POSET EMBEDDINGS OF HILBERT FUNCTIONS

GIULIO CAVIGLIA AND MANOJ KUMMINI

ABSTRACT. For a standard graded algebra R, we consider embeddings of the poset of Hilbert functions of R-ideals into the poset of R-ideals, as a way of classification of Hilbert functions. There are examples of rings for which such embeddings do not exist. We describe how the embedding can be lifted to certain ring extensions, which is then used in the case of polarization and distraction. A version of a theorem of Clements–Lindström is proved. We exhibit a condition on the embedding that ensures that the classification of Hilbert functions is obtained with images of lexicographic segment ideals.

1. INTRODUCTION

Let R be a *standard graded algebra* over a field $\Bbbk$, i.e., $R \simeq \bigoplus_{d \geq 0} R_d$ as $\Bbbk$-vector-spaces with $R_0 = \Bbbk$, $R = \Bbbk[R_1]$ and $\dim_\Bbbk R_1 < \infty$. When R is a polynomial ring, a theorem of F. Macaulay (see, e.g., [BH93, Section 4.2]) provides a classification of the Hilbert functions of homogeneous R-ideals; more precisely, a function $H : \mathbb{N} \to \mathbb{N}$ is the Hilbert function of some homogeneous ideal if and only if it is the Hilbert function of an ideal generated by LEX-segments. (LEX denotes the graded lexicographic monomial order on a polynomial ring.) LEX-segment ideals in polynomial rings have been extensively studied. It is known that such ideals have several extremal properties; see [Big93, Hul93, Par96, Sba01, Con04, CHH04].

Let $A = \Bbbk[x_1, \ldots, x_n]$ be a polynomial ring, $\mathfrak{a} = (x_1^{e_1}, \ldots, x_n^{e_n})$ and $R = A/\mathfrak{a}$. J. Kruskal [Kru63] and G. Katona [Kat68] (for the case $e_1 = \cdots = e_n = 2$) and G. Clements and B. Lindström [CL69] (more generally, for all $2 \leq e_1 \leq \cdots \leq e_n \leq \infty$) proved that every homogeneous R-ideal has the same Hilbert function as the image (in R) of a LEX-segment A-ideal. The following conjecture of D. Eisenbud, M. Green and J. Harris furthered the interest in studying images of LEX-segment ideals in quotient rings: let A be the polynomial ring, as above, and let I be a homogeneous A-ideal containing an A-regular sequence of homogeneous polynomials $f_1, \ldots, f_n$ of degrees $e_1 \leq \cdots \leq e_n$; then there exists a LEX-segment ideal L such that the Hilbert functions of $L + (x_1^{e_1}, \ldots, x_n^{e_n})$ and I are identical. (See [EGH96, Conjecture CB12] for the original formulation, and [FR07] for a more recent survey.) In a similar vein, V. Gasharov, N. Horwitz, J. Mermin, S. Murai and I. Peeva studied algebras $R = A/\mathfrak{a}$ (where $\mathfrak{a}$ is graded A-ideal) for which every possible Hilbert function is attained by the images (in R) of LEX-segment A-ideals: quotients by compressed-monomial-plus-powers ideals [MP06], rational normal curves [GHP08], Veronese rings [GMP11] and quotients by coloured square-free monomial ideals [MM10]. In these papers, such rings are called Macaulay-lex, to emphasize the fact that every Hilbert function is attained by the image of a LEX-segment ideal, analogous to the theorem of Macaulay. Mermin, however, showed that most monomial complete intersections fail to be Macaulay-lex, even after a reordering of variables [Mer10, Theorem 4.4].

Motivated by these results, we consider two problems:

(i) Is there another approach to classification of Hilbert functions in quotient rings?
(ii) What is the significance of LEX-segment ideals?

To study this, we look at certain embeddings of the poset of Hilbert functions into the poset of R-ideals. When such embeddings exist, they induce a filtration of the R_d by $\Bbbk$-subspaces; for some R, this results in a degree-wise total order (which we call an *embedding order*) on a standard basis of R. Further, the embedding order respects multiplication precisely when all the Hilbert functions are given by images of LEX-segment ideals. We now describe this approach in detail.

By $\mathbb{N}$ we mean the set of non-negative integers. Let $\mathcal{I}_R = \{J : J \text{ is a homogeneous } R\text{-ideal}\}$, considered as a poset under inclusion. For $I \in \mathcal{I}_R$ and $t \in \mathbb{N}$, we will write I_t for the $\Bbbk$-vector-space of the homogeneous elements of I of degree t. (Note that $I \simeq \bigoplus_{t \in \mathbb{N}} I_t$, as $\Bbbk$-vector-spaces.) The *Hilbert function* of I is the function $\mathbb{N} \to \mathbb{N}, t \mapsto \dim_{\Bbbk} I_t$ the *Hilbert series* of I is the formal power series

$$H_I(\mathfrak{z}) = \sum_{t \in \mathbb{N}} (\dim_{\Bbbk} I_t)\, \mathfrak{z}^t \in \mathbb{Z}[[\mathfrak{z}]].$$

For $H \in \mathbb{Z}[[\mathfrak{z}]]$, we write H^t for the coefficient of $\mathfrak{z}^t$ in H, so $H = \sum_t H^t \mathfrak{z}^t$. The *poset of Hilbert series of R-ideals* is the set $\mathcal{H}_R = \{H_J : J \in \mathcal{I}_R\}$ endowed with the partial order: $H \succcurlyeq H' \in \mathcal{H}_R$ if, for all $t \in \mathbb{N}$, $H^t \geq (H')^t$. For the sake of convenience, we work with $\mathcal{H}_R$ instead of the analogous poset of Hilbert functions of R-ideals.

Question 1.1. Is there an (order-preserving) embedding $\epsilon : \mathcal{H}_R \hookrightarrow \mathcal{I}_R$, as posets, such that $\mathbf{H} \circ \epsilon = \mathrm{id}_{\mathcal{H}_R}$, where $\mathbf{H} : \mathcal{I}_R \longrightarrow \mathcal{H}_R$ is the function $J \mapsto H_J$?

We will say that $\mathcal{H}_R$ *admits an embedding into* $\mathcal{I}_R$ (and often, by abuse of terminology, merely that $\mathcal{H}_R$ *admits an embedding*) if Question 1.1 has an affirmative answer. For example, if every Hilbert series in $\mathcal{H}_R$ is attained by the image of a LEX-segment A-ideal, then $\mathcal{H}_R$ admits an embedding, as is the case when $R = A$ (Macaulay) or $R = A/(x_1^{e_1}, \ldots, x_n^{e_n})$ (Kruskal, Katona, Clements–Lindström) or R is belongs to one of the classes of examples studied by Gasharov, Horwitz, Mermin, Murai and Peeva, listed earlier. In Proposition 2.7, we prove a necessary condition for $\mathcal{H}_R$ to admit an embedding, using which we exhibit a few algebras whose posets of Hilbert series fail to admit embeddings. Proposition 2.16 describes when the ideals that appear as images of a given embedding are LEX-segment R-ideals. In Section 3, we prove that an embedding can be lifted to certain ring extensions (Theorem 3.3), and use it for a special case of distractions (Proposition 3.17) and for polarization (Theorem 3.23). Theorem 3.25 is an analogue, in the situation of embeddings, of the result of Clements–Lindström mentioned above. In Section 4, we prove some lemmas about obtaining stable ideals, which can be read independently of the previous sections, and are used in Section 3.

Notation and terminology. We use [Eis95] as a general reference. In this paper, R will always denote a standard graded algebra over a field $\Bbbk$. The homogeneous maximal ideal of R is $\mathfrak{m} = \bigoplus_{d \geq 1} R_d$. By A, we mean a (standard graded) polynomial ring over $\Bbbk$ that has a surjective homogeneous $\Bbbk$-algebra homomorphism $A \xrightarrow{\phi} R$ of degree 0. We fix this homomorphism. Let $\mathfrak{a} = \ker \phi$. We will further assume that the embedding dimensions of A and R are the same; equivalently, $\mathfrak{a}_1 = 0$. In particular, if R is a polynomial ring, then $A = R$.

Definition 1.2. (The following definitions depend on the choice of basis of A_1.) Fix a basis $x_1, \ldots, x_n$ of A_1. We write $\mathrm{Mon}(A)$ for the set of monomials in the x_i in A. By a *monomial* of R, we mean the image of a monomial of A under ϕ. A *standard basis* of R is a set $\mathcal{B} \subseteq \mathrm{Mon}(A)$ such that $\{\phi(f) : f \in \mathcal{B}\}$ forms a $\Bbbk$-basis of R. Let $\mathcal{B}$ be a standard basis of R; for $d \in \mathbb{N}$, we write $\mathcal{B}_d$ for the set of monomials of $\mathcal{B}$ of degree d. For a $\Bbbk$-subspace V of R_d, we write $|V|$ for $\dim_{\Bbbk} V$. For a subset V of R, we write $(V)R$ for the ideal generated by V in the ring R. A *graded total order* on R is a pair $(\mathcal{B}, \tau)$ consisting of a standard basis $\mathcal{B}$ of R and a total order τ on $\mathcal{B}$ such that $m \prec_\tau m'$ if $\deg m < \deg m'$. For $d \in \mathbb{N}$, the *τ-segment of R_d of dimension r* is the $\Bbbk$-vector-space generated by the images in R of the first r monomials in $\mathcal{B}_d$ in the order $\prec_\tau$. We say that a graded total order $(\mathcal{B}, \tau)$ is a *monomial order* if (i) for all $f \in \mathcal{B}$ and for all $g \mid f$, $\frac{f}{g} \in \mathcal{B}$, and, (ii) for all $f, f' \in \mathcal{B}$ and for all $g \mid \gcd(f, f')$, $f \prec_\tau f'$ if and only if $\frac{f}{g} \prec_\tau \frac{f'}{g}$. Suppose that $\mathfrak{a}$ is a monomial ideal. Then $\mathcal{B} = \mathrm{Mon}(A) \smallsetminus \mathfrak{a}$; therefore, while referring to any graded total order, we will drop the reference to the standard basis. On the polynomial ring A, we will also need to use *weight orders* on the set of monomials, induced by assigning weights inside $\mathbb{N}$ to the x_i. Weight orders need not be total orders, in general. □

