BOUNDS ON NUMBER OF GENERATORS OF PRIME IDEALS

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ABSTRACT. Let S be a standard graded polynomial ring over an algebraically closed field $\Bbbk$, and let $P \subseteq S$ be a non-degenerate homogeneous prime ideal. Inspired by a classical theorem of Castelnuovo on the maximal number of linearly independent quadrics contained in P, we prove that the number of minimal generators of a given degree j of a graded prime ideal can be bounded above by a doubly-exponential function that only depends on j and on the height of P. In addition to providing a bound for generators in any degree j, not just for quadrics, our techniques allow us to drop the geometric assumption that $\Bbbk$ is algebraically closed. By means of standard techniques, we also obtain analogous upper bounds on higher graded Betti numbers of a prime ideal.

1. INTRODUCTION

This article deals with the following fundamental question: *"How many minimal generators of a given degree does a prime ideal in a polynomial ring have?"*

A classical theorem of Guido Castelnuovo states that, if P is a non-degenerate (i.e., containing no linear form) homogeneous prime ideal of height h in a polynomial ring S over an algebraically closed field $\Bbbk$, then the number of linearly independent quadrics contained in P is at most $\binom{h+1}{2}$. What is relevant to observe is that the bound is independent of the number of variables of S, which for the purposes of this paper should be thought of as an unknown integer $n \gg 0$. The modern proof of Castelnuovo's theorem is very much geometric: a repeated application of Bertini's theorem guarantees that a general hyperplane section of the variety defined by P is still non-degenerate, irreducible and reduced. In algebraic terms, going modulo a linear regular element and saturating yields an ideal $\overline{P}$ in a polynomial ring in one less variable which is still prime, and does not contain any linear forms. For the latter, the assumption that $\Bbbk$ is algebraically closed is crucial. Castelnuovo's theorem supports a more general philosophy which suggests that, sometimes under some reasonable geometric assumptions, prime ideals are forced to have better behavior than all other ideals. We point out, for instance, that this bound is completely false for general ideals, even radical; for instance the height-one ideal $(x_1 x_j \mid j = 2, \ldots, n) \subseteq \Bbbk[x_1, \ldots, x_n]$ is radical, but it contains $n-1$ linearly independent quadrics.

Castelnuovo's theorem resembles, in its spirit, two fundamental problems involving numerical invariants, both of which have recently been settled: the Eisenbud-Goto conjecture and Stillman's conjecture. The Eisenbud-Goto conjecture [EG84] states that, in the same geometric assumptions of Castelnuovo's theorem, the sum of the regularity of S/P and the height of P can be bounded above by the multiplicity of S/P. Recently, McCullough and Peeva [MP18] constructed a family of examples, based on two new constructions called step-by-step homogenization and Rees-like algebras and a series of examples due to Mayr-Meyer, which fail spectacularly in satisfying the bound predicted by

the Eisenbud-Goto conjecture. In fact, they show that the regularity of non-degenerate homogeneous prime ideals cannot be bounded by any polynomial function of the multiplicity. See also [CCM+19] for some further developments in this direction. While the conjecture is still open in several important cases, such as when P defines a smooth projective variety, the striking result of McCullough and Peeva has put prime ideals under a completely new perspective.

Stillman's conjecture, on the other hand, states that if $I \subseteq S$ is a homogeneous ideal generated by t forms of degrees $d_1, \ldots, d_t$, then the length of a graded minimal free resolution of I over S can be bounded above by a constant which only depends on $d_1, \ldots, d_t$ (see [PS09]). This conjecture has recently been proved in full generality by Ananyan and Hochster [AH20], with the use of a fundamental and extremely useful new notion, that they call strength. After them, several other authors have been able to provide a proof utilizing various techniques; for instance, see [ESS19]. The known upper bounds either come from finiteness conditions related to Noetherianity and, as such, are not explicit (as in [AH20] and [ESS19]), or are expressed as huge towers of exponentials, with several layers. Even if the estimates are either not explicit or typically far from being optimal, the remarkable fact is that they exist in the first place.

