SOME IDEALS WITH LARGE PROJECTIVE DIMENSION

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Abstract. For an ideal I in a polynomial ring over a field, a monomial support of I is the set of monomials that appear as terms in a set of minimal generators of I. Craig Huneke asked whether the size of a monomial support is a bound for the projective dimension of the ideal. We construct an example to show that, if the number of variables and the degrees of the generators are unspecified, the projective dimension of I grows at least exponentially with the size of a monomial support. The ideal we construct is generated by monomials and binomials.

1. Introduction

Let R be a polynomial ring over a field k and let $I \subseteq R$ be a homogeneous ideal. Two measures of the complexity of I are its projective dimension, and its (Castelnuovo-Mumford) regularity; see [Eis95].

There have been attempts to obtain uniform bounds for projective dimension and regularity based on numerical invariants of the ideal. Bounds on regularity are discussed in [Eis05]. M. Stillman asked if there is a bound for the projective dimension of an ideal having minimal generators in degrees $d_1 \leq d_2 \leq \cdots \leq d_r$, when the number of variables in the ring is not fixed. Only partial answers to this question are known; see [Eng05].

Related to Stillman's question, C. Huneke asked the following: is the size of a monomial support of an ideal a bound for its projective dimension? Here, by *a monomial support* of I, we mean the collection of monomials that appear as terms in a set of minimal generators of I. Note that an ideal can have different monomial supports. If I is a monomial ideal, generated by N monomials, then $\operatorname{pd} R/I \leq N$; this follows from the Taylor resolution of R/I which has length at most N [Eis95, Ex. 17.11].

We answer Huneke's question in the negative; in Sec. 2, we construct a binomial ideal to show that the projective dimension can grow exponentially with the size of a monomial support. Motivated by this example, we wonder:

Question 1. *Suppose $I \subseteq R$ has a monomial support of N monomials, counted with multiplicity. Then what is a good upper bound for $\operatorname{pd} R/I$?*

Let $n \geq 2, d \geq 2$ be arbitrary. The ideal we construct in the next section has a support of $2(n-1)(d-1)+n$ monomials counted with multiplicity and projective dimension n^d. Using this example, we show that for any positive integer N, the maximum of the projective dimension of an ideal I with a support of N monomials, counted with multiplicity, is at least $2^{\frac{N}{2}}$. Therefore any answer to Question 1 should be at least exponential. If the number of variables of R is not fixed, as in Stillman's question, the existence of any bound is still unknown.

Our decision of taking the multiplicity into account while counting the monomials in the support of I is only a matter of exposition. For example, let $m_1, \ldots, m_N$ be N distinct monomials, all of the same degree, and let $f_1, \ldots, f_r$, with $f_i = \sum_{j=i}^{N} a_{ij} m_j$ and a_j's in K, be a minimal system of generators for an ideal I. By doing an elimination, analogous to the one used in computing a reduced Gröbner basis, we can find a system of generators $g_1, \ldots, g_r$, $I = (g_1, \ldots, g_r)$, such that the initial monomial of g_i does not belong to the monomial support of g_j when $j \neq i$. In this way we get a monomial support for I of at most $\sum_{i=0}^{r-1}(N-2i) = -r^2 + r(N+1)$ monomials, counted with multiplicity. The maximum value of it, as a function of r, is $\lfloor (\frac{N+1}{2})^2 \rfloor$, which occurs when $r = \lfloor (N+1)/2 \rfloor$.

In general, if we have N distinct monomials in a monomial support of an homogeneous ideal I, then we would have at most $\lfloor (\frac{N+1}{2})^2 \rfloor$ of them when counted with multiplicity, this is because the above function is quadratic and the worst possible case happens precisely when I is generated by forms having the same degree.