Remark 1.3. Note that whether $\mathcal{H}_R$ admits an embedding or not does not depend on any choice of basis of R_1. As we will see in Proposition 2.4, the existence of an embedding is equivalent to a the existence of a certain filtration of R as a $\Bbbk$-vector-space. However, in Discussion 2.15 and what follows, we consider total orders on rings defined by monomial ideals and on certain semigroup rings that correspond to embeddings; in those cases, we have the basis of R_1 given by the images of the variables of A in mind. □

Remark 1.4. Suppose that R is defined by a monomial ideal or that it is an affine semigroup algebra all of whose $\Bbbk$-algebra generators are of the same degree (i.e., there is a injective $\Bbbk$-algebra homomorphism $\xi : R \longrightarrow A'$ for some polynomial ring A' such that the $\xi(\phi(x_i))$ are monomials in A', in some fixed basis of A'_1, of the same degree). If $\mathcal{H}_R$ admits an embedding ϵ, then for all $H \in \mathcal{H}_R$, we may take $\epsilon(H)$ to be a monomial ideal. To see this, write $R = A/\mathfrak{a}$. We can find a weight order ω on A such that $\mathfrak{a} = \operatorname{in}_\omega(\mathfrak{a})$ (the initial ideal of $\mathfrak{a}$ with respect to the weight order ω) and such that for every homogeneous ideal I with $\mathfrak{a} \subseteq I$, $\operatorname{in}_\omega(I)$ is generated by $\mathfrak{a}$ and monomials. When R is defined by a monomial ideal, this is immediate. For the details of the latter case, see [GHP08, Theorem 2.5]. ☐

2. Generalities

The poset $\mathcal{I}_R$ of R-ideals is a *lattice*; if $I, J \in \mathcal{I}_R$, then their *join*, or least upper bound, is $I \vee J = I + J$ and their *meet*, or greatest lower bound, is $I \wedge J = I \cap J$. (We use [Sta97] as the reference on lattices.) We will see that if $\mathcal{H}_R$ admits an embedding, then it is a lattice with specific meet and join functions. Using this criterion, we show that for certain standard graded algebras R, $\mathcal{H}_R$ admits no embedding. First, we see in the next lemma that an embedding (if it exists) can be done degree-by-degree.

Lemma 2.1. *Suppose that $\mathcal{H}_R$ admits an embedding ϵ. Let $H, \tilde{H} \in \mathcal{H}_R$. Then for all $d \in \mathbb{N}$, if $H^d \leq \tilde{H}^d$ then $(\epsilon(H))_d \subseteq (\epsilon(\tilde{H}))_d$.*

Proof. Let I be an R-ideal such that $H_I = H$. Define J to be the R-ideal generated by I and all the forms of degree $d+1$, and let K be the ideal generated by all the forms of J of degree greater than or equal to d. The fact that $H_J \succcurlyeq H_I$ and $H_J \succcurlyeq H_K$ gives $\epsilon(H_J) \supseteq \epsilon(H_I)$ and $\epsilon(H_J) \supseteq \epsilon(H_K)$, while the equalities $H_I^d = H_J^d = H_K^d$ imply that $(\epsilon(H_I))_d = (\epsilon(H_J))_d = (\epsilon(H_K))_d$. Let I' be an R-ideal such that $H_{I'} = \tilde{H}$, and define the ideals J' and K' in a way analogous to above. Since $H_{K'} \succcurlyeq H_K$ we get the desired inequality. ☐

Remark 2.2. For a $\Bbbk$-vector-space $V \subseteq R_d$, we write $\epsilon(V)$ for $(\epsilon(H_{(V)R}))_d$. (This depends only on $|V|$ by Lemma 2.1.) Hence for all $I \in \mathcal{I}_R$, $\epsilon(H_I) = \bigoplus_{d \in \mathbb{N}} \epsilon(I_d)$.

Definition 2.3. An *embedding filtration* of R is a collection of filtrations $\{0 = V_{d,0} \subsetneq V_{d,1} \subsetneq \cdots \subsetneq V_{d,|R_d|} = R_d : d \in \mathbb{N}\}$ of R into $\Bbbk$-vector-spaces that satisfies, for all $d \in \mathbb{N}$ and for all $0 \leq r \leq |R_d|$,

(i) $R_1 V_{d,r} = V_{d+1,s}$ for some $0 \leq s \leq |R_{d+1}|$, and,
(ii) for all $W \subseteq R_d$, $|R_1 V_{d,|W|}| \leq |R_1 W|$.

Proposition 2.4. *Let R be a standard graded algebra. Then $\mathcal{H}_R$ admits an embedding into $\mathcal{I}_R$ if and only if R has an embedding filtration.*

Proof. Suppose that $\mathcal{H}_R$ admits an embedding ϵ. Let $\mathcal{V} = \{0 = V_{d,0} \subsetneq V_{d,1} \subsetneq \cdots \subsetneq V_{d,|R_d|} = R_d : d \in \mathbb{N}\}$ be a collection of filtrations of R into $\Bbbk$-vector-spaces. For all d and for all $0 \leq r \leq |R_d|$, replace $V_{d,r}$ by $\epsilon(V_{d,r})$. We will show that $\mathcal{V}$ is an embedding filtration of R.

Let $W \subseteq R_d$ be a $\Bbbk$-subspace. Let $V = V_{d,|W|}$. By Lemma 2.1, $V = (\epsilon(H_{(W)R}))_d$, which gives $R_1 V \subseteq (\epsilon(H_{(W)R}))_{d+1}$. Note that $|(\epsilon(H_{(W)R}))_{d+1}| = |R_1 W|$, so, $|R_1 V| \leq |R_1 W|$. Now, applying the above calculation with $W = V$, we get $R_1 V \subseteq (\epsilon(H_{(V)R}))_{d+1}$. By Remark 2.2, $\epsilon(H_{(V)R})_d = V$ and $\epsilon(H_{(V)R})_{d+1} = \epsilon(R_1 V)$. Hence $R_1 V \subseteq \epsilon(R_1 V)$, so $R_1 V = \epsilon(R_1 V) = V_{d+1,|R_1 V|}$.

Conversely, given an embedding filtration $\{0 = V_{d,0} \subsetneq V_{d,1} \subsetneq \cdots \subsetneq V_{d,|R_d|} = R_d : d \in \mathbb{N}\}$, we define an embedding $\epsilon : \mathcal{H}_R \longrightarrow \mathcal{I}_R$ by setting $\epsilon(H) = \bigoplus_{d \in \mathbb{N}} V_{d,H^d}$. It follows, from the definition of embedding filtrations, that $\epsilon(H)$ is an ideal. ☐

Remark 2.5. For a graded R-module M, its *graded Betti numbers* are $\beta_{i,j}^R(M) = \dim_\Bbbk \operatorname{Tor}_i^R(M, \Bbbk)_j$. Let $\epsilon : \mathcal{H}_R \longrightarrow \mathcal{I}_R$ be an embedding. Let I be an R-ideal. Then there is an inequality $\beta_{1,j}^R(R/I) \leq \beta_{1,j}^R(R/\epsilon(H_I))$. We see this as follows. Note that for any homogeneous R-ideal J, $\beta_{1,j}^R(R/J) = (|J_j| - |R_1 J_{j-1}|)$. Now let $J = \epsilon(H_I)$. Let $\{0 = V_{d,0} \subsetneq V_{d,1} \subsetneq \cdots \subsetneq V_{d,|R_d|} = R_d : d \in \mathbb{N}\}$ be the embedding filtration of R given by ϵ, as in the proof of Proposition 2.4. Then, for all $d \geq 1$, $J_d = V_{d,|I_d|}$. Hence $|R_1 J_{d-1}| \leq |R_1 I_{d-1}|$, which implies that $\beta_{1,j}^R(R/I) \leq \beta_{1,j}^R(R/J)$. Note, also, that the same argument shows that $\beta_{1,j}^R(R/\epsilon(H_I))$ depends only on H_I and not on ϵ. For which i and j is the inequality $\beta^R_{i,j}(R/I) \leq \beta^R_{i,j}(R/\epsilon(H_I))$ true? Mermin and Murai [MM10, Proposition 3.2] showed that in general $\beta^A_{i,j}(R/I) \not\leq \beta^A_{i,j}(R/\epsilon(H_I))$. □

Definition 2.6. Let $H, \tilde{H} \in \mathbb{Z}[[\mathfrak{z}]]$. Define $\max(H, \tilde{H})$ and $\min(H, \tilde{H})$ in $\mathbb{Z}[[\mathfrak{z}]]$ by setting, for all $t \in \mathbb{N}$, $(\max(H, \tilde{H}))^t = \max\{H^t, \tilde{H}^t\}$ and $(\min(H, \tilde{H}))^t = \min\{H^t, \tilde{H}^t\}$.

Note that the usual total order on $\mathbb{Z}$ makes $\mathbb{Z}[[\mathfrak{z}]]$ into a distributive lattice. We now derive a necessary criterion so that we have an embedding.

