REGULARITY OF PRIME IDEALS

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Abstract. We answer some natural questions which arise from the recent paper [MP] of McCullough and Peeva providing counterexamples to the Eisenbud-Goto Regularity Conjecture. We give counterexamples using Rees algebras, and also construct counterexamples that do not rely on the Mayr-Meyer construction. Furthermore, examples of prime ideals for which the difference between the maximal degree of a minimal generator and the maximal degree of a minimal first syzygy can be made arbitrarily large are given. Using a result of Ananyan-Hochster we show that there exists an upper bound on regularity of prime ideals in terms of the multiplicity alone.

1. **Introduction**

Regularity is a numerical invariant that measures the complexity of the structure of homogeneous ideals in a polynomial ring. It has been studied in Algebraic Geometry and Commutative Algebra; see the expository paper [Ch]. We consider a standard graded polynomial ring $U = k[z_1, \ldots, z_p]$ over a field k, where all variables have degree one. Let L be a homogeneous ideal in the ring U, and let $\beta_{ij}(L) = \dim_k \operatorname{Tor}_i^U(L,k)_j$ be its graded Betti numbers. The *(Castelnuovo-Mumford) regularity* of L is

$$\operatorname{reg}(L) = \max\left\{ j \,\middle|\, \beta_{i,\,i+j}(L) \neq 0 \right\} .$$

Alternatively, regularity can be defined using local cohomology. Papers of Bayer-Mumford, Bayer-Stillman, and Koh, give examples of families of ideals attaining doubly exponential regularity. In contrast, Bertram-Ein-Lazarsfeld, Chardin-Ulrich, and Mumford have proved that there are nice bounds on the regularity of the ideals of smooth (or nearly smooth) projective varieties; see

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the expository paper [Ch2]. As discussed in the influential paper [BM] by Bayer and Mumford (1993), the biggest missing link between the general case and the smooth case is to obtain a decent bound on the regularity of all prime ideals (the ideals that define irreducible projective varieties). The long standing Eisenbud-Goto Regularity Conjecture predicts an elegant linear bound, in terms of the degree of the variety:

The Regularity Conjecture 1.1. (Eisenbud-Goto [EG], 1984) *Suppose that the field k is algebraically closed. If $L \subset (z_1, \dots, z_p)^2$ is a homogeneous prime ideal in U, then*

$$\mathrm{reg}(L) \ \leq \ \deg(U/L) - \mathrm{codim}(L) + 1\,,$$

where $\deg(U/L)$ *is the multiplicity of U/L (also called the degree of U/L, or the degree of $X = \mathrm{Proj}(U/L)$), and* $\mathrm{codim}(L)$ *is the codimension (also called height) of L.*

The conjecture is proved for curves by Gruson-Lazarsfeld-Peskine, for smooth surfaces by Lazarsfeld and Pinkham, for most smooth 3-folds by Ran and Kwak, if U/L is Cohen-Macaulay by Eisenbud-Goto, and in many other special cases.

Recently, McCullough and Peeva [MP] introduced two new techniques and used them to provide many counterexamples to the Eisenbud-Goto Regularity Conjecture. In this note we answer some natural questions which arise from the paper [MP].

The counterexamples in [MP] come from Rees-like algebras, which were introduced in [MP, Section 3]. Rees-like algebras, unlike the usual Rees algebras, have well-structured defining equations and minimal free resolutions. The properties of Rees algebras are of high interest and can be quite intricate (see for example [Hu], [KPU]). Several mathematicians have asked us if the defining ideals of Rees algebras contain counterexamples as well or whether the Regularity Conjecture holds for them. In Sections 3 and 4 we provide counterexamples using Rees algebras. In the latter section we study standard graded Rees algebras that arise as Rees algebras of ideals generated in one degree.

The main theorem in [MP] shows that the regularity of prime ideals is not bounded by any polynomial function of the multiplicity. It is natural to ask if there exists a bound on regularity in terms of the multiplicity alone. Such a bound does not exist for primary ideals (Example 5.5). However, we prove in Section 5 that the recent work of Ananyan-Hochster [AH] (who solved Stillman's Conjecture) implies the existence of the desired bound for prime ideals.

In the counterexamples in [MP] the multiplicity is smaller than the maximal degree of a minimal generator of a prime ideal. One may wonder whether there are prime ideals for which the difference between the maximal degree of a minimal generator and the maximal degree of a minimal (first) syzygy can be made arbitrarily large. In Section 6 we show that such prime ideals exist. We obtain them by starting with Ullery's designer ideals (which are not prime) [Ul] and applying to them the method by McCullough-Peeva in order to get prime ideals.

In Section 7 we construct a family of three-generated ideals whose regularity grows faster than the product of the degrees of the generators. To our knowledge, this is the only known such family other than those based on the Mayr-Meyer construction. Applying the construction in [MP] we use this family to construct an infinite family of counterexamples to the Eisenbud-Goto Regularity Conjecture that do not rely on the Mayr-Meyer construction.