The solution of Stillman's conjecture implies uniform upper bounds for several other invariants, including the Castelnuovo-Mumford regularity. Castelnuovo's theorem resembles Stillman's conjecture in what it provides a bound on a given invariant, the number of minimal quadratic generators of a non-degenerate prime $P \subseteq \Bbbk[x_1, \ldots, x_n]$, which does not depend on $\Bbbk$ (as long as it is algebraically closed), or on the number of variables n of the ambient ring.

It is well-known that the bound of $\binom{h+1}{2}$ produced by Castelnuovo's theorem fails if $\Bbbk$ is not assumed to be algebraically closed (for instance, see [DMV19, Example 5.7], or Example 4.4). To the best of our knowledge, if no geometric assumptions such as $\Bbbk$ being algebraically closed are involved, then there is no known upper bound for the number of quadratic minimal generators of a prime ideal just in terms of its height. One difficulty is that the usual approach of using Bertini's theorem fails. In fact, as pointed out before, linear forms can be introduced in the process of taking hyperplane sections and saturating the the ideal P, and this forces one to account for the number of possibly minimal quadratic generators killed by these linear forms. Such a number could very well depend on n.

Even less seems to be known if one is not merely interested in quadrics, but wants to study the number of minimal homogeneous generators of P of any degree $j \geqslant 2$. That is, if one wants to provide an upper bound on the $(0, j)$-th graded Betti number $\beta_{0,j}(P)$ just in terms of j and of the height of P. Our main result answers precisely this question, with no assumptions on the base field.

Theorem A (see Theorem 4.2). Let S be a standard graded polynomial ring over a field $\Bbbk$, and $P \subseteq S$ be a homogeneous prime ideal of height h. For every $j \geqslant 0$ we have that

$$\beta_{0,j}(P) \leqslant h^{2^{j+1}-3}.$$

Our methods rely on a version developed in [CV15] of the well-known Buchberger's algorithm, which allows a direct computation of a Gröbner basis with respect to any given weight-order without first refining it to a monomial one (see Algorithm 2.1), and Lemma

2.2, which allows to extract a *minimal* Gröbner basis from it. Our use of this algorithm is more theoretical than computational. What is relevant to us is that it yields Proposition 2.4, which is the key for the inductive step in the proof of Theorem A. Another important observation involved in the proof is the fact that the minimal number of generators of a prime ideal in degree j equals the number of minimal relations of degree $D + j$ of an almost complete intersection of degrees $d_1 \leqslant d_2 \leqslant \ldots \leqslant d_h \ll D$ (see Lemma 4.1). It is important to point out that, while the validity of Stillman's conjecture allows to bound the number of minimal relations of an ideal in terms of the degrees of its generators, this observation cannot be applied to our scenario. In fact, no information on the degrees $d_1, \ldots, d_h$ of the almost complete intersection or on its multiplicity can be deduced when constructing it in Lemma 4.1.

Theorem A exhibits an explicit upper bound of doubly-exponential type. While this estimate seems far from being optimal, especially for large values of j or h, a doubly-exponential behavior is inevitable given our methods. Nevertheless, just like for the current known bounds for Stillman's conjecture, the main purpose of our result is to prove that a bound depending only on j and h actually exists.

In the case of quadrics, an *ad hoc* analysis of the quantities involved allows us to obtain a significantly more accurate upper bound than the one of Theorem A. This estimate, like the one of Castelnuovo's theorem, is quadratic in the height of the prime.

Theorem B (see Theorem 4.5). Let S be a standard graded polynomial ring over a field $\Bbbk$, and $P \subseteq S$ be a non-degenerate homogeneous prime ideal of height h. The number of linearly independent quadrics contained in P is at most $2h^2 + h$.