Proposition 2.7. *If $\mathcal{H}_R$ admits an embedding then it is a sublattice of $\mathbb{Z}[[\mathfrak{z}]]$.*

Proof. Let $H, \tilde{H} \in \mathcal{H}_R$. Let $\epsilon : \mathcal{H}_R \longrightarrow \mathcal{I}_R$ be the embedding. Let $I, \tilde{I} \in \mathcal{I}_R$, $H = H_I$ and $\tilde{H} = H_{\tilde{I}}$. Without loss of generality, we may assume that $I = \epsilon(H)$ and $\tilde{I} = \epsilon(\tilde{H})$. Fix an embedding filtration $\{0 = V_{d,0} \subsetneq V_{d,1} \subsetneq \cdots \subsetneq V_{d,|R_d|} = R_d : d \in \mathbb{N}\}$ of R. Write $I = \bigoplus_d V_{d,r_d}$ and $\tilde{I} = \bigoplus_d V_{d,\tilde{r}_d}$. Note that $H_{(I+\tilde{I})} = H \vee \tilde{H}$ and $H_{(I\cap\tilde{I})} = H \wedge \tilde{H}$. □

Remark 2.8. Suppose that $\mathcal{H}_R$ admits an embedding ϵ. Let $I = \epsilon(H)$ for some $H \in \mathcal{H}_R$. Then $\mathcal{I}_{R/I} \simeq \{J \in \mathcal{I}_R : I \subseteq J\}$ and $\mathcal{H}_{R/I} \simeq \{H_J : J \in \mathcal{I}_R, I \subseteq J\}$. In particular, ϵ induces an embedding of $\mathcal{H}_{R/I}$ into $\mathcal{I}_{R/I}$. Thus, if $\mathfrak{a}$ is a LEX-segment A-ideal then the embedding by LEX-segment A-ideals gives an embedding filtration on A; the images (in R) of LEX-segment A-ideals give an embedding filtration of R. □

If $\mathcal{H}_R$ admits an embedding, then we obtain a complete flag, and hence a basis, of R_1. The next example illustrates this. In the three examples that follow, we have represented the rings R with a given basis of R_1, only for the sake of concreteness. The assertion that the posets $\mathcal{H}_R$ in those examples do not admit an embedding is an inherent statement, independent of the bases.

Example 2.9. Let $R = A/\mathfrak{a}$ be an Artinian Gorenstein $\mathbb{k}$-algebra such that $\mathfrak{a}$ is generated by quadratic forms in A and $H_R = 1 + 4\mathfrak{z}^1 + 4\mathfrak{z}^2 + \mathfrak{z}^3$. We show that R has an embedding filtration. In the process of proving that R is Koszul, A. Conca, M. Rossi and G. Valla observed that there exists a linear form $l_1 \in R_1$ such that $|R_1 l_1| = 2$ [CRV01, p.118], moreover for every other linear form l the inequality $|R_1 l| \geq 2$ holds. They further showed (see [CRV01, Lemma 6.14]) (i) that there exists a 2-dimensional $\mathbb{k}$-vector-space $V \subseteq R_1$ such that $l_1 \in V$, and $|R_1 V| = 3$, and, (ii) for every 2-dimensional $\mathbb{k}$-vector-spaces $W \subseteq R_1$ the inequality $|R_1 W| \geq 3$ holds. Let $l_2 \in V$ be such that l_1, l_2 form a $\mathbb{k}$-basis of V. Pick $l_3 \notin V$, if such a linear form exists, such that $R_1 V = R_1 V + R_1 l_3$; otherwise, let $l_3 \notin V$ be any linear form. Let l_4 be any linear form such that $l_1, \ldots, l_4$ is a $\mathbb{k}$-basis for R_1. Choose $q_1, \ldots, q_4$ a $\mathbb{k}$-basis of R_1 such that $\{q_1, q_2\}$ is a $\mathbb{k}$-basis of $R_1 l_1$ and $\{q_1, q_2, q_3\}$ is a $\mathbb{k}$-basis of $R_1 V$. Let s be any generator of the socle of R (which is a one-dimensional $\mathbb{k}$-vector-space). Then $\mathbb{k} l_1 \subsetneq \mathbb{k}\{l_1, l_2\} \subsetneq \mathbb{k}\{l_1, l_2, l_3\} \subsetneq \mathbb{k}\{l_1, \ldots, l_4\}$, $\mathbb{k} q_1 \subsetneq \mathbb{k}\{q_1, q_2\} \subsetneq \mathbb{k}\{q_1, q_2, q_3\} \subsetneq \mathbb{k}\{q_1, \ldots, q_4\}$, $\mathbb{k} s$ is an embedding filtration of R. □

Example 2.10 (Strongly stable ideals). If $\mathfrak{a}$ is a strongly stable A-ideal (also called 0-Borel) and $R = A/\mathfrak{a}$, then $\mathcal{H}_R$ need not admit an embedding; contrast this with LEX-segment ideals, in Remark 2.8. Take $A = \mathbb{k}[x_1, x_2, x_3]$, $\mathfrak{a} = (x_1, x_2)^2 (x_1, x_2, x_3)^2$. Consider $I = (x_1^2, x_1 x_2, x_2^2)R$ and $I' = (x_1^2, x_1 x_2, x_1 x_3)R$. Then $H_I = 3\mathfrak{z}^2 + 7\mathfrak{z}^3$ and $H_{I'} = 3\mathfrak{z}^2 + 6\mathfrak{z}^3 + \sum_{t \geq 4} \mathfrak{z}^t$. Hence, if $\mathcal{H}_R$ admitted an embedding, then there would be an R-ideal J with $H_J = 3\mathfrak{z}^2 + 6\mathfrak{z}^3$; however, such an ideal does not exist. (We see this as follows. Let J be any homogeneous R-ideal such that $|J_2| = 3$ and $|J_4| = 0$. In particular, $|J_2 \mathfrak{m}^2| = 0$ so $J_2 \subseteq (0 :_R \mathfrak{m}^2)_2 = I_2$. Hence $J_2 = I_2$. Since I is generated by I_2, $I_3 = I_2 \mathfrak{m}$. Therefore $|J_3| \geq |I_3| = 7$.) □

Example 2.11 (Gröbner flags). Let $A = \mathbb{k}[x_1, \ldots, x_6]$ and $\mathfrak{a} = x_1(x_1, \ldots, x_4) + (x_2^2, x_2 x_3) + (x_3^2) + x_4(x_4, x_5) + x_5(x_5, x_6)$. Let $R = A/\mathfrak{a}$. Then $0 \subseteq \mathbb{k}\{x_1\} \subseteq \mathbb{k}\{x_1, x_2\} \subseteq \cdots \subseteq R_1$ is a Gröbner flag for R; see [CRV01] for the definition and properties. In particular R is Koszul. Note that $H_{(x_1)} = \mathfrak{z}^1 + 2\mathfrak{z}^2 + \sum_{t=3}^{\infty} \mathfrak{z}^t$ and that $H_{(x_5)} = \mathfrak{z}^1 + 3\mathfrak{z}^2$. There is, however, no R-ideal I with $H_I = \mathfrak{z}^1 + 2\mathfrak{z}^2$. (For, if I is such that $|I_1| = 1$ and $|I_3| = 0$, then $I \subseteq (0 :_R \mathfrak{m}^2) = (x_5)$. Hence $I_1 = \mathbb{k} x_5$, and, therefore, $|I_2| \geq 3$.) □

Example 2.12 (Tensor products). Let $A = \mathbb{k}[x, y, z]$ and $\mathfrak{a} = (x, y)^3 + (z^2)$. Then $H_{(x)} = \mathfrak{z}^1 + 3\mathfrak{z}^2 + 2\mathfrak{z}^3$ and $H_{(z)} = \mathfrak{z}^1 + 2\mathfrak{z}^2 + 3\mathfrak{z}^3$. However, there is no monomial R-ideal I with $H_{R/I} = \mathfrak{z}^1 + 2\mathfrak{z}^2 + 2\mathfrak{z}^3$, for, if I is any monomial ideal with $I_1 \neq 0$, then $H_I \succcurlyeq H_{(x)} = H_{(y)}$ or $H_I \succcurlyeq H_{(z)}$. Notice that $R \simeq$

$\Bbbk[x,y]/(x,y)^3 \otimes_\Bbbk \Bbbk[z]/(z^2)$; both $\mathcal{H}_{\frac{\Bbbk[x,y]}{(x,y)^3}}$ and $\mathcal{H}_{\frac{\Bbbk[z]}{(z^2)}}$ admit embedding. (It suffices to consider monomial ideals; see Remark 1.4.) $\square$

Remark 2.13 (Veronese subrings, due to A. Conca). Embeddings of Hilbert functions restrict to Veronese subrings. Let R be a standard graded algebra such that $\mathcal{H}_R$ admits an embedding ϵ. Let $m \in \mathbb{N}$. Let S be the mth Veronese subring of R, i.e., $S = \bigoplus_{i\in\mathbb{N}} S_i$ with $S_i = R_{im}$ for all $i \in \mathbb{N}$. We define $\bar{\epsilon} : \mathcal{H}_S \longrightarrow \mathcal{I}_S$ as follows. Let $H \in \mathcal{H}_S$. Let $\{0 = V_{d,0} \subsetneq V_{d,1} \subsetneq \cdots \subsetneq V_{d,|R_d|} = R_d : d \in \mathbb{N}\}$ be the embedding filtration of R given by ϵ. In degree $d \in \mathbb{N}$, we define $\bar{\epsilon}(H)$ by setting $(\bar{\epsilon}(H))_d = V_{md,H^d}$. Note that $\bigoplus_{d\in\mathbb{N}} V_{md,H^d}$ is an ideal of S, since $R_m V_{md,H^d} \subseteq V_{m(d+1),H^{d+1}}$. Hence $\{0 = V_{md,0} \subsetneq V_{md,1} \subsetneq \cdots \subsetneq V_{md,|R_{md}|} = S_d : d \in \mathbb{N}\}$ is an embedding filtration of S. $\square$

Remark 2.14 (Gotzmann property). Suppose that $R = A$, a polynomial ring. Then the Gotzmann Persistence Theorem [BH93, Theorem 4.3.3] asserts that for all homogeneous R-ideals I, if I is generated minimally in degrees less than or equal to d and $|\mathfrak{m}I_d| \leq |\mathfrak{m}V|$ for all $\Bbbk$-vector-spaces $V \subseteq R_d$ satisfying $|V| = |I_d|$, then for all $t \geq d$, $|\mathfrak{m}I_t| \leq |\mathfrak{m}W|$ for all $\Bbbk$-vector-spaces $W \subseteq R_t$ satisfying $|W| = |I_t|$. Now suppose that R is any standard graded algebra, such that $\mathcal{H}_R$ admits an embedding ϵ. Reinterpreting the conclusion of the Gotzmann Persistence Theorem, we say that R has the *Gotzmann property* (with respect to ϵ) if for all homogeneous R-ideals I, if I is minimally generated in degrees than or equal to d and $\epsilon(H_I)$ has no minimal generators in degree $d+1$, then $\epsilon(H_I)$ is generated minimally in degrees less than or equal to d as well. Note that having the Gotzmann property is independent of the chosen embedding: as we observed in Remark 2.5, the degrees and number of minimal generators of $\epsilon(I)$ depends only on H_I.