2. **Multiplicity of prime ideals**

Throughout this section, we consider a polynomial ring $W = k[w_1, \dots, w_p]$ over an arbitrary field k and positively graded with $\deg(w_i) \in \mathbb{N}$ for every i. Suppose $c_i := \deg(w_i) > 1$ for $i \leq q$ and $\deg(w_i) = 1$ for $i > q$ (for some $q \leq p$).

A function $Q : \mathbb{Z} \longrightarrow \mathbb{Q}$ is a *quasipolynomial (over* $\mathbb{Q}$*)* of degree r if

$$Q(n) = a_r(n)n^r + a_{r-1}(n)n^{r-1} + \cdots + a_1(n)n + a_0(n),$$

where $a_i : \mathbb{Z} \longrightarrow \mathbb{Q}$ is a periodic function for each $i = 0, \dots, r$ and $a_r \neq 0$. A natural number v is called a *period* of Q if

$$a_i(n+v) = a_i(n) \text{ for all } n \in \mathbb{Z} \text{ and for all } i = 0, \dots, r.$$

Let M be a homogeneous ideal in the polynomial ring W. The *Hilbert function* $h_{W/M} : \mathbb{Z}_{\geq 0} \longrightarrow \mathbb{Z}_{\geq 0}$ of W/M is $h_{W/M}(n) = \dim_k (W/M)_n$. It is often studied via the *Hilbert series* $\mathrm{Hilb}_{W/M}(u) = \sum_{n \geq 0} u^n \dim_k (W/M)_n$. By a theorem of Hilbert-Serre, there is a quasipolynomial $Q(n)$ of degree $\dim(W/M) - 1$ and period $\mathrm{lcm}(c_1, \dots, c_p)$ such that

$$h_{W/M}(n) = Q(n) \text{ for } n \gg 0.$$

For a proof, see for example [BI].

Set $d := \gcd(c_1, \dots, c_p)$, and observe that $Q(dj + t) = 0$ for $0 < t < d$.

Proposition 2.1. *If M is a prime ideal, then $a_r(dj)$ is a constant (independent of the parameter j), which we denote a_r.*

Proof. We may easily reduce to the case $d = 1$ by dividing the degrees of the variables by their greatest common divisor.

Assume the opposite. Set $a(n) = a_r(n)$. Let m and $m+s$ be two different integers for which the Hilbert function agrees with the quasipolynomial Q and such that $a(m) > a(m+s)$. Since $\gcd(c_1, \ldots, c_p) = 1$, there exist $\ell_i \in \mathbb{Z}$ such that $s = \sum_1^q \ell_i c_i$. Hence, $m + s = m + \sum_1^q \ell_i c_i$. Adding a large positive multiple of $b := \operatorname{lcm}(c_1, \ldots, c_p)$ to the righthand-side, we get

$$a(m+s) = a\left(m + \sum_{i=1}^{q} \ell_i c_i + vb\right) = a\left(m + \sum_{i=1}^{q} \ell'_i c_i\right)$$

where each ℓ'_i is positive. Foir each i, as w_i is a non-zerodivisor, we have an inclusion $w_i(W/M)_j \subseteq (W/M)_{j+c_i}$ and thus $\dim_k (W/M)_j \leq \dim_k (W/M)_{j+c_i}$ for every $j \geq 0$. Hence,

$$a(m+s) = a\left(m + \sum_{i=1}^{q} \ell'_i c_i\right) \geq a(m)\,,$$

which is a contradiction. □

If M is a prime ideal in W, then we call

$$e_{Hilb}(M) := r! a_r$$

the *Hilbert multiplicity* of S/M or of M, and also denote it by $\deg(M)$ or $e_{Hilb}(W/M)$.

On the other hand, recall the construction of the Euler polynomial:

Notation 2.2. Fix a finite graded complex $\mathbf{V}$ of finitely generated W-free modules and with $V_i = 0$ for $i < 0$. We may write $V_i = \oplus_{j \in \mathbf{Z}} W(-j)^{b_{ij}}$. The *Euler polynomial* of $\mathbf{V}$ is

$$\mathrm{E}_{\mathbf{V}} = \sum_{i \geq 0} \sum_{j \in \mathbf{Z}} (-1)^i b_{ij} u^j\,.$$

Let N be a graded finitely generated W-module, and let $\mathbf{V}$ be a finite graded free resolution of N. Since every graded free resolution of N is isomorphic to the direct sum of the minimal graded free resolution and a trivial complex, it follows that the Euler polynomial does not depend on the choice of the resolution, so we call it the *Euler polynomial* of N. We factor out a maximal possible power of $1-u$ and write

$$\mathrm{E}_{\mathbf{V}} = (1-u)^c\, h_{\mathbf{V}}(u)\,,$$

where $h_{\mathbf{V}}(1) \neq 0$.

We set $N = W/M$ and in the notation above, we call

$$e_{Euler}(M) := h_{\mathbf{V}}(1)$$

the *Euler multiplicity* of S/M or of M, and also denote it by $e_{Euler}(W/M)$.