Finally, we extend Theorem A and produce explicit upper bounds for Betti numbers $\beta_{i,j}(I)$ of any radical ideal I in terms of i, j, and the bigheight of I (see Theorem 4.7). In order to achieve this, we use Theorem A and the strategies involved in its proof to show that, for *any* monomial order $\leqslant$ and any unmixed radical ideal I of height h, there is doubly-exponential upper bound for $\beta_{i,j}(\mathrm{in}_{\leqslant}(I))$ only depending on i, j and the height of I (see Proposition 4.6).

Notation and setup. Throughout this article, $\Bbbk$ is a field, and $S = \Bbbk[x_1, \ldots, x_n] = \bigoplus_{j \geqslant 0} S_j$ is a graded polynomial ring over $\Bbbk$, with $\deg(x_i) = 1$ for every $i = 1, \ldots, n$. We will refer to this as the standard grading on S. Given an integer $1 \leqslant h \leqslant n$, which will be either specified or clear from the context, we let $\overline{S}$ denote the subring $\Bbbk[x_1, \ldots, x_h]$ of S. Given a finitely generated graded S-module $M = \bigoplus_j M_j$, we let $\beta_{i,j}^S(M) = \dim_{\Bbbk}(\mathrm{Tor}_i^S(M, \Bbbk)_j)$ be the (i, j)-th graded Betti number of M as an S-module, and we let $\beta_{i, \leqslant j}^S(M) = \sum_{t \leqslant j} \beta_{i,j}^S(M)$. We will drop the superscript when the ring over which we are computing Betti numbers is clear from the context.

2. General coordinates and relative Buchberger's algorithm

Given $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{N}^n$ and a monomial $\mathbf{X^u} = x_1^{u_1} \cdots x_n^{u_n} \in S$, we let the weight of $\mathbf{X^u}$ to be $\omega \cdot \mathbf{u} = \sum_{i=1}^n \omega_i u_i$. This naturally induces a total preorder on monomials: if $\mathbf{X^u}$ and $\mathbf{X^v}$ are monomials in S, then $\mathbf{X^u} \leqslant_{\omega} \mathbf{X^v}$ if and only if $\omega \cdot \mathbf{u} \leqslant \omega \cdot \mathbf{v}$. If $f \in S$ is a polynomial, we can write it uniquely as a sum of monomials, with coefficients in $\Bbbk$. We then define the initial form $\mathrm{in}_{\omega}(f)$ of f as the sum, with coefficients, of the monomials in the support

of f with maximal weight with respect to $\preccurlyeq_\omega$. The initial ideal of I with respect to $\preccurlyeq_\omega$ is $\mathrm{in}_\omega(I) = (\mathrm{in}_\omega(g) \mid g \in I)$, the ideal generated by the initial forms of elements of I. If $g_1, \ldots, g_t$ is any system of generators of I, then clearly $(\mathrm{in}_\omega(g_i) \mid i = 1, \ldots, t) \subseteq \mathrm{in}_\omega(I)$, and it is well-known that the containment can be strict. We say that a collection of homogeneous generators $g_1, \ldots, g_t$ of a given ideal I is a Gröbner basis with respect to $\preccurlyeq_\omega$ if equality holds above. Finally, we say that $g_1, \ldots, g_t$ is a *minimal* Gröbner basis of I if $\mathrm{in}_\omega(g_1), \ldots, \mathrm{in}_\omega(g_t)$ are a minimal set of generators of $\mathrm{in}_\omega(I)$.

2.1. Minimal Gröbner bases for weight preorders.

We now focus on how to compute a minimal Gröbner basis of a given homogeneous ideal $I \subseteq S$ with respect to a given weight. If $\preccurlyeq$ is a monomial order (that is, $\mathrm{in}_{\preccurlyeq}(f)$ is a monomial for every $f \in S$), then a standard way to compute a Gröbner basis of I with respect to $\preccurlyeq$ is the well-known Buchberger's algorithm. This process involves the computation of the so-called S-pairs, which are obtained from syzygies between initial forms with respect to $\preccurlyeq$ of pairs of elements of I, and the calculation of remainders of certain divisions. In a finite number of steps, the Buchberger's algorithm produces a Gröbner basis of I with respect to $\preccurlyeq$.