2. Main Example

The following example is a slight generalization of the ideal mentioned in the introduction. Let $d \geq 2$ and let $n_i \geq 2$, $1 \leq i \leq d$ be positive integers. Denote by $\mathcal{I}$ the index set $\{1, \cdots, n_1\} \times \cdots \times \{1, \cdots, n_d\}$. Let $X := \{x_\nu : \nu \in \mathcal{I}\}$ be a d-dimensional array of variables and let $R = k[X]$. Let

$$s_{ij} := \prod_{\substack{\nu \in \mathcal{I} \\ \nu_i = j}} x_\nu, \ 1 \leq j \leq n_i, \ 1 \leq i \leq d.$$

We will call s_{ij} *the jth slice in the ith direction.* Fig. 1 illustrates the above definitions for a $3 \times 4 \times 2$ array. ($\ell_{.}$ in the figure will defined later.)

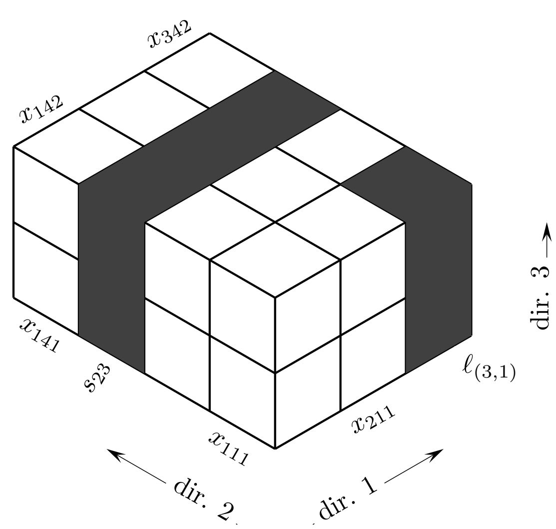

Figure 1. Slices of a $3 \times 4 \times 2$ array

Let $I = (s_{i1} - s_{ij} : 2 \le j \le n_i, 1 \le i \le d-1) + (s_{dj} : 1 \le j \le n_d)$. Then:

Proposition 2. *With notation as above,* $\operatorname{depth} R/I = 0$.

Proof. Write $\mathfrak{m}$ for the homogeneous maximal ideal of R and let

$$s := \prod_{i=1}^{d-1} \prod_{j=2}^{n_i} s_{ij}$$

s is the product of the variables not appearing in the first slices in each of the directions $1, \cdots, d-1$. We claim that $s \in (I : \mathfrak{m}) \setminus I$. Indeed, if $(I : \mathfrak{m}) \ne I$, then $\mathfrak{m}$ is an associated prime of R/I, so $\operatorname{depth} R/I = 0$.

We first reduce the proof to the case when $\operatorname{char} k = 0$, as follows. Since I is generated by monomials and binomials with ± 1 as coefficients, a Gröbner basis for I, and hence the ideal membership problem $s \in (I : \mathfrak{m}) \setminus I$ are independent of the characteristic of the field. See [Eis95] for the definition of a Gröbner basis and the ideal membership problem. We assume, from now on, that $\operatorname{char} k = 0$.

Let $\nu \in \mathcal{I}$. Using the binomial relations in I, we can write

$$s \equiv \prod_{i=1}^{d-1} s_{i1} \cdots \widehat{s_{i\nu_i}} \cdots s_{in_i} \mod I$$

where $\widehat{\cdot}$ denotes omitting the variable from the product. Consider the slice $s_{d\nu_d} = \prod_{\substack{\mu \in \mathcal{I} \\ \mu_d = \nu_d}} x_\mu$. If $\mu \ne \nu \in \mathcal{I}$ is such that $\mu_d = \nu_d$, then there exists $1 \le i \le d-1$ such that $\mu_i \ne \nu_i \implies x_\mu | (s_{i1} \cdots \widehat{s_{i\nu_i}} \cdots s_{in_i}) \implies s_{d\nu_d} | ((\prod_{i=1}^{d-1} s_{i1} \cdots \widehat{s_{i\nu_i}} \cdots s_{in_i}) x_\nu) \implies ((\prod_{i=1}^{d-1} s_{i1} \cdots \widehat{s_{i\nu_i}} \cdots s_{in_i}) x_\nu) \in I$. Hence $s \in (I : \mathfrak{m})$.