There are standard graded algebras R such that R does not have the Gotzmann property while $\mathcal{H}_R$ admits an embedding. As an example, consider $A = \Bbbk[x]$ and $\mathfrak{a} = (x^3)$. It is immediate that the quotient $R = A/\mathfrak{a}$ has the Gotzmann property and that $\mathcal{H}_R$ admits an embedding. Let $S = R[y]$ and notice that it has an embedding induced by lexicographic order on $A[y]$. On the other hand S fails to have the Gotzmann property since both $I = (y)S$ and $\epsilon(I) = (x, y^3)S$ have no generators in degree 2. $\square$

Discussion 2.15. Suppose that R is defined by a monomial ideal or that it is an affine semigroup algebra all of whose generators are of the same degree. Suppose that R has an embedding filtration. Then, as in Remark 1.4, we can take initial ideals, and obtain a graded total order $(\mathcal{B}, \tau)$ that satisfies, for all $d \in \mathbb{N}$ and for all τ-segments V of R_d, (i) R_1V is a τ-segment of R_{d+1}, and, (ii) $|R_1V| \leq |R_1W|$ for all $\Bbbk$-subspaces $W \subseteq R_d$ with $|W| = |V|$. We call such a graded total order an *embedding order* on R. Conversely, any embedding order gives rise to an embedding filtration. We thus conclude that $\mathcal{H}_R$ admits an embedding if and only if there exists an embedding order on R. $\square$

When $R = A$ or $R = A/(x_1^2, \ldots, x_n^2)$ the embedding orders on R are enumerated in [Mer06].

Proposition 2.16. *Let $(\mathcal{B}, \tau)$ be an embedding order of R. Suppose that it is a monomial order. Then there exists a graded lexicographic order* LEX *on A such that for all $f, f' \in \mathcal{B}$, if $f \prec_\tau f'$, then $f \prec_{\text{LEX}} f'$.*

Proof. Without loss of generality, we may assume (since $\mathfrak{a}_1 = 0$) that $x_1 \prec_\tau \cdots \prec_\tau x_n$. Let LEX be the graded lexicographic order on A with $x_1 \prec_{\text{LEX}} \cdots \prec_{\text{LEX}} x_n$. Assume the contrary, and let t be the smallest degree such that there exist monomials f and f' in A with $t = \deg f = \deg f'$ such that $f \prec_\tau f'$ and $f' \prec_{\text{LEX}} f$. By the choice of t, we see that $\gcd(f, f') = 1$.

Let i be the smallest index such that x_i divides at least one of f and f'. By going modulo $(x_1, \ldots, x_{i-1})$ and using Remark 2.8, we can assume, without loss of generality, that $i = 1$. Since $f' \prec_{\text{LEX}} f$, $x_1 \mid f'$.

Let $\{0 = V_{d,0} \subsetneq V_{d,1} \subsetneq \cdots \subsetneq V_{d,|R_d|} = R_d : d \in \mathbb{N}\}$ be the embedding filtration of R induced by τ. Since $V_{1,1}$ is spanned by $\phi(x_1)$, there exists s such that $V_{t,s}$ is the span of the images of all the degree-t monomials in $\mathcal{B}$ divisible by x_1. Therefore $\phi(f') \in V_{t,s}$, which implies that $f' \prec_\tau f$, a contradiction. $\square$

Remark 2.17. If R has an embedding order τ, then for all τ-segment R-ideals I, R/I inherits the embedding order τ (see Remark 2.8). Furthermore, $(0 :_R \mathfrak{m})$ is a τ-segment ideal. For $d \in \mathbb{N}$, let $s_d = \dim_\Bbbk(0 :_R \mathfrak{m})_d$. Let $\{0 = V_{d,0} \subsetneq V_{d,1} \subsetneq \cdots \subsetneq V_{d,|R_d|} = R_d : d \in \mathbb{N}\}$ be the embedding filtration of R induced by τ. Then $|R_1V_{d,s_d}| \leq |R_1(0 :_R \mathfrak{m})_d| = 0$, so $V_{d,s_d} \subseteq (0 :_R \mathfrak{m})_d$; since $|V_{d,s_d}| = |(0 :_R \mathfrak{m})_d|$, we see further that $V_{d,s_d} = (0 :_R \mathfrak{m})_d$. Hence $(0 :_R \mathfrak{m}) = \oplus_d (0 :_R \mathfrak{m})_d$ is a τ-segment ideal. $\square$

Example 2.18 (Embedding with no monomial orders). Let $A = \Bbbk[w, x, y, z]$, with homogeneous maximal ideal $\mathfrak{n}$. Let $\mathfrak{a} = (wxy, wxz, wyz, xyz) + \mathfrak{n}^4$ and $R = A/\mathfrak{a}$. Without loss of generality, we order R_1 by $w \prec_\tau x \prec_\tau y \prec_\tau z$. Note that $|(w^2R)_3| = 4 > 2 = |(wxR)_3|$, so $wx \prec_\tau w^2$. In fact, there is no embedding order in which w^2 precedes all the other monomials in R_2. On the other hand, we have an embedding order $wx \prec_\tau wy \prec_\tau w^2 \prec_\tau wz \prec_\tau xy \prec_\tau x^2 \prec_\tau xz \prec_\tau y^2 \prec_\tau yz \prec_\tau z^2$. Since $\mathfrak{n}^4 \subseteq \mathfrak{a}$ we can give any order on R_3, provided R_1V is a τ-segment of R_3 for every τ-segment V of R_2. For example, we may take $w^2x \prec_\tau wx^2 \prec_\tau w^2y \prec_\tau wy^2 \prec_\tau w^3 \prec_\tau w^2z \prec_\tau wz^2 \prec_\tau x^2y \prec_\tau xy^2 \prec_\tau x^3 \prec_\tau x^2z \prec_\tau xz^2 \prec_\tau y^3 \prec_\tau y^2z \prec_\tau yz^2 \prec_\tau z^3$. □

3. Extension rings

For certain R-free quotient rings S of $R[z]$, we determine a sufficient condition (which we call *z-stability*, see Definition 3.2) for extending an embedding filtration of R to S. We use it to extend embeddings under polarization and distraction, and prove the following analogue of a theorem of Clements and Lindström.

Notation 3.1. In this section z is an indeterminate over A and R. Let $t \in \mathbb{N} \cup \{\infty\}$ and $S = R[z]/(z^t)$. If $t = \infty$, the $R[z]$-ideal (z^t) denotes the zero ideal. As usual $R = A/\mathfrak{a}$, where A is a polynomial ring. We will denote by $\mathfrak{I}$ the set $\{i \in \mathbb{Z} | 0 \leq i < t\}$. Let $B = A[z]$; treat S as a quotient ring of B. Treat B and S as *multigraded* using the natural decomposition $B = \oplus_{i \in \mathbb{N}} \oplus_{j \in \mathbb{N}} A_i z^j$ as $\Bbbk$-vector-spaces. By $\mathcal{F} := \{0 = V_{d,0} \subsetneq V_{d,1} \subsetneq \cdots \subsetneq V_{d,|R_d|} = R_d : d \in \mathbb{N}\}$, we mean the embedding filtration of R that corresponds to the embedding of $\mathcal{H}_R$ (by Proposition 2.4).

Definition 3.2. Let $W \subseteq S_d$ be a multigraded $\Bbbk$-vector-space. The *R-coefficient sequence* of W is the sequence $(W_{d-i})_{i \in \mathfrak{I}}$ of $\Bbbk$-subspaces $W_{d-i} \subseteq R_{d-i}$ defined by the $\Bbbk$-vector-space decomposition $W = \bigoplus_{i \in \mathfrak{I}} W_{d-i} z^i$. We say that W is *z-stable* if $R_1 W_{d-i} \subseteq W_{d-i+1}$ for all positive $i \in \mathfrak{I}$. Let $I \subseteq S$ be an ideal; we say that I is *z-stable* if I is multigraded and I_d is z-stable for all $d \geq 0$.

Theorem 3.3. *Let $t \in \mathbb{N} \cup \{\infty\}$ and $S = R[z]/(z^t)$. Suppose that $\mathcal{H}_R$ admits an embedding and that for all $H \in \mathcal{H}_S$, there exists a z-stable S-ideal I such that $H_I = H$. Then $\mathcal{H}_S$ admits an embedding.*

Definition 3.4. Let $d \in \mathbb{N}$. A *segment* of S_d is a z-stable $\Bbbk$-vector-space $\bigoplus_{i \in \mathfrak{I}} V_{d-i, r_{d-i}} z^i$ such that $V_{d-i,r_{d-i}} \subseteq R_{j-i} V_{d-j, \min\{1+r_{d-j}, |R_{d-j}|\}}$, for all $i < j \in \mathfrak{I}$; its *length* is $\sum_{i \in \mathfrak{I}} r_{d-i}$.