If $\omega \in \mathbb{N}^n$ is a weight, the Buchberger's algorithm as stated could fail to produce a Gröbner basis of I with respect to $\preccurlyeq_\omega$. In [CV15, Section 4], the first author and Varbaro produce a variant of Buchberger's algorithm which works for a weight order as well without refining it to a monomial order. In [CV15] the authors point out that they are more interested in the theoretical aspects of the algorithm they describe, rather than in the computational ones. The same is true for us. However, for our purposes, we need to revise [CV15, Algorithm 4.2] and show how to obtain a *minimal* Gröbner basis from it. We start by recalling some notation used in [CV15] to describe the algorithm.

Let $\widetilde{S} = S[y]$, and let $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{N}^n$ be a weight. We give bi-degrees $\deg(x_i) = (1, \omega_i)$ and $\deg(y) = (0, 1)$ to the variables of $\widetilde{S}$. Given a non-zero polynomial $g = \sum_{\mathbf{u}} c_{\mathbf{u}} \mathbf{X}^{\mathbf{u}} \in S$ with $c_{\mathbf{u}} \in \Bbbk$, let $d = \max\{\omega \cdot \mathbf{u} \mid c_{\mathbf{u}} \neq 0\}$ be the largest weight of a monomial in its support. We let $\widetilde{g} = y^d \sum_{\mathbf{u}} c_{\mathbf{u}} y^{-\omega \cdot \mathbf{u}} \mathbf{X}^{\mathbf{u}}$ be its homogenization in $\widetilde{S}$. Given $f \in \widetilde{S}$, we let $\underline{f} \in S$ be its evaluation at $y = 0$. Given a non-zero element $f \in \widetilde{S}$, we let $\deg_x(f)$ be its total degree in the variables $x_1, \ldots, x_n$. If $g \in S$ has degree d, then $\deg_x(\widetilde{g}) = d$, and it coincides with the degree of $\widetilde{g} \in S$. Given an ideal $I \subseteq S$, we let $\widetilde{I} = (\widetilde{g} \mid g \in I)$ be the ideal of $\widetilde{S}$ generated by the homogenization of all elements in I. On the other hand, given an ideal $J \subseteq \widetilde{S}$, we let $\underline{J}$ be the ideal of S obtained as the image of J under the evaluation map at $y = 0$. It is well-known that, given $I \subseteq S$, if we evaluate $\widetilde{I}$ at $y = 1$ we get back the ideal I. On the other hand, evaluating at $y = 0$ one gets the initial ideal of I with respect to ω, that is, $\underline{(\widetilde{I})} = \mathrm{in}_\omega(I)$. For a reference of these facts, see for example [MS05, Proposition 8.26]. Finally, if $I = (g_1, \ldots, g_t) \subseteq S$ is a homogeneous ideal, then $\widetilde{I} = (\widetilde{g_1}, \ldots, \widetilde{g_t}) : y^\infty$.

For the convenience of the reader, we briefly recall [CV15, Algorithm 4.2], which returns a Gröbner basis with respect to any weight order $\omega \in \mathbb{N}^n$.