Let A be the tableau

$$\begin{array}{lcl} a_{11} & \cdots & a_{1n_1} \\ a_{21} & \cdots & a_{2n_2} \\ & \ddots & \\ a_{(d-1)1} & \cdots & a_{(d-1)n_{(d-1)}} \end{array}$$

of non-negative integers. We use *tableau* loosely here; we only mean that the rows of A possibly have different number of elements. For such a tableau A, we say it *satisfies row condition* $(c_1, \cdots, c_t)$ if the sum of the elements on the ith row is $c_i - 1$.

Let $\mathcal{P} := \{1, \cdots, n_1\} \times \cdots \times \{1, \cdots, n_{d-1}\}$. For each $p \in \mathcal{P}$, we define a monomial

$$\ell_p := \prod_{\substack{\nu \in \mathcal{I} \\ \nu_i = p_i, 1 \le i \le d-1}} x_\nu,$$

See Fig. 1 for an illustration of $\ell_{(3,1)}$ in the $3 \times 4 \times 2$ case. Further, write $|p|_A$ for $\sum_{i=1}^{d-1} a_{ip_i}$

Let

$$F = \sum_{A : A \text{ satisfies } (n_1, \cdots, n_{d-1})} \left\{ \prod_{p \in \mathcal{P}} \frac{1}{(|p|_A!)^{n_d}} \ell_p^{|p|_A} \right\} \tag{1}$$

We let R act on itself by partial differentiation with respect to the variables. We show below that, under this action, $s \notin (0 :_R F)$ while $I \subseteq (0 :_R F)$ from which we conclude that $s \notin I$, thus proving the proposition.

For any tableau A that satisfies the row condition $(n_1, \cdots, n_{d-1})$, write τ_A for the corresponding monomial term that appears in F (see (1)). Let

$$A_s := \begin{matrix} 0 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ & & \ddots & \\ 0 & 1 & \cdots & 1 \end{matrix}$$

Then $s = \alpha\tau_{A_s}$ for some non-zero rational number α If $A \neq A_s$, then s contains a variable that τ_A does not contain, so $s \circ F = s \circ \tau_{A_s} = 1$. Hence $s \notin (0 \colon {}_R F)$.

For any $1 \leq j \leq n_d$, $s_{dj} \circ F = 0$. For, any A that appears in the summation of (1) has at least one $p_A \in \mathcal{P}$ such that $|p_A|_A = 0$. Hence the variables in ℓ_{p_A} do not appear in τ_A. However, s_{dj} contains one such variable, and hence, $s_{dj} \circ \tau_A = 0 \implies s_{dj} \circ F = 0$.

Observe that any slice $s_{ij}, 1 \leq i \leq d-1, 1 \leq j \leq n_i$ can be written as a product of $\ell_p, p \in \mathcal{P}$ as follows:

$$s_{ij} = \prod_{\substack{\nu \in \mathcal{I} \\ \nu_i = j}} x_\nu = \prod_{\substack{1 \leq \nu_{i'} \leq n_{i'} \\ 1 \leq i' \leq d-1 \\ \nu_i = j}} \underbrace{\left[\prod_{1 \leq \nu_d \leq n_d} x_\nu \right]}_{\ell_{(\nu_1, \cdots, \nu_{d-1})}} = \prod_{\substack{p \in \mathcal{P} \\ p_i = j}} \ell_p$$

Let $\mathcal{P}_{ij} = \{p \in \mathcal{P} : p_i = j\}$. Then $s_{ij} \circ F = \left(\prod_{p \in \mathcal{P}_{ij}} \ell_p \right) \circ F$. Therefore to differentiate with respect to s_{ij}, we may differentiate with respect to all $\ell_p, p \in \mathcal{P}$, sequentially.