Observation 3.5. Let $W \subseteq S_d$ be a z-stable $\Bbbk$-vector-space, with R-coefficient sequence $(W_{d-i})_{i \in \mathfrak{I}}$. Then

$$S_1 W = R_1 W_d \bigoplus \bigoplus_{\substack{i \in \mathfrak{I} \\ i > 0}} W_{d-i+1} z^i$$

If, further, W is a segment of S_d, $S_1 W$ is a segment of S_{d+1}.

Definition 3.6. For a multigraded $\Bbbk$-vector-space $W \subseteq S_d$ with R-coefficient sequence $(W_{d-i})_{i \in \mathfrak{I}}$, let

$$d_R(W) = \left(\sum_{j=0}^{i} |W_{d-j}| \right)_{i \in \mathfrak{I}}.$$

Let $\Lambda_d = \{d_R(W) : W \text{ is a multigraded } \Bbbk\text{-vector-space of } S_d\}$. Give a partial order $\lessdot$ on Λ_d by setting $(a_i)_{i \in \mathfrak{I}} \lessdot (b_i)_{i \in \mathfrak{I}}$ if $a_i \leq b_i$ for all i. For all $(a_i)_{i \in \mathfrak{I}} \in \Lambda_d$, $a_i = a_d$ for all $i \geq d$ and $a_d \leq |S_d|$; hence Λ_d is a finite set.

Lemma 3.7. *Let $W = \bigoplus_{i \in \mathfrak{I}} V_{d-i, r_{d-i}} z^i \subseteq S_d$ be a z-stable $\Bbbk$-vector-space such that $d_R(W)$ is minimal in $(\Lambda_d, \lessdot)$. Then W is a segment.*

Proof. By way of contradiction assume that W is not a segment. Pick $i < j \in \mathfrak{I}$ such that $V_{d-i,r_{d-i}} \not\subseteq R_{j-i} V_{d-j, \min\{1+r_{d-j}, |R_{d-j}|\}}$. We may assume that $j - i$ is minimal with this property. As $V_{d-j,|R_{d-j}|} = R_{d-j}$, we see that $r_{d-j} < |R_{d-j}| - 1$. Hence $V_{d-i,r_{d-i}} \not\subseteq R_{j-i} V_{d-j,1+r_{d-j}}$; since these vector-spaces belong to $\mathcal{F}$, we observe that

$$R_{j-i} V_{d-j,1+r_{d-j}} \subsetneq V_{d-i,r_{d-i}}. \tag{3.8}$$

Define

$$\tilde{W} = \bigoplus_{h \in \mathfrak{I} \smallsetminus \{i,j\}} V_{d-h,r_{d-h}} z^h \bigoplus V_{d-i,r_{d-i}-1} z^i \bigoplus V_{d-j,r_{d-j}+1} z^j.$$

It is immediate that $d_R(\tilde{W}) \lessdot d_R(W)$ and that they are not equal to each other. Hence it suffices to show that $\tilde{W}$ is z-stable. We consider two cases.

If $j = i+1$, then we need to show that $R_1 V_{d-i-1,1+r_{d-i-1}} \subseteq V_{d-i,r_{d-i}-1}$, which is immediate from (3.8). Otherwise, we first show that, for all i', $i < i' < j$, $V_{d-i',r_{d-i'}} = R_{j-i'} V_{d-j,1+r_{d-j}}$. Both the vector-spaces belong to $\mathcal{F}$, so they are comparable (with respect to inclusion). Minimality of $j-i$ implies $V_{d-i',r_{d-i'}} \subseteq R_{j-i'} V_{d-j,1+r_{d-j}}$. If $V_{d-i',r_{d-i'}} \subsetneq R_{j-i'} V_{d-j,1+r_{d-j}}$, then $V_{d-i',1+r_{d-i'}} \subseteq R_{j-i'} V_{d-j,1+r_{d-j}}$, so, using (3.8), we see that $R_{i'-i} V_{d-i',1+r_{d-i'}} \subseteq R_{j-i} V_{d-j,1+r_{d-j}} \subsetneq V_{d-i,r_{d-i}}$, contradicting the minimality of $j-i$. Taking $i' = i+1$ and $i' = j-1$, now, completes the proof of the assertion that $\tilde{W}$ is z-stable. ☐

Lemma 3.9. *Let W, W' be segments in S_d. If $|W| \leq |W'|$, then $W \subseteq W'$. In particular, for every $1 \leq s \leq |S_d|$, there exists a unique segment of length s.*

Proof. Write $W = \bigoplus_{i \in \mathfrak{I}} V_{d-i,r_{d-i}} z^i$ and $W' = \bigoplus_{i \in \mathfrak{I}} V_{d-i,s_{d-i}} z^i$. We need to show that $r_{d-i} \leq s_{d-i}$ for all $i \in \mathfrak{I}$. Assume, by way of contradiction, that there exists i such that $r_{d-i} > s_{d-i}$. Then we observe that $1 + s_{d-i} \leq |R_{d-i}|$ and, hence, that

$$V_{d-i,s_{d-i'}} \subseteq R_{i-i'} V_{d-i,1+s_{d-i}} \subseteq R_{i-i'} V_{d-i,r_{d-i}} \subseteq V_{d-i',r_{d-i'}},$$

for all $i' < i$. Hence $s_{d-i'} \leq r_{d-i'}$ for all $i' < i$. However, since $\sum_{j \in \mathfrak{I}} r_{d-j} = \sum_{\substack{j \in \mathfrak{I} \\ j \leq d}} r_{d-j} \leq \sum_{\substack{j \in \mathfrak{I} \\ j \leq d}} s_{d-j} = \sum_{j \in \mathfrak{I}} s_{d-j}$, there exists $j > i$ such that $s_{d-j} > r_{d-j}$. Repeating the above argument, now reversing the roles of W and W', we see that $s_{d-i} > r_{d-i}$, contradicting our assumption. Hence $r_{d-i} \leq s_{d-i}$ for all $i \in \mathfrak{I}$.

Let $1 \leq s \leq |S_d|$. The existence of a segment of length s follows from Lemma 3.7. Uniqueness is immediate from the first assertion of this lemma. ☐

Proof of Theorem 3.3. Let $H \in \mathcal{H}_S$. Let I be a z-stable S-ideal such that $H_I = H$. Let $d \in \mathbb{N}$. Let J_d be the (unique) segment of S_d of length $|I_d|$, which exists by Lemma 3.9. By Observation 3.5, $S_1 J_d$ is a segment of S_{d+1}. By the minimality of $d_R(J_d)$ (among all the z-stable subspaces of S_d of length $|I_d|$) and Observation 3.5, we see that $|S_1 J_d| \leq |S_1 I_d| \leq |I_{d+1}| = |J_{d+1}|$ and that $S_1 J_d$ is a segment of S_{d+1}. Hence, by Lemma 3.9, $S_1 J_d \subseteq J_{d+1}$. Therefore $J = \bigoplus_d J_d$ is an S-ideal. Now the map $\epsilon : \mathcal{H}_S \longrightarrow \mathcal{I}_R$ sending $H \mapsto J$ is an embedding. ☐

We note that Lemmas 3.7 and 3.9 makes no reference to the hypothesis of Theorem 3.3. Consequently, applying them and arguing as in the proof of Theorem 3.3, we obtain the following:

Theorem 3.10. *Let $t \in \mathbb{N} \cup \{\infty\}$ and $S = R[z]/(z^t)$. Suppose that $\mathcal{H}_R$ admits an embedding ϵ. Let I be a z-stable S-ideal and $J = \epsilon(H_I)$. Then $H_{(I,z^i)} \succcurlyeq H_{(J,z^i)}$ for all $i \in \mathfrak{I}$.*

Proof. Let I be a z-stable S-ideal such that $H_I = H$. We construct the ideal J by minimizing the function $d_R(J_d)$ for all d as in the proof of Theorem 3.3; see the proof of Lemma 3.7. Note that J depends only on H. Let $i \in \mathfrak{I}$ and $d \in \mathbb{N}$. We want to show that $|(I, z^i)_d| \geq |(J, z^i)_d|$. Let $(W_{d-j})_{j \in \mathfrak{I}}$ and $(V_{d-j,r_{d-j}})_{j \in \mathfrak{I}}$ be, respectively, the R-coefficient sequences of I_d and J_d. Then $|(I, z^i)_d| \geq |(J, z^i)_d|$ if and only if

$$\sum_{\substack{j \in \mathfrak{I} \\ j < i}} |W_{d-j}| \geq \sum_{\substack{j \in \mathfrak{I} \\ j < i}} r_{d-j}$$

which, indeed, is true by the construction of J_d. ☐

In Discussion 2.15 we noted that if R is defined by a monomial ideal that it is an affine semigroup algebra all of whose generators are of the same degree, $\mathcal{H}_R$ admits an embedding if and only if there exists an embedding order on R. In this situation, we can strengthen Theorem 3.3, to conclude that if the embedding order on R is induced by a graded lexicographic order on A (in the sense of Proposition 2.16), then the extension to S is induced by a graded lexicographic order on B.

Definition 3.11. Let $(\mathcal{B}, \sigma)$ be an embedding order on R. Let $\mathcal{B}' = \{fz^i : f \in \mathcal{B}, i \in \mathfrak{I}\}$. Define a total order τ on $\mathcal{B}'$ as follows. Let $fz^a, gz^b \in \mathcal{B}'$ with $f, g \in \mathcal{B}$, $\deg f + a = \deg g + b$ and $a \leq b$. Set $fz^a \prec_\tau gz^b$ if there exists $g' \in \mathcal{B}$ with $\deg g' = \deg g$ such that $f \in A_{b-a}g'$ and $g' \prec_\sigma g$. Otherwise set $gz^b \prec_\tau fz^a$.