A prime ideal M is called *non-degenerate* if $M \subset (w_1, \ldots, w_p)^2$.

Theorem 2.3. *If M is a non-degenerate homogeneous prime ideal in W, then*

$$e_{Euler}(M) = e_{Hilb}(M) \prod_{i=1}^{p} \deg(w_i)\,.$$

The proof uses the technique of step-by-step homogenization introduced in [MP]. Theorem 2.3 is an immediate corollary of Theorem 2.5.

The following result from [MP] describes the step-by-step homogenization technique:

Step-by-step Homogenization Theorem 2.4. [MP] *Let M be a homogeneous non-degenerate prime ideal, and let $\mathcal{K}$ be a minimal set of homogeneous generators of M. Consider the homogenous map (of degree 0)*

$$\begin{aligned} \nu : W = k[w_1, \ldots, w_p] &\longrightarrow W' := k[w_1, \ldots, w_p, v_1, \ldots, v_q] \\ w_i &\longmapsto w_i v_i^{\deg_W(w_i)-1} \quad \text{for } 1 \leq i \leq q\,, \end{aligned}$$

where $v_1, \ldots, v_q$ are new variables and W' is standard graded. The ideal $M' \subset W'$ generated by the elements of $\nu(\mathcal{K})$ is a homogeneous non-degenerate prime ideal in W'. Furthermore, the graded Betti numbers of W'/M' over W' are the same as those of W/M over W.

We say that M' is obtained from M by *step-by-step homogenization* or by *relabeling*. The relation between the multiplicities of M and M' is given in the next result:

Theorem 2.5. *In the notation and under the assumptions of Theorem 2.4 we have*

$$\begin{aligned} e_{Euler}(M') &= e_{Hilb}(M') \\ e_{Euler}(M') &= e_{Euler}(M) \\ e_{Hilb}(M') &= e_{Hilb}(M) \prod_{i=1}^{p} \deg(w_i)\,. \end{aligned}$$

Proof. The first equality holds because M' is a homogeneous ideal in a standard graded ring. The second equality holds because step-by-step homogenization preserves the graded Betti numbers by Theorem 2.4. We will prove the third equality.

We have the same Euler polynomial $\mathrm{E}(u) := \mathrm{E}_{\mathbf{M}} = \mathrm{E}_{\mathbf{M}'}$ by Theorem 2.4. We get the Hilbert series

$$\mathrm{Hilb}_{W/M}(u) = \frac{\mathrm{E}(u)}{(1-u)^{p-q}\prod_{i=1}^{q}\left(1-u^{\deg(w_i)}\right)}$$
$$\mathrm{Hilb}_{W'/M'}(u) = \frac{\mathrm{E}(u)}{(1-u)^{p+q}}\,.$$

Therefore,

$$\begin{aligned}\mathrm{Hilb}_{W'/M'}(u) &= \frac{\prod_{i=1}^{q}\left(1-u^{\deg(w_i)}\right)}{(1-u)^{2q}}\,\mathrm{Hilb}_{W/M}(u)\\ &= \frac{\prod_{i=1}^{q}\left(1+u+\cdots+u^{\deg(w_i)-1}\right)}{(1-u)^{q}}\,\mathrm{Hilb}_{W/M}(u)\,.\end{aligned}$$

Note that the factor in front of $\mathrm{Hilb}_{W/M}(u)$ is a series with positive coefficients. Apply Lemma 2.6. ☐

If S/M has Hilbert multiplicity a, we say that its Hilbert series has it too.

Lemma 2.6. *Let h be a Hilbert function given by a quasipolynomial $Q(n)$ of degree r and with constant leading coefficient $a := r!a_r$. Let g be the Hilbert series of h.*

(1) *The Hilbert multiplicity of the Hilbert series $\dfrac{g}{1-u}$ is the same.*
(2) *The Hilbert multiplicity of the Hilbert series $\left(1+u+\cdots+u^b\right)g$ is $(b+1)a$.*

Proof. We have $g = \sum_{n\geq 0} Q(n)u^n$.

(1) Since

$$(1+u+u^2+\cdots)g = \sum_{n\geq 0} u^n\left(Q(0)+Q(1)+\cdots+Q(n)\right)$$

the quasipolynomial for the considered Hilbert series is

$$\begin{aligned}Q(n)+\cdots+Q(0) &= a_r\left(n^r+(n-1)^r+\cdots+1\right)\\ &\quad + a_{r-1}\left(n^{r-1}+(n-1)^{r-1}+\cdots+1\right)\\ &\quad + \cdots\\ &= \frac{a_r}{r+1}n^{r+1} + \text{terms of lower degree}\,.\end{aligned}$$

Thus the Hilbert multiplicity is $(r+1)!\frac{a_r}{r+1} = r!a_r$.

