representation of $\dim R_d$, and define $S = R/(L_1, \ldots, L_r)$. Then*

$$dim_K S_d \leq \binom{k_d - r}{d} + \binom{k_{d-1} - r}{d-1} + \cdots + \binom{k_1 - r}{1}$$

Proof: Note that R_L is isomorphic to $R/(L)$ and by Theorem 1.4 one deduces $dim_K(R/(L))_d \leq \binom{k_d-1}{d} + \binom{k_{d-1}-1}{d-1} + \cdots + \binom{k_1-1}{1}$. On the other hand by Remark 1.3 we can apply Green's Theorem again and obtain the result by induction. $\qquad\square$

2   The Eakin-Sathaye Theorem

We now prove Theorem 2.1. First of all note that since $v(I^i)$ is finite, without loss of generality we can assume that I is also finitely generated: in fact if $J \subseteq I$ is a finitely generated ideal such that $J^i = I^i$, the result for J implies the one for I. Moreover, by the use of Nakayama's Lemma, we can replace I by the homogeneous maximal ideal of the fiber cone $S = \bigoplus_{i \geq 0} I^i/mI^i$. Note that S is a standard graded algebra finitely generated over the infinite field $R/m = K$.

Theorem 2.1 can be rephrased as:

THEOREM 2.1 (E-S) *Let R be a standard graded algebra finitely generated over an infinite field K. Let i and r be positive integers such that*

$$\dim_K(R_i) < \binom{i+r}{r}.$$

Then there exist homogeneous linear forms $h_1, \ldots, h_r$ such that $(R/(h_1, \ldots, h_r))_i$ is equal to zero.

Proof: First of all note that $\dim_K R_i \leq \binom{i+r}{r} - 1 = \binom{i+r}{i} - 1 = \binom{i+r-1}{i} + \binom{i+r-2}{i-1} + \cdots + \binom{i+r-j}{i-j+1} + \cdots + \binom{r}{1}$. This can be proved directly or by using Remark 1.1. In fact one can first order the array of Macaulay coefficients using the lexicographic order and then note that the previous array of $(i+r, 0, \ldots, 0)$ is given by $(i+r-1, i+r-2, \ldots, r)$.

Let $L_1, \ldots, L_r$ be generic linear forms. By Corollary 1.5 we have

$$\dim_K (R/(L_1, \ldots, L_r))_i \leq \binom{i-1}{i} + \binom{i-2}{i-1} + \cdots + \binom{0}{1}$$

The term on the right hand side is zero and therefore the theorem is proved. $\qquad\square$

References

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