Theorem 3.12. *With notation as above, $(\mathcal{B}', \tau)$ is an embedding order for S; the embedding of $\mathcal{H}_S$ from Theorem 3.3 is induced by $(\mathcal{B}', \tau)$. Moreover, if $x_1, \ldots, x_n$ are the variables of A and $(\mathcal{B}, \sigma)$ is a monomial order with $x_1 \prec_\sigma \cdots \prec_\sigma x_n$ then $(\mathcal{B}', \tau)$ is a monomial order with $x_1 \prec_\tau \cdots \prec_\tau x_n \prec_\tau z$.*

Proof. Indeed, $\mathcal{B}'$ is a standard basis for S. To show that $(\mathcal{B}', \tau)$ is an embedding order, it suffices to show that $(\mathcal{B}', \tau)$ induces the embedding from Theorem 3.3.

Recall that $B = A[z]$; write ϕ for the surjective homomorphism $B \to S$. Consider $f, g \in \mathcal{B}$, $f \neq g$ and $0 \leq a \leq b < t$ with $\deg f + a = \deg g + b = d$. By Lemma 3.9, there exist segments $W \subsetneq W' \subseteq S_d$ such that $W' = W + \Bbbk \cdot \langle \phi(gz^b) \rangle$. As in Definition 3.2 write $W = \bigoplus_{i \in \mathfrak{I}} V_{d-i, r_{d-i}} z^i$ (with $R_{j-i} V_{d-j, r_{d-j}} \subseteq V_{d-i, r_{d-i}} \subseteq R_{j-i} V_{d-j, 1+r_{d-j}}$, for all $0 \leq i < j < t$). Hence

$$W' = \bigoplus_{i=0}^{b-1} V_{d-i, r_{d-i}} z^i \bigoplus V_{d-b, 1+r_{d-b}} z^b \bigoplus \bigoplus_{\substack{i \in \mathfrak{I} \\ i > b}} V_{d-i, r_{d-i}} z^i.$$

Since W' is a segment, we see that $V_{d-i, r_{d-i}} = R_{b-i} V_{d-b, 1+r_{d-b}}$ for all $0 \leq i \leq b-1$ and that $V_{d-b, 1+r_{d-b}} \subseteq R_{j-b} V_{d-j, 1+r_{d-j}}$, for all $b+1 \leq j < t$. Note that $V_{d-b, 1+r_{d-b}} = \Bbbk \cdot \langle \phi(g) \rangle \bigoplus V_{d-b, r_{d-b}}$.

Since $a \leq b$, $\phi(fz^a) \in W$ if and only if $f \in A_{b-a} g'$ for some monomial $g' \in \mathcal{B}$ with $\deg g' = \deg g$ and $g' \prec_\sigma g$, i.e., if and only if $fz^a \prec_\tau gz^b$, so the embedding of $\mathcal{H}_S$ is induced by τ. Now, in order to prove that $(\mathcal{B}', \tau)$ is a monomial order if $(\mathcal{B}, \sigma)$ is, consider, as above, $f, g \in \mathcal{B}$ and $0 \leq a \leq b < t$ with $\deg f + a = \deg g + b = d$. When $fz^a \prec_\tau gz^b$, $h \in \mathcal{B}$ and $0 \leq c + a < t$, it follows directly from Definition 3.11 that $fhz^{a+c} \prec_\tau ghz^{b+c}$. On the other hand if $gz^b \prec_\tau fz^a$ we know that whenever $g' \in \mathcal{B}$ with $\deg g' = \deg g$ and $f \in A_{b-a} g'$ we also must have $g \prec_\sigma g'$, hence $ghz^{b+c} \prec_\tau fhz^{a+c}$. □

Distractions. A distraction is a $\Bbbk$-linear automorphism ϕ of a polynomial ring that has the property that if m and n are monomials and m divides n, , then $\phi(m)$ divides $\phi(n)$. We follow the formulation of [BCR05], and show that embedding the Hilbert functions can be extended to distractions. (For earlier work using distractions, see [BCR05, Introduction].) We consider a special case, and describe how embeddings of Hilbert series can be extended to distractions.

Notation 3.13. Let $\Bbbk$ be an infinite field and $A = \Bbbk[x_1, \ldots, x_n]$, with $\mathbb{N}^2$-grading given by $\deg x_1 = (1, 0)$ and $\deg x_i = (0, 1)$, for all $2 \leq i \leq n$. Let $\mathfrak{a}$ be a multigraded ideal. As earlier, $R = A/\mathfrak{a}$.

Definition 3.14 ([BCR05, Definitions 2.1, 2.2])**.** Let $N \in \mathbb{N}$. A *distraction matrix* is an infinite matrix $L = (l_{i,j} : 1 \leq i \leq n, j \in \mathbb{N})$ such that (i) $l_{i,j} \in A_1$ for all i, j, (ii) for all $j_1, \ldots, j_n \in \mathbb{N}$, $\{l_{1,j_1}, \ldots, l_{n,j_n}\}$ spans A_1 (as a $\Bbbk$-vector-space), and, (iii) there exists $N \in \mathbb{N}$ such that $l_{i,j} = l_{i,N}$ for all $j \geq N$. The *distraction* associated to L, D_L, is the $\Bbbk$-vector-space morphism $D_L : A \longrightarrow A$ such that $\prod_{i=1}^n x_i^{a_i} \mapsto \prod_{i=1}^n \prod_{j=1}^{a_i} l_{i,j}$.

Remark 3.15. We observe that, for all $d \in \mathbb{N}$, $D_L|_{A_d} : A_d \to A_d$ is an isomorphism of vector spaces. Therefore for all subspaces V, V' of A_d with $V \cap V' = 0$, $D_L(V + V') = D_L(V) \oplus D_L(V')$. In particular, D_L preserves Hilbert functions.

In [BCR05], the authors consider monomial ideals. We, however, need the results in bigraded ideals. The following lemma can be proved following their ideas, but for the sake of completeness, we include a proof. We will use this lemma, again, in the proof of Theorem 3.23.

Lemma 3.16. *Let $\mathfrak{a}$ be a multigraded ideal, and $L = (l_{i,j} : 1 \leq i \leq n, j \in \mathbb{N})$ be a distraction matrix such that $l_{i,j} = x_i$ for all $2 \leq i \leq n$ and for all $j \in \mathbb{N}$. Then $D_L(\mathfrak{a})$ is an ideal.*

Proof. Let $\mathfrak{a}_{(j)} = ((I :_A x_1^j) \cap \Bbbk[x_2, \ldots x_n]), j \in \mathbb{N}$. They are ideals in $\Bbbk[x_2, \ldots x_n]$, and, as $\Bbbk$-vector-spaces, $\mathfrak{a} = \bigoplus_{j \in \mathbb{N}} \mathfrak{a}_{(j)} x_1^j$. Hence, by Remark 3.15, $D_L(\mathfrak{a}) = \bigoplus_{j \in \mathbb{N}} \left(\mathfrak{a}_{(j)} \prod_{t=1}^j l_{1,t} \right)$. Since the $\mathfrak{a}_{(j)}$ are ideals, note that $x_i D_L(\mathfrak{a}) \subseteq D_L(\mathfrak{a})$ for $2 \leq i \leq n$. To finish the proof, it suffices to show that $x_1 \mathfrak{a}_{(j)} \prod_{t=1}^j l_{1,t} \subseteq D_L(\mathfrak{a})$. Let

$f \in \mathfrak{a}_{(j)}$. We want to show that $fx_1 \prod_{t=1}^{j} l_{1,t} \in D_L(\mathfrak{a})$. Since $\mathfrak{a}_{(j)} \subseteq \mathfrak{a}_{(j+1)}$, we know that $f \prod_{t=1}^{j+1} l_{1,t} \in D_L(\mathfrak{a})$. Moreover, $fx_i \prod_{t=1}^{j} l_{1,t} \in D_L(\mathfrak{a})$ for all $2 \leq i \leq n$ (since $x_i \mathfrak{a}_{(j)} \subseteq \mathfrak{a}_{(j)}$). Therefore $fx_1 \prod_{t=1}^{j} l_{1,t} \in D_L(\mathfrak{a})$. ☐

Proposition 3.17. *Let $\mathfrak{a}$ be a multigraded ideal, and $L = (l_{i,j} : 1 \leq i \leq n, j \in \mathbb{N})$ be a distraction matrix such that $l_{i,j} = x_i$ for all $2 \leq i \leq n$ and for all $j \in \mathbb{N}$. If $\mathcal{H}_R$ admits an embedding, then $\mathcal{H}_{\frac{A}{D_L(\mathfrak{a})}}$ admits an embedding.*

Proof. (Indeed, by Lemma 3.16, $D_L(\mathfrak{a})$ is an ideal.) Write $S = \frac{A}{D_L(\mathfrak{a})}$. Let ω be the weight order with $w(x_1) = 1$ and $w(x_i) = 0$ for all $2 \leq i \leq n$. Then $\operatorname{in}_\omega(D_L(\mathfrak{a})) = \mathfrak{a}$. (To see this, note that it is enough to show that $\mathfrak{a} \subseteq \operatorname{in}_\omega(D_L(\mathfrak{a}))$. Let $f \in \mathfrak{a}$ be a multigraded element with $\deg f = (a, b)$. Write $f = x_1^a g$, with g a homogeneous polynomial of degree b in $x_2, \ldots, x_n$. Then $D_L(f) = (\prod_{j=1}^{a} l_{1,j}) \cdot g$. Note that x_1^a appears with a non-zero coefficient in $\prod_{j=1}^{a} l_{1,j}$. Hence $f = \operatorname{in}_\omega(D_L(f)) \in \operatorname{in}_\omega(D_L(\mathfrak{a}))$.) Let $H \in \mathcal{H}_S$. Let $I \in \mathcal{I}_A$ be such that $D_L(\mathfrak{a}) \subseteq I$ and $H_{IS} = H$. Then $\mathfrak{a} \subseteq \operatorname{in}_\omega(I)$. Define $\epsilon' : \mathcal{H}_S \longrightarrow \mathcal{I}_{A/D_L(\mathfrak{a})}$ by sending $H \mapsto (D_L(\epsilon(\operatorname{in}_\omega(I))))S$; this is an embedding. ☐