(2) Since

$$(1+u+\cdots+u^b)g = \sum_{n\geq 0} u^n\big(Q(n)+\cdots+Q(n-b)\big)$$

the quasipolynomial for the considered Hilbert series is

$$\begin{aligned} Q(n)+\cdots+Q(n-b) &= a_r\big(n^r+(n-1)^r+\cdots+(n-b)^r\big) \\ &\quad + a_{r-1}\big(n^{r-1}+(n-1)^{r-1}+\cdots+(n-b)^{r-1}\big) \\ &\quad + \cdots \\ &= a_r(b+1)n^r + \text{terms of lower degree}\,. \end{aligned}$$

□

3. Multiplicities of Rees algebras and Rees-like algebras

In this section we provide counterexamples to the Eisenbud-Goto Regularity Conjecture [EG] using Rees algebras.

Notation 3.1. We follow the notation in [MP]. Consider the polynomial ring

$$S = k[x_1,\ldots,x_n]$$

over a field k with a standard grading defined by $\deg(x_i)=1$ for every i. Let I be a homogeneous ideal minimally generated by forms $f_1,\ldots,f_m$ of degrees $a_1,\ldots,a_m$, where $m\geq 2$.

Theorem 3.2. *Consider the Rees algebra $S[It]$ and the Rees-like algebra $S[It,t^2]$. We have:*

$$\begin{aligned} e_{Hilb}(S[It]) &\leq e_{Hilb}(S[It,t^2]) = 1 \\ e_{Euler}(S[It]) &\leq \frac{1}{2}e_{Euler}(S[It,t^2]) = \prod_{i=1}^{m}(\deg(f_i)+1)\,. \end{aligned}$$

Denote by Q and L the defining ideals of $S[It,t^2]$ and $S[It]$ respectively. Then

$$\deg L' \leq \frac{1}{2}\deg Q'\,,$$

where L' and Q' (Q' is denoted by P in [MP]*) are the respective step-by-step homogenizations of L and Q.*

Proof. Note that

$$\dim\left(S[t]\right) = \dim\left(S[It]\right) = \dim\left(S[It,t^2]\right) = n+1$$

since they are domains of the same transcendence degree. Since $S[It] \subset S[It, t^2] \subset S[t]$ we conclude

$$0 < e_{Hilb}(S[It]) \leq e_{Hilb}(S[It, t^2]) \leq e_{Hilb}(S[t]) = 1\,.$$

The first and second equalities below come from Theorem 2.3:

$$\begin{aligned} e_{Euler}(S[It]) &= e_{Hilb}(S[It]) \prod_{i=1}^{m} \deg(y_i) \leq e_{Hilb}(S[It, t^2]) \prod_{i=1}^{m} \deg(y_i) \\ &= \frac{1}{2} e_{Euler}(S[It, t^2]) = \frac{1}{2} e_{Euler}(Q') = \prod_{i=1}^{m} (\deg(f_i) + 1)\,. \end{aligned}$$

The factor $\frac{1}{2}$ comes from the variable z which has degree 2. The last equality holds by [MP, Theorem 1.6(2)], and the equality before holds by Theorem 2.5.

The inequality $\deg L' \leq \frac{1}{2} \deg Q'$ now follows from Theorem 2.5. □

For a graded ideal N (in a positively graded polynomial ring), we denote by $\text{maxdeg}(N)$ the maximal degree of an element in a minimal system of homogeneous generators of N.

Theorem 3.3. *For $r \in \mathbf{N}$ we consider the step-by-step homogenization L'_r of the defining ideal L_r of the Rees algebra $S[I_r t]$, where I_r is the Koh ideal used in Counterexample 1.8(1) in* [MP]. *Then multiplicity and maxdeg of the prime ideal L'_r satisfy*

$$\deg L'_r \leq 2 \times 3^{22r-3}$$

$$\text{maxdeg}\, L'_r \geq 2^{2^{r-1}} + 1\,.$$

Thus it is a counterexample to the Regularity Conjecture (see [MP, 1.2]*) for $r \geq 10$.*

Proof. Note that L_r contains all the y-linear minimal generators listed in [MP, (3.4)]. They are minimal generators of L_r by [MP, Proposition 2.9] and since L_r cannot contain any elements in which no $y_1, \ldots, y_m$ appears.

Let P_r be the prime ideal used in Counterexample 1.8(1) in [MP]. Then

$$\deg L'_r \leq \frac{1}{2} \deg P_r \leq 2 \times 3^{22r-3}$$

by Theorem 3.2 and [MP, Counterexample 1.8(1)]. □

Similarly, Counterexample 1.8(2) in [MP] leads to Rees-algebra counterexamples to the Regularity Conjecture.

4. **Standard graded Rees Algebras**

In this section, we provide a different view than Theorem 3.3 on using Rees Algebras to produce examples of large regularity. We focus on standard graded Rees Algebras, which arise as the Rees algebras of ideals generated in one degree.

First, we observe how to reduce to the case of ideals generated in one degree.