Algorithm 2.1. Let $I \subseteq S$ be a homogeneous ideal, and $g_1, \ldots, g_t$ be a system of homogeneous generators of I. Let $J_0 = (\widetilde{g_1}, \ldots, \widetilde{g_t})$. First, a free presentation over S of $S/\underline{J_0}$ is lifted

to a composition of maps

$$\widetilde{S}^s \xrightarrow{\ \ \Phi\ \ } \widetilde{S}^t \xrightarrow{\ [\widetilde{g_1},\dots,\widetilde{g_t}]\ } \widetilde{S}.$$

Observe that the above might not even be a complex. The columns of Φ represent lifts to $\widetilde{S}$ of syzygies $\sigma_1,\dots,\sigma_s$ of $g_1,\dots,g_t$. As shown in [CV15, Algorithm 4.2], we can write $[\widetilde{g_1},\dots,\widetilde{g_t}] \circ \Phi(\widetilde{S}^s)$ as the ideal $(y^{a_1}\psi_1,\dots,y^{a_s}\psi_s)$ for some bi-homogeneous elements $\psi_i \in \widetilde{S} \setminus (y)\widetilde{S}$, with $a_i > 0$ for all $i = 1,\dots,s$. We will say that the element ψ_i is obtained by *pushing forward* the syzygy σ_i. Moreover, if we let $Q = (\psi_1,\dots,\psi_s)$, then either y is a non-zero divisor on $\widetilde{S}/J_0$, in which case $g_1,\dots,g_t$ was already a Gröbner basis of I, or $Q \nsubseteq J_0$. In the latter case, set $J_1 = J_0 + Q$. We will refer to the above process as *one iteration* of the algorithm.

Performing more iterations, we obtain an increasing chain of ideals $J_0 \subseteq J_1 \subseteq \dots \subseteq \widetilde{S}$ which, since $\widetilde{S}$ is Noetherian, must eventually stabilize at J_m for some m. The same argument used in [CV15, Algorithm 4.2] shows that $J_m = \widetilde{I}$, and thus $\mathrm{in}_\omega(I) = J_m$. By construction, the set of generators of J_m obtained in this process, when evaluated at $y = 1$, produces a Gröbner basis of I. Furthermore, when evaluated at $y = 0$, it produces a set of generators of $\mathrm{in}_\omega(I)$, not necessarily minimal.

Finally, to obtain a minimal Gröbner basis from [CV15, Algorithm 4.2], we will need the following reduction lemma.

Lemma 2.2. Let $f_1,\dots,f_t \in \widetilde{S} \setminus (y)\widetilde{S}$ be bi-homogeneous elements, and let $I = (f_1,\dots,f_t)$. Given a bi-homogeneous element $f \in \widetilde{S}$, there exists a bi-homogeneous element $g \in \widetilde{S} \setminus (y)\widetilde{S}$ such that $(I,f) : y^\infty = (I,g) : y^\infty$, and either $g = 0$ or $\deg_{\underline{x}}(g) = \deg_{\underline{x}}(f)$ and $\underline{g} \notin \underline{I}$.

Proof. If $(I,f) : y^\infty = I : y^\infty$ then we can set $g = 0$. For the rest of the proof, assume that $(I,f) : y^\infty \supsetneq I : y^\infty$. Among all bi-homogeneous elements $g \in \widetilde{S} \setminus (y)\widetilde{S}$ such that $(I,f) : y^\infty = (I,g) : y^\infty$ and $\deg_{\underline{x}}(g) = \deg_{\underline{x}}(f)$, we choose one such that $\underline{g}$ has minimal weight. Note that our current assumptions guarantee that $g \neq 0$. We claim that $\underline{g} \notin \underline{I}$. In fact, if $\underline{g} \in \underline{I}$, we would be able to find homogeneous elements $s_1,\dots,s_t \in S$ such that $\underline{g} + \sum_{i=1}^{t} s_i \underline{f_i} = 0$, $(\widetilde{s_i}) = s_i$, and either $s_i = 0$ or $s_i f_i$ has the same weight as $\underline{g}$. Lifting this relation to $\widetilde{S}$, we get that $g + \sum_{i=1}^{t} s_i f_i \in (y)\widetilde{S}$, that is, there would exist a bi-homogeneous element $g' \in \widetilde{S} \setminus (y)\widetilde{S}$ such that $g + \sum_{i=1}^{t} s_i f_i = y^N g'$ for some $N \geqslant 1$. In particular, note that $\deg_{\underline{x}}(g') = \deg_{\underline{x}}(g) = \deg_{\underline{x}}(f)$, but $\underline{g}'$ has weight strictly smaller than $\underline{g}$. However, the above relation gives that $(I,g') : y^\infty = (I,g) : y^\infty = (I,f) : y^\infty$, which contradicts our minimal choice for the weight of $\underline{g}$. $\qquad\square$