Let $1 \leq i \leq d-1, 1 \leq j \leq n_i$ and $q \in \mathcal{P}_{ij}$. Then

$$\begin{aligned} \ell_q \circ F = \ell_q \circ & \sum_{A : A \text{ satisfies } (n_1, \cdots, n_{d-1})} \left\{ \prod_{p \in \mathcal{P}} \frac{1}{(|p|_A!)^{n_d}} \ell_p^{|p|_A} \right\} \\ = & \sum_{A : A \text{ satisfies } (n_1, \cdots, n_{d-1})} \left\{ \frac{(|q|_A)^{n_d}}{(|q|_A!)^{n_d}} \ell_q^{(|q|_A - 1)} \prod_{\substack{p \in \mathcal{P} \\ p \neq q}} \frac{1}{(|p|_A!)^{n_d}} \ell_p^{|p|_A} \right\} \end{aligned}$$

Therefore,

$$s_{ij} \circ F = \sum_{A : A \text{ satisfies } (n_1, \cdots, n_{d-1})} \left\{ \prod_{p \in \mathcal{P}_{ij}} \frac{(|p|_A)^{n_d}}{(|p|_A!)^{n_d}} \ell_p^{(|p|_A - 1)} \prod_{p \notin \mathcal{P}_{ij}} \frac{1}{(|p|_A!)^{n_d}} \ell_p^{|p|_A} \right\} \tag{2}$$

We can write $\{A : A \text{ satisfies } (n_1, \cdots, n_{d-1})\} = \{A : a_{ij} = 0\} \bigcup \{A : a_{ij} \neq 0\}$. Every row of A contains at least one zero. If $a_{ij} = 0$, then there is a $p \in \mathcal{P}_{ij}$ such that $|p|_A = 0$. Therefore there is no contribution from those A with $a_{ij} = 0$ in the RHS of (2). Moreover, $a_{ij} \neq 0 \implies |p|_A \neq 0$. Hence

$$s_{ij} \circ F = \sum_{\substack{A \text{ satisfies } (n_1, \cdots, n_{d-1}) \\ a_{ij} \neq 0}} \left\{ \prod_{p \in \mathcal{P}_{ij}} \frac{1}{[(|p|_A - 1)!]^{n_d}} \ell_p^{(|p|_A - 1)} \prod_{p \notin \mathcal{P}_{ij}} \frac{1}{(|p|_A!)^{n_d}} \ell_p^{|p|_A} \right\}$$

There is a 1-1 correspondence between $\{A : A \text{ satisfies } (n_1, \cdots, n_{d-1}), a_{ij} \neq 0\}$ and $\{A : A \text{ satisfies } (n_1, \cdots, n_i - 1, \cdots, n_{d-1})\}$. Using this we can write

$$s_{ij} \circ F = \sum_{A \text{ satisfies } (n_1, \cdots, n_i - 1, \cdots, n_{d-1})} \left\{ \prod_{p \in \mathcal{P}} \frac{1}{(|p|_A!)^{n_d}} \ell_p^{|p|_A} \right\} \tag{3}$$

Note that this representation of $s_{ij} \circ F$ is independent of j; hence $(s_{i1} - s_{ij}) \circ F = 0$ for all $1 \leq i \leq d-1$ and $2 \leq j \leq n_i$. Hence $I \subseteq (0 :_R F)$. □

It now follows from the Auslander-Buchsbaum formula (see, *e.g*, [Eis95, Theorem 19.9] that

Corollary 3. *With notation as above,* $\operatorname{pd} R/I = n_1 \cdots n_d$. □

Parenthetically, we note that the ideal we construct has $n_i - 1$ generators of degree $n_1 \cdots \widehat{n_i} \cdots n_d$, for $1 \leq i \leq d-1$ and n_d generators of degree $n_1 \cdots n_{d-1}$.

Consider the case when $n_1 = \cdots = n_d = n$. Then the ideal is generated by $(n-1)(d-1)$ binomials and n monomials, and, hence, has a monomial support of $2(n-1)(d-1) + n$.

Corollary 4. *Any upper bound for projective dimension of an ideal supported on N monomials counted with multiplicity is at least $2^{N/2}$.*

Proof. Given a positive integer N, choose $n = 2$ variables in each of $d = \frac{N}{2}$ dimensions, and construct R and I as above. Then $\operatorname{pd} R/I = 2^{N/2}$. □

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