Construction 4.1. We follow the notation in [MP]. Consider the standard graded polynomial ring $S = k[x_1, \dots, x_n]$ over a field k. Let I be a homogeneous ideal minimally generated by forms $f_1, \dots, f_m$ of degrees $a_1, \dots, a_m$, where $m \geq 2$. Set $d = \max_i\{a_i\}$. Consider a new ideal $\widetilde{I}$ generated by the forms $\{x^{d-a_i} f_i\}$ of degree d in the polynomial ring $\widetilde{S} = S[x]$. We bigrade $\widetilde{S}$ by $\deg(x_i) = (1,1)$ for every i and $\deg(x) = (0,1)$. The ideal $\widetilde{I}$ is bigraded, and therefore $\widetilde{S}/\widetilde{I}$ has a bigraded minimal free resolution $\widetilde{\mathbf{U}}$ over $\widetilde{S}$. The regularity $\tilde{r}$ of $\widetilde{S}/\widetilde{I}$ (assuming standard grading) is equal to the regularity of $\widetilde{\mathbf{U}}$ with respect to the second coordinate of the bigrading. It follows that it is bigger than the regularity r' of $\widetilde{\mathbf{U}}$ with respect to the first coordinate of the bigrading since $\deg(x) = (0,1)$. Observe that $x - 1$ is a non-zerodivisor (for degree reasons using the second coordinate of the bigrading) on $\widetilde{S}/\widetilde{I}$. Therefore, $\widetilde{\mathbf{U}} \otimes \widetilde{S}/(x-1)$ is a graded (posibly non-minimal) free resolution of S/I over S. Hence, the regularity r of S/I is smaller than r'. We showed that

$$\operatorname{reg}_S(I) \leq \operatorname{reg}_{\widetilde{S}}(\widetilde{S}/\widetilde{I})\,.$$

In fact, we have such an inequality in every homological degree, that is,

$$\operatorname{maxdeg}(\operatorname{Syz}_i^S(S/I)) \leq \operatorname{maxdeg}(\operatorname{Syz}_i^{\widetilde{S}}(\widetilde{S}/\widetilde{I}))\,,$$

where $\operatorname{maxdeg}(N)$ stands for the maximal degree in a system of minimal homogeneous generators of a graded finitely generated module N.

Now, we consider the Rees Algebra $R := \widetilde{S}[\widetilde{I}t]$ as a standard graded ring. Its prime graded (with respect to the standard grading) defining ideal T satisfies

$$\operatorname{maxdeg}(T) \geq \operatorname{maxdeg}(\operatorname{Syz}_1^{\widetilde{S}}(\widetilde{I})) - (d-1) \geq \operatorname{maxdeg}(\operatorname{Syz}_1^S(I)) - (d-1)\,.$$

Example 4.2. We will apply Construction 4.1 to Koh's examples based on the Mayr-Meyer [MM] construction. For $r \geq 1$, Koh constructed in [Ko] an ideal I_r generated by $22r - 3$ quadrics and one linear form in a polynomial ring with $22r - 1$ variables, and such that $\operatorname{maxdeg}(\operatorname{Syz}_1(I_r)) \geq 2^{2^{r-1}}$. The construction above produces an ideal $\widetilde{I}_r$ generated by $22r - 2$ quadrics in a polynomial ring with $22r$ variables, and such that $\operatorname{maxdeg}(T_r) \geq 2^{2^{r-1}} - 1$. On the other hand, by Theorem 4.3, $\deg(T_r) \leq 2^{\min\{22r-2,\, 22r\}} - 1 = 2^{22r-2} - 1$. Thus, $\deg(T_r) < \operatorname{maxdeg}(T_r)$ for $r \geq 10$.

Now, we turn to Rees Algebras.

Theorem 4.3. *Let M be an ideal generated by $m \geq 1$ forms of the same degree $d \geq 2$ in $R = k[X_1, \ldots, X_n]$, and $R_M \cong R[Mt]$ be the Rees Algebra of M which is considered as a standard graded quotient of the polynomial ring $R[Y_1, \ldots, Y_m]$. Then,*

$$\deg(R_M) \leq \frac{d^{\min\{m,n\}} - 1}{d - 1},$$

and equality holds if further M is $(X_1, \ldots, X_n)$-primary or its m generators form a regular sequence.

Proof. Call $g_1, \ldots, g_m$ the given m generators of M.

First give bidegree $(1, 0)$ to the X_i's and bidegree $(0, 1)$ to the Y_i's.

Notice that each bigraded component $(R_M)_{p,j}$ for this bigrading has at most a vector space dimension equal to the one of the Rees algebra associated to generic forms (ones with indeterminate coefficients). Indeed, this dimension can be computed as the rank of a Sylvester matrix associated to the collection of elements $g_{i_1} \cdots g_{i_j}$ with $i_1 \leq \cdots \leq i_j$ in the degree $p + jd$.

This inequality in turn shows that the Hilbert function of R_M is bounded above by the one given by generic forms (over the extension of k generated by the coefficients).