Remark 3.18. The same proof will work for a more general distraction matrix, in which, for all i and j, $l_{i,j}$ is a linear form in $x_i, x_{i+1}, \ldots, x_n$, with x_i appearing with a non-zero coefficient. However, unlike polarization (discussed below), where working with one variable generalizes to the general case, the distraction matrix in Proposition 3.17 is not general. ☐

Polarization. We use Theorem 3.3 to show that polarization preserves embeddability of Hilbert functions. Let $A = \Bbbk[x_1, \ldots, x_n, y]$ be a polynomial ring. Polarization [MS05, Exercise 3.15] is an operation that converts an A-ideal $\mathfrak{a}$ to an $A[z]$-ideal $\mathfrak{b}$. We will show that every embedding of $\mathcal{H}_{A/\mathfrak{a}}$ gives rise to an embedding of $\mathcal{H}_{A[z]/\mathfrak{b}}$. Any polarization can be achieved by repeatedly applying partial polarizations, so we will restrict our discussion to this case.

Notation 3.19. Let A and z be as above. We give A the $\mathbb{N}^2$-grading with $\deg x_i = (1, 0)$, for all i, and $\deg y = (0, 1)$. Write $B = A[z]$, graded with $\deg x_i = (1, 0, 0)$ for all i, $\deg y = (0, 1, 0)$ and $\deg z = (0, 0, 1)$. Homogeneous elements, ideals and modules in these gradings will be referred to as *multigraded*. For a multigraded element f (of A or B), we will denote its degrees by $\deg_{\mathbf{x}} f$, $\deg_y f$ and $\deg_z f$. By $\mathfrak{a}$, we will mean a multigraded A-ideal.

Definition 3.20. A *polarization* is a $\Bbbk$-vector-space morphism $p^A_{y,d,z} : A \longrightarrow A[z]$, for some $d \in \mathbb{N}$, such that, for all homogeneous forms $f \in A$ with $\deg_y f = 0$,

$$p^A_{y,d,z} : fy^i \mapsto \begin{cases} fy^i, & \text{if } i < d, \\ fy^{i-1}z, & \text{otherwise.} \end{cases}$$

Remark 3.21. Let $\mathfrak{a}$ be a multigraded A-ideal, and let $\mathfrak{b} = (p^A_{y,d,z}(\mathfrak{a}))A[z]$. Then $\mathfrak{b}$ is a multigraded B-ideal. Moreover, $y - z$ is a non-zero-divisor on $B/\mathfrak{b}$. Hence $H_{A/\mathfrak{a}}(\mathfrak{z}) = (1 - \mathfrak{z})H_{B/\mathfrak{b}}(\mathfrak{z})$. ☐

Lemma 3.22. *Let ω be the weight vector on B with $\omega(x_i) = 1$ for all i, $\omega(y) = 1$ and $\omega(z) = 0$. Let $g : B \longrightarrow B$ be the $\Bbbk$-algebra morphism induced by the change of coordinates $x_i \mapsto x_i$ for all i, $y \mapsto y$ and $z \mapsto y + z$. Then $\operatorname{in}_\omega(g(p_{y,d,z}(\mathfrak{a}))) = \mathfrak{a}B$.*

Proof. It follows from the definition that $\mathfrak{a}B \subseteq \operatorname{in}_\omega(g(p_{y,d,z}(\mathfrak{a})))$. Observe that the Hilbert series of $p_{y,d,z}(\mathfrak{a})$ and of $\operatorname{in}_\omega(g(p_{y,d,z}(\mathfrak{a})))$ are identical. It is easy to see, from Remark 3.21, that the Hilbert series of $\mathfrak{a}B$ and of $p_{y,d,z}(\mathfrak{a})$ are identical. Therefore $\mathfrak{a}B = \operatorname{in}_\omega(g(p_{y,d,z}(\mathfrak{a})))$. ☐

Theorem 3.23. *If $\mathcal{H}_R$ admits an embedding, then $\mathcal{H}_{B/(p^A_{y,d,z}(\mathfrak{a}))}$ admits an embedding.*

Proof. Write $S = B/(p^A_{y,d,z}(\mathfrak{a}))$. First, for every homogeneous B-ideal I containing $\mathfrak{a}B$, there exists a homogeneous B-ideal J containing $\mathfrak{a}B$ such that $J(B/\mathfrak{a}B)$ is z-stable; see Lemma 4.1. Now, by Theorem 3.3, there is an embedding $\epsilon' : \mathcal{H}_{B/\mathfrak{a}B} \longrightarrow \mathcal{I}_{B/\mathfrak{a}B}$. Let $H \in \mathcal{H}_S$. Let I be an B-ideal such that $p^A_{y,d,z}(\mathfrak{a}) \subseteq I$ and $H = H_{IS}$. Applying Lemma 3.22, we find $I' \subseteq B$ such that $\mathfrak{a}B \subseteq I'$ and $H_{I'(B/\mathfrak{a}B)} = H$. We may assume that $I'(B/\mathfrak{a}B) = \epsilon'(H)$. Taking the initial ideal with respect to a suitable weight order, we may further assume that I' is multigraded (in the grading of B).

Let L be the following distraction matrix (Definition 3.14):

$$\begin{array}{c} \\ x_1 \\ \vdots \\ x_n \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & \cdots & d-1 & d & d+1 & \cdots \end{array} \\ \begin{pmatrix} x_1 & x_1 & \cdots & x_1 & x_1 & x_1 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\ x_n & x_n & \cdots & x_n & x_n & x_n & \cdots \\ y & y & \cdots & y & y+z & y & \cdots \\ z & z & \cdots & z & z & z & \cdots \end{pmatrix} \end{array}$$

By Lemma 3.16 we see that both $D_L(\mathfrak{a}B)$ and $D_L(I')$ are B-ideals; additionally, $D_L(\mathfrak{a}B) \subseteq D_L(I')$. Let ω be a weight order with $\omega(x_i) = 1$ for all i, $\omega(y) = 0$ and $\omega(z) = 0$. Then $\operatorname{in}_\omega(D_L(\mathfrak{a}B)) = p^A_{y,d,z}(\mathfrak{a})$. Define $\epsilon : \mathcal{H}_S \longrightarrow \mathcal{I}_S$ by setting $\epsilon : H \mapsto \operatorname{in}_\omega(D_L(I'))S$. ☐

Remark 3.24. Mermin showed that if a monomial complete intersection $R = A/\mathfrak{a}$ (i.e., $\mathfrak{a}$ is generated by an A-regular sequence of monomials) has the property that every Hilbert function is attained by the image of a LEX-segment ideal, then $\mathfrak{a} = (x_1^{e_1}, \dots, x_r^{e_r-1}x_i)$ for some $e_1 \leq \cdots \leq e_r$ and $i \geq r$ [Mer10, Theorem 4.4]. Theorem 3.23 shows that if we allow for other graded term orders, then $\mathcal{H}_R$ admits an embedding for every monomial complete intersection R. ☐

A Clements–Lindström type theorem for embeddings. We prove an analogue of the following theorem of Clements and Lindström: If $A = \Bbbk[x_1, \dots, x_n]$ and $\mathfrak{a} = (x_1^{e_1}, \dots, x_n^{e_n})$ with $2 \leq e_1 \leq \cdots \leq e_n \leq \infty$, then for every homogeneous A-ideal I with $\mathfrak{a} \subseteq I$, there exists a LEX-segment ideal L such that Hilbert functions of $L + \mathfrak{a}$ and I are identical. If $t = \infty$, then the theorem below (even without the hypothesis that $\mathfrak{a}$ is a monomial ideal) follows from the argument that proved the existence of the embedding ϵ' in the beginning of the proof of Theorem 3.23. Note that, in that context, we may take $\mathfrak{a}$ to be homogeneous in the standard grading of A to apply Lemma 4.1 and Theorem 3.3.

Theorem 3.25. *Let $A = \Bbbk[x_1, \dots, x_n]$ and $t \in \mathbb{Z} \cup \{\infty\}$. Let $\mathfrak{a}$ be a monomial A-ideal such that $x_i^t \in \mathfrak{a}$ for all $1 \leq i \leq n$. Let $B = A[z]$, where z is an indeterminate. If $\mathcal{H}_R$ admits an embedding, then $\mathcal{H}_{B/(\mathfrak{a}B, z^t)}$ admits an embedding.*

Proof. Since $\mathfrak{a}$ is a monomial ideal, in order to to study embeddings, we need to consider only monomial ideals (Remark 1.4); hence we may assume that $\Bbbk = \mathbb{C}$. Write $S = B/(\mathfrak{a}B, z^t)$. For every $H \in \mathcal{H}_S$, there exists a z-stable S-ideal I such that $H_I = H$; see Lemma 4.2. By Theorem 3.3 $\mathcal{H}_S$ admits an embedding. ☐

4. Stabilization

The results of this section do not depend on the previous sections, and are used in Section 3.

We adopt the following notation for this section. Let $A = \Bbbk[x_1, \dots, x_n]$ be a standard graded polynomial ring. Let $\mathfrak{a}$ be an A-ideal. Let $B = A[z]$, with $\mathbb{N}^2$-grading given by $\deg x_i = (1, 0)$, for all $1 \leq i \leq n$ and $\deg z = (0, 1)$. Let ω be the weight vector on B with $\omega(x_i) = 1$ for all i and $\omega(z) = 0$.