Definition 2.3. Given a list $\Gamma = \{g_1,\dots,g_t\}$ of homogeneous polynomials in S of degrees $d_1,\dots,d_t$, we let

$$\mathrm{syz}(\Gamma) = \ker\left(\bigoplus_{i=1}^{t} S(-d_i) \xrightarrow{[g_1,\dots,g_t]} S \right)$$

be the S-module of the syzygies of $g_1,\dots,g_t$.

We warn the reader that $\mathrm{syz}_S(\Gamma)$ does not necessarily coincide with the first module of syzygies of the ideal generated by $g_1, \ldots, g_t$, since we are not assuming any minimality conditions on such polynomials.

We now apply Algorithm 2.1 and Lemma 2.2 to reach the main goal of this section.

Proposition 2.4. Let $\omega \in \mathbb{N}^n$, and $I \subseteq S$ be a homogeneous ideal. Let $g_1, \ldots, g_t$ be a set of homogeneous generators of I of degrees $d_1 \leqslant \ldots \leqslant d_t$, such that $(\mathrm{in}_\omega(g_i) \mid i = 1, \ldots, t) = \mathrm{in}_\omega(I)_{\leqslant d_t}$. If we let $\Gamma = \{\mathrm{in}_\omega(g_i) \mid i = 1, \ldots, t\}$, then

$$\max\{\beta_{0,d_t+1}(\mathrm{in}_\omega(I)), \beta_{1,d_t+1}(\mathrm{in}_\omega(I))\} \leqslant \beta_{0,d_t+1}(\mathrm{syz}_S(\Gamma)).$$

Proof. Performing one iteration of Algorithm 2.1 to $g_1, \ldots, g_t$, with the same notation used therein we obtain bi-homogeneous elements $\psi_1, \ldots, \psi_s \in S \setminus (y)S$. Let $\delta_i = \deg_{\underline{x}}(\psi_i)$. By possibly relabeling such elements, we may assume that

$$\delta_1 \leqslant \ldots \leqslant \delta_r \leqslant d_t < \delta_{r+1} = \ldots = \delta_{r+u} = d_t + 1 < \delta_{r+u+1} \leqslant \ldots \leqslant \delta_s.$$

Let $J = (\widetilde{g_1}, \ldots, \widetilde{g_t})$, and we apply Lemma 2.2 to J together with each element ψ_i, for $1 \leqslant i \leqslant r$. We then find elements $\varphi_1, \ldots, \varphi_r \in \widetilde{S}$ such that $(J, \psi_j) : y^\infty = (J, \varphi_i) : y^\infty$. Moreover, either $\varphi_i = 0$, or $\deg_{\underline{x}}(\varphi_i) = \deg_{\underline{x}}(\psi_i)$ and $\varphi_i \notin J$. Since by assumption $\mathrm{in}_\omega(g_1), \ldots, \mathrm{in}_\omega(g_t)$ already generate $J = \mathrm{in}_\omega(I)_{\leqslant d_t}$, we must have $\varphi_i = 0$ for all i. Note that the condition that $(J, \psi_i) : y^\infty = (J, \varphi_i) : y^\infty = J : y^\infty$ implies that the elements ψ_i for $1 \leqslant i \leqslant r$ can be disregarded in a subsequent iterations of Algorithm 2.1.