If $m \leq n$ the generic forms are providing a regular sequence. And any complete intersection has its Rees algebra resolved by the Eagon-Northcott complex of the $(2 \times m)$-matrix with maximal minors $Y_s g_r - Y_r g_s$.

Let $\operatorname{Hilb}_{R_M}(u)$ be the Hilbert series of R_M. Whenever $m \leq n$ this shows the (term by term) inequality :

$$\operatorname{Hilb}_{R_M}(u) \leq \frac{(1-u^d)^{m-1} + u^d\big(\sum_{i=0}^{m-2}(1-u^d)^i(1-u)^{m-i-1}\big)}{(1-u)^{m+n}}.$$

As $M \neq 0$, the dimension of the Rees algebra of M is the same as the one of a complete intersection and hence the inequality above shows that the degree of R_M is bounded above by the one corresponding to a complete intersection, whose value is $d^{m-1} + d^{m-2} + \cdots + d + 1$ by the above formula for the complete intersection case.

Notice further that any graded ideal generated in degree d is a subideal of $J := (X_1, \ldots, X_n)^d$. It follows that the Hilbert series of R_M is bounded above by the one of R_J. Again, recall that R_M and R_J have same dimension, hence $\deg(R_M) \leq \deg(R_J) = \frac{d^n-1}{d-1}$.

We computed the multiplicity in the complete intersection case, and if M is $(X_1, \ldots, X_n)$-primary it contains an ideal M' generated by a regular sequence of forms of degree d. The degree of R_M is hence bounded below by the degree of $R_{M'}$ and above by the one R_J; these are both equal to $\frac{d^n-1}{d-1}$ and the conclusion follows. □

5. Regularity is bounded in terms of multiplicity

In this section we show that an upper bound on regularity of non-degenerate prime ideals in terms of the multiplicity alone follows from the recent work of Ananyan and Hochster [AH], who solved Stillman's Conjecture. From now on, the polynomial rings occurring in the paper are standard graded.

Lemma 5.1. *Let L_1 and L_2 be two homogeneous ideals of $U = k[X_1, \ldots, X_N]$ whose number and degrees of generators are bounded by a constant c. Then the number and the degrees of the generators of $L_1 \cap L_2$ are bounded by a constant depending only on c.*

Proof. Looking at the exact sequence

$$0 \longrightarrow \frac{U}{L_1 \cap L_2} \longrightarrow \frac{U}{L_1} \oplus \frac{U}{L_2} \longrightarrow \frac{U}{L_1 + L_2} \longrightarrow 0$$

we infer the following inequality for any $j \in \mathbb{Z}$:

$$\dim_k \operatorname{Tor}_0^U \left(\frac{U}{L_1 \cap L_2}, k \right)_j \leq \dim_k \operatorname{Tor}_0^U \left(\frac{U}{L_1}, k \right)_j + \dim_k \operatorname{Tor}_0^U \left(\frac{U}{L_2}, k \right)_j + \dim_k \operatorname{Tor}_1^U \left(\frac{U}{L_1 + L_2}, k \right)_j .$$

By [AH, Theorem D (a)] the regularity and the graded Betti numbers of $\frac{U}{L_1+L_2}$ are bounded by a constant depending only on c, so we get the desired property. □

Theorem 5.2. *Let e and h be positive integers and k be a field. There exist constants, depending only on e and h, bounding respectively the projective dimension, regularity, and the graded Betti numbers of every homogeneous unmixed radical ideal of multiplicity e and height h in a standard graded polynomial ring over k.*

Proof. We may assume $e \geq 2$. Let L be a homogeneous unmixed radical ideal of $U = k[X_1, \ldots, X_N]$ of multiplicity e and height h.

First, we will show that the ideal L contains a regular sequence $g_1, \dots, g_h$ of forms of degrees less than or equal to e. Choose a Noether normalization $k[X_1, \dots, X_d]$ (this may need a finite extension of the base field to change coordinates if $|k| \leq e$, but this extension keeps L unmixed and radical and does not affect the invariants we are bounding). Then the generators of $L \cap k[X_1, \dots, X_d, X_{d+i}] = (g_i)$ for $i = 1, \dots h$ form a regular sequence of forms of degrees at most e.

Let $\mathfrak{b} := (g_1, \dots, g_h)$ and

$$m := \sum_{i=1}^{h} (\deg g_i - 1) = \operatorname{reg}(U/\mathfrak{b})\,.$$

If $L = \mathfrak{b}$ the assertion is clear, so suppose $L \neq \mathfrak{b}$. If $\mathfrak{p}_i$ is a minimal prime of $\mathfrak{b}$, then there exists a form f_i of degree m such that $\mathfrak{p}_i = \mathfrak{b} : (f_i)$ (by [Ch3, 4.1] or [CU, 1.2], for instance). Hence, if $L = \cap_{i=1}^{a} \mathfrak{p}_i$, then $L = \mathfrak{b} : (f)$ with $f := \sum_{i=1}^{a} f_i$. The exact sequence

$$0 \longrightarrow U/L(-m) \longrightarrow U/\mathfrak{b} \longrightarrow U/(\mathfrak{b} + (f)) \longrightarrow 0$$

then shows that $\operatorname{reg}(U/L) = \operatorname{reg}\big(U/(\mathfrak{b} + (f))\big) - m + 1$. Hence the regularity of L is bounded by the one of an ideal generated in degrees $(e, \dots, e, m)$ (e repeated h times, and $m \leq h(e-1)$).