Lemma 4.1. *Let I be an B-ideal such that $\mathfrak{a}B \subseteq I$. Then there exists B-ideal J such that $\mathfrak{a}B \subseteq J$, $H_I = H_J$ and $J(B/\mathfrak{a}B)$ is z-stable.*

Proof. Write $S = B/\mathfrak{a}B$. Let ω be the weight vector on B with $\omega(x_i) = 1$ for all i and $\omega(z) = 0$. By replacing I by $\operatorname{in}_\omega(I)$, we may assume that I is a multigraded B-ideal.

For $1 \leq l \leq n$, let L_l be the distraction matrix

$$\begin{array}{c} \\ x_1 \\ \vdots \\ x_n \\ z \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & \cdots & t & t+1 & \cdots \end{array} \\ \begin{pmatrix} x_1 & x_1 & \cdots & x_1 & x_1 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ x_n & x_n & \cdots & x_n & x_n & \cdots \\ x_l + z & z & \cdots & z & z & \cdots \end{pmatrix} \end{array}.$$

Write $\Phi = \operatorname{in}_\omega \circ D_{L_1} \circ \cdots \circ \operatorname{in}_\omega \circ D_{L_n}$. Let $J^{(r)} = \Phi^r(I)$. We will show that $J^{(r)} = J^{(r+1)}$ for all $r \gg 0$. Note that $\Phi(\mathfrak{a}B) = \mathfrak{a}B$. Write $J^{(r)} = \bigoplus_{d \in \mathbb{N}} \bigoplus_{i \in \mathbb{N}} J^{(r)}_{d,d-i} z^i$ as $\Bbbk$-vector-spaces. For all $d \in \mathbb{N}$ and all $i \in \mathbb{N}$ (equivalently, $0 \le i \le d$), we have $\sum_{j=0}^{i} |J^{(r+1)}_{d,d-j}| \ge \sum_{j=0}^{i} |J^{(r)}_{d,d-j}|$; equality holds for all $i \in \mathbb{N}$ (equivalently, $0 \le i \le d$) if and only if $J^{(r)}_d$ is z-stable.

Since $|J^{(r+1)}_d| = |J^{(r)}_d|$, we see that there exists r such that $J^{(r)}_d$ is z-stable. Let r_d be such that $J^{(r_d)}_t$ is z-stable for all $0 \le t \le d$. For $d \ge 0$, let $\mathfrak{b}^{(d)}$ be the ideal generated by $\bigoplus_{t=0}^{d} J^{(r_d)}_t$. Since S is Noetherian, the ascending chain $\mathfrak{b}^{(1)} \subseteq \mathfrak{b}^{(2)} \subseteq \cdots$ stabilizes, so $J^{(r)} = J^{(r+1)}$ for all $r \gg 0$. Set J to be the stable value. $\square$

Lemma 4.2. *Let $t > 1$ be an integer, and suppose that $\Bbbk$ contains a primitive tth root of unity ζ. Assume that $\mathfrak{a}$ is an A-ideal such that $x_i^t \in \mathfrak{a}$ for all $1 \le i \le n$. Let I be an B-ideal such that $(\mathfrak{a}B, z^t) \subseteq I$. Then there exists B-ideal J such that $(\mathfrak{a}B, z^t) \subseteq J$, $H_I = H_J$ and $J(B/(\mathfrak{a}B, z^t))$ is z-stable.*

Proof. Let ω be the weight vector on B with $\omega(x_i) = 1$ for all i and $\omega(z) = 0$. Replacing I by $\operatorname{in}_\omega(I)$, we may assume that I is multigraded. Let L_j be the distraction matrix:

$$\begin{array}{c|c} & \begin{array}{cccccc} 1 & 2 & \cdots & t & t+1 & \cdots \end{array} \\ \begin{array}{c} x_1 \\ \vdots \\ x_n \\ z \end{array} & \begin{pmatrix} x_1 & x_1 & \cdots & x_1 & x_1 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ x_n & x_n & \cdots & x_n & x_n & \cdots \\ x_j - z & x_j - \zeta z & \cdots & x_j - \zeta^{t-1} z & z & \cdots \end{pmatrix} \end{array}$$

From [MM11, Lemma 3.6] we see the following: (i) $D_{L_j(z^t)} = x_j^t - z^t$ (ii) For all $fz^i \in B$, with $f \in A$ and $i \le t$, $x_j z^{i-1}$ appears with a nonzero coefficient in $D_{L_j}(fz^i)$). Moreover, $(\operatorname{in}_\omega \circ D_{L_j})(\mathfrak{a}B, z^t) = (\mathfrak{a}B, z^t)$. Let $J^{(r)} = (\operatorname{in}_\omega \circ D_{L_j})^r(I)$. Then, arguing as in the proof of Lemma 4.1, we see that for all $r \gg 0$, $J^{(r)} = J^{(r+1)}$ and that for all $fz^i \in J^{(r)}$ with $z \nmid f$ and $i \le t$, $fx_j z^{i-1} \in J^{(r)}$. Repeating this argument for all $1 \le j \le n$, we complete the proof. $\square$

Acknowledgements

We thank A. Conca and the referee for helpful comments. The computer algebra system `Macaulay2` [M2] provided valuable assistance in studying examples.

References

[BCR05] A. M. Bigatti, A. Conca, and L. Robbiano, *Generic initial ideals and distractions*, Comm. Algebra **33** (2005), no. 6, 1709–1732. ↑8

[BH93] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. ↑1, 5

[Big93] A. M. Bigatti, *Upper bounds for the Betti numbers of a given Hilbert function*, Comm. Algebra **21** (1993), no. 7, 2317–2334. ↑1

[CHH04] A. Conca, J. Herzog, and T. Hibi, *Rigid resolutions and big Betti numbers*, Comment. Math. Helv. **79** (2004), no. 4, 826–839. ↑1

[CL69] G. F. Clements and B. Lindström, *A generalization of a combinatorial theorem of Macaulay*, J. Combinatorial Theory **7** (1969), 230–238. ↑1

[Con04] A. Conca, *Koszul homology and extremal properties of Gin and Lex*, Trans. Amer. Math. Soc. **356** (2004), no. 7, 2945–2961. ↑1

[CRV01] A. Conca, M. E. Rossi, and G. Valla, *Gröbner flags and Gorenstein algebras*, Compositio Math. **129** (2001), no. 1, 95–121. ↑4

[EGH96] D. Eisenbud, M. Green, and J. Harris, *Cayley-Bacharach theorems and conjectures*, Bull. Amer. Math. Soc. (N.S.) **33** (1996), no. 3, 295–324. ↑1

[Eis95] D. Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995. With a view toward algebraic geometry. ↑2

[FR07] C. A. Francisco and B. P. Richert, *Lex-plus-powers ideals*, Syzygies and Hilbert functions, 2007, pp. 113–144. ↑1

[GHP08] V. Gasharov, N. Horwitz, and I. Peeva, *Hilbert functions over toric rings*, Michigan Math. J. **57** (2008), 339–357. Special volume in honor of Melvin Hochster. ↑1, 3

[GMP11] V. Gasharov, S. Murai, and I. Peeva, *Hilbert schemes and maximal Betti numbers over Veronese rings*, Math. Z. **267** (2011), no. 1-2, 155–172. ↑1

[M2] D. R. Grayson and M. E. Stillman, *Macaulay 2, a software system for research in algebraic geometry*, 2006. Available at http://www.math.uiuc.edu/Macaulay2/. ↑11

[Hul93] H. A. Hulett, *Maximum Betti numbers of homogeneous ideals with a given Hilbert function*, Comm. Algebra **21** (1993), no. 7, 2335–2350. ↑1

[Kat68] G. Katona, *A theorem of finite sets*, Theory of graphs (Proc. Colloq., Tihany, 1966), 1968, pp. 187–207. ↑1

[Kru63] J. B. Kruskal, *The number of simplices in a complex*, Mathematical optimization techniques, 1963, pp. 251–278. ↑1

[Mer06] J. Mermin, *Lexlike sequences*, J. Algebra **303** (2006), no. 1, 295–308. ↑5

[Mer10] ———, *Monomial regular sequences*, Proc. Amer. Math. Soc. **138** (2010), no. 6, 1983–1988 (electronic). ↑1, 10

[MM10] J. Mermin and S. Murai, *Betti numbers of lex ideals over some macaulay-lex rings*, J. Algebraic Combin. **31** (2010), no. 2, 299–318. ↑1, 4

[MM11] ———, *The lex-plus-powers conjecture holds for pure powers*, Adv. Math. **226** (2011), no. 4, 3511–3539. ↑11

[MP06] J. Mermin and I. Peeva, *Lexifying ideals*, Math. Res. Lett. **13** (2006), no. 2-3, 409–422. ↑1

[MS05] E. Miller and B. Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2005. ↑9

[Par96] K. Pardue, *Deformation classes of graded modules and maximal Betti numbers*, Illinois J. Math. **40** (1996), no. 4, 564–585. ↑1

[Sba01] E. Sbarra, *Upper bounds for local cohomology for rings with given Hilbert function*, Comm. Algebra **29** (2001), no. 12, 5383–5409. ↑1

[Sta97] R. P. Stanley, *Enumerative combinatorics. Vol. 1*, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 1997. With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original. ↑3

Department of Mathematics, Purdue University, West Lafayette IN 47907
E-mail address: `gcavigli@math.purdue.edu`

Department of Mathematics, Purdue University, West Lafayette IN 47907
Current address: Chennai Mathematical Institute, Siruseri, Tamilnadu 603103. India.
E-mail address: `mkummini@cmi.ac.in`