By [AH, Theorem D (a)] it follows that the projective dimension, regularity, and Betti numbers of L are bounded as well by constants depending only on e and h. □

Corollary 5.3. *Let e be a positive integer and k be an algebraically closed field. There exist constants, depending only on e, bounding the projective dimension, regularity and graded Betti numbers of every homogeneous non-degenerate prime ideal in a polynomial ring over k of multiplicity e.*

Proof. The claim follows immediately by Theorem 5.2 since the height of a homogeneous non-degenerate prime ideal in a polynomial ring over an algebraically closed field is less than its multiplicity. □

Remarks 5.4.

(1) This corollary also holds for reduced, equidimensionnal ideals that are connected in codimension 1 (or connected in codimension c, for any given c). Also the conclusion concerning the regularity holds without the condition of being non-degenerate.

(2) Notice that applying regularity bounds in terms of degrees of defining equations, following works of Galligo [Ga] and Giusti[Gi], Bayer-Mumford [BM], Caviglia-Sbarra [CS], one gets from the proof of Theorem 5.2 a bound on the regularity of U/L that depends upon e, h and the dimension of this

quotient (alternatively on its projective dimension). For instance, using the estimate given in [CFN, 3.5 (ii)] one has with notations as in the proof, and if $\dim(U/L) \geq 2$,

$$\operatorname{reg}\bigl(U/(\mathfrak{b} + (f))\bigr) \leq \Bigl[2h^2(e-1)^2 e^{h-1}\Bigr]^{2^{\dim(U/L)-2}},$$

and hence

$$\operatorname{reg}(U/L) < \Bigl[2h^2 e^{h+1}\Bigr]^{2^{\dim(U/L)-2}}.$$

Examples 5.5. We give three examples. They show that Corollary 5.3 cannot be generalized to radical ideals, nor to primary ideals.

(1) Fix $n \in \mathbb{N}$ and let

$$M = (x_1, \ldots, x_n) \cap (y_1, \ldots, y_n) \subseteq U = k[x_i, y_i \,|\, i = 1, \ldots, n]\,.$$

Then M is a non-degenerate unmixed radical ideal of multiplicity 2, but

$$\operatorname{projdim} U/M = 2n - 1\,.$$

(2) Fix $n \in \mathbb{N}$ and let

$$M = (x^2,\ xy,\ y^2,\ xa^n + yb^n) \subseteq U = k[x, y, a, b]\,.$$

It follows from [En, Lemma 10] that

(i) M is a non-degenerate ideal of regularity equal to $n + 1$;
(ii) $e(U/M) = 2$;
(iii) M is (x, y)-primary.

(3) Fix $n, e \in \mathbb{N}$ with $e \geq 3$. By [HMMS, Theorem 1.2] there exists an ideal M in a polynomial ring U over k such that:

(i) M is a non-degenerate ideal of projective dimension at least n;
(ii) $e(U/M) = e$;
(iii) M is (x, y)-primary, where x and y are independent linear forms.

These examples still leave open the question whether there exists a bound on the regularity of unmixed radical ideals (over an algebraically closed field) in terms of the multiplicity alone.

6. **Prime ideals from designer ideals**

In this section, we apply the method in [MP] to the designer ideals constructed by Ullery in [Ul].

First we define notation related to maximal shifts. Let $T = k[x_1, \dots, x_n]$ and let M be a finitely generated T-module. Set

$$t_i^T(M) = \max\{j \mid \operatorname{Tor}_i^T(M,k)_j \neq 0\} = \max\{j \mid \beta_{ij}^T(M) \neq 0\}.$$

Thus $t_0^T(M)$ is the maximal degree of an element in a minimal generating set of M and $t_1^T(M)$ is the maximal degree of a minimal first syzygy of M. The *maximal shifts* $t_i^T(M)$ are related to regularity by

$$\operatorname{reg}(M) = \max_{0 \leq i \leq \operatorname{projdim}(M)} \big\{t_i^T(M) - i\big\}.$$

We state a version of a result of Ullery:

Theorem 6.1 ([Ul, Theorem 1.3])**.** *Let $T = k[x_1, \dots, x_n]$ and let M be a finitely generated T-module generated in a single non-negative degree with strictly increasing sequence of maximal graded shifts $\big(t_0^T(M), t_1^T(M), \dots, t_r^T(M)\big)$. Set $a = t_0^T(M)$ and fix a positive integer N such that the number of elements in a minimal homogeneous generating set of M is $\leq \binom{N+a-1}{a}$. Then there exists an ideal J_M in $S = T[y_1, \dots, y_N]$ such that*

$$t_i^S(J_M) = \begin{cases} t_{i+1}^T(M) & \text{if } 0 \leq i \leq r-1 \\ t_r^T(M) + i - r + 1 & \text{if } r \leq i \leq N + r - 1. \end{cases}$$

In particular, if we pick M to be generated in degree $a = 1$, and N sufficiently large, then there exists an ideal J_M generated by homogeneous quadrics with any strictly increasing sequence as an initial sequence of its maximal graded shifts.

We now take the step-by-step homogenization of the defining prime ideal of the Rees-like algebra of J_M to produce prime ideals over any field with generators in degree at most 6 and arbitrarily large degree of first syzygies:

Theorem 6.2. *Fix a positive integer $s \geq 9$ and field k. There exists a non-degenerate prime ideal P in a polynomial ring R over k with $t_0^R(P) = 6$ and $t_1^R(P) = s$.*

Proof. Take $T = k[x_1, x_2, x_3]$ and $M = \operatorname{Ext}_S^3\big(T/(x_1, x_2, x_3)^{s-3}, S\big)(4-s)$ (where $(4-s)$ denotes a shift of degrees). Then M is a Cohen-Macaulay module with Betti table of the form:

	0	1	2	3
1:	$*$	$*$	$*$	-
2:	-	-	-	-
3:	-	-	-	-
$\vdots$	-	-	-	-
s-4:	-	-	-	$*$

where "∗" denotes a non-zero entry and "-" denotes a zero entry. Thus,

$$t_0^T(M) = 1,\ t_1^T(M) = 2,\ t_2^T(M) = 3,\ t_4^T(M) = s - 1\,.$$

By Theorem 6.1, there is an ideal J in a larger polynomial ring S with

$$t_0^S(J) = 2,\ t_1^S(J) = 3,\ t_2^S(J) = s - 1$$

and $t_i^S(J) = s + i - 3$ for $3 \le i \le \operatorname{projdim} J$.

Now let P be the step-by-step homogenization of the defining prime ideal of the Rees-like algebra of J in a larger polynomial ring R, as constructed in [MP]. By [MP, Theorem 1.6],

$$t_0^R(P) = \max\left\{2\big(t_0^S(J)+1\big),\, t_1^S(J)+1\right\} = \max\{6,4\} = 6.$$

The structure of the minimal free resolution of P in [MP, Theorem 3.10] implies that

$$t_1^R(P) = \max\left\{3\big(t_0^S(J)+1\big),\, t_1^S(J)+t_0^S(J)+2,\, t_2^S(J)+1\right\} = \max\{9,7,s\} = s\,.$$

□

7. Ideals with Large Regularity

In this section we provide an infinite family of counterexamples to the Eisenbud-Goto Regularity Conjecture that do not rely on the Mayr-Meyer construction.

Proposition 7.1. *Let T be a polynomial ring over a field k and let $J = (f, g, h)$ be a homogeneous ideal of T such that f, g, h all have the same degree. Let x, y be new variables and set $S = T[x, y]$. Let*

$$I = (x^3, y^3, x^2 f + xyg + y^2 h)\,.$$

Then

$$\operatorname{reg}_S(S/I) \ge \operatorname{reg}_T(T/J) + 4\,.$$

Proof. Note that S/I is finitely generated as a T-module. In fact, $x^2y^2(S/I) \cong (T/J)(-4)$ is a T-direct summand of S/I. Since S is faithfully flat over T, we have

$$\operatorname{reg}_S(S/I) \ge \operatorname{reg}_T(x^2y^2(S/I)) = \operatorname{reg}_T(T/J) + 4.$$

□

In [Ca, Example 4.2.1] Caviglia showed that if $T = k[z_1, z_2, z_3, z_4]$ and

$$J = (z_1^d,\ z_2^d,\ z_1 z_3^{d-1} - z_2 z_4^{d-1})$$

with $d \geq 2$, then $\operatorname{reg}(T/J) = d^2 - 2$. We set $S = T[x, y]$ and

$$I = \left(x^3,\, y^3,\, x^2 z_1^d + xy(z_1 z_3^{d-1} - z_2 z_4^{d-1}) + y^2 z_2^d\right).$$

By the previous proposition we see that $\operatorname{reg}(S/I) \geq d^2 + 2$ for $d \geq 2$, while the degrees of the three generators of I are $3, 3$, and $d+2$. By [MP, Theorem 1.6] we obtain the following result:

Theorem 7.2. *Let P be the step-by-step homogenization of the Rees-like algebra of the ideal I above, in the polynomial ring R (as constructed in* [MP]*). Then*

$$\deg(R/P) = 32(d+3)$$
$$\operatorname{reg}(R/P) \geq d^2 + d + 12\,.$$

In particular, the Eisenbud-Goto conjecture fails when $d \geq 34$.

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