We now apply Lemma 2.2 to J and each element $\psi_{r+1}, \ldots, \psi_{r+u}$. We can therefore find elements $\varphi_1, \ldots, \varphi_v \in \widetilde{S} \setminus (y)\widetilde{S}$, with $v \leqslant u$, such that $\mathrm{in}_\omega(g_1), \ldots, \mathrm{in}_\omega(g_t), \varphi_1, \ldots, \varphi_v$ are minimal generators of the ideal they generate. Furthermore, Lemma 2.2 guarantees that $(J, \varphi_1, \ldots, \varphi_v) : y^\infty = (J, \psi_{r+1}, \ldots, \psi_{r+u}) : y^\infty$.

We can now repeat Algorithm 2.1 with the elements $g_1, \ldots, g_t, \varphi_1, \ldots, \varphi_v, \psi_{r+u+1}, \ldots, \psi_s$ as input. This returns the same elements $\psi_1, \ldots, \psi_s$ obtained before, together with new elements $\theta_1, \ldots, \theta_r \in \widetilde{S} \setminus (y)\widetilde{S}$ obtained by the Algorithm by pushing forward syzygies that involve at least one of the elements $\varphi_1, \ldots, \varphi_v, \psi_{r+u+1}, \ldots, \psi_s$. We observe, however, that by minimality of $\mathrm{in}_\omega(g_1), \ldots, \mathrm{in}_\omega(g_t), \varphi_1, \ldots, \varphi_v$ we must have that $\deg_{\underline{x}}(\theta_i) > d_t + 1$ for every i. Using Lemma 2.2 as before, we see once again that $\psi_1, \ldots, \psi_r$ can be disregarded in a subsequent iteration of Algorithm 2.1. Moreover, it is now clear that $(J, \varphi_1, \ldots, \varphi_v) : y^\infty = (J, \varphi_1, \ldots, \varphi_v, \psi_{r+i}) : y^\infty$ for every $1 \leqslant i \leqslant u$, and thus also the elements $\psi_{r+1}, \ldots, \psi_{r+u}$ can be disregarded in the next iteration. We now see that any further iteration of Algorithm 2.1, together with the considerations we just made, does not return any new element in $\underline{x}$-degree at most $d_t + 1$. However, as the algorithm must eventually returns a Gröbner basis of $\mathrm{in}_\omega(I)$, we conclude that $\mathrm{in}_\omega(g_1), \ldots, \mathrm{in}_\omega(g_t), \varphi_1, \ldots, \varphi_v$ must generate $\mathrm{in}_\omega(I)$ in degree up to $d_t + 1$. Recall that the elements $\varphi_1, \ldots, \varphi_v$ come from pushing forward v syzygyes $\sigma_1, \ldots, \sigma_v \in \mathrm{syz}_S(\Gamma)$ of degree $d_t + 1$. Since a $\Bbbk$-linear relation among $\sigma_1, \ldots, \sigma_v$ would give rise to a $\Bbbk$-linear relation among $\varphi_1, \ldots, \varphi_v$, and hence among $\varphi_1, \ldots, \varphi_v$, we conclude that the syzygies $\sigma_1, \ldots, \sigma_v$ must be $\Bbbk$-linearly independent. In particular, we have that $\beta_{0,d_t+1}(\mathrm{syz}_S(\Gamma)) \geqslant v = \beta_{0,d_t+1}(\mathrm{in}_\omega(I))$.

Now, if σ is any minimal generator of $\mathrm{syz}_S(\mathrm{in}_\omega(I))$ of degree $d_t + 1$, then for degree reasons σ must be a syzygy between minimal generators of $\mathrm{in}_\omega(I)$ of degree at most d_t. Because of our assumptions, we therefore have that $\sigma \in \mathrm{syz}_S(\Gamma)$. If σ was not a minimal

generator of $\mathrm{syz}_S(\Gamma)$, a fortiori it would not be a minimal generator of $\mathrm{syz}_S(\mathrm{in}_\omega(I))$. This shows that $\beta_{1,d_t+1}(\mathrm{in}_\omega(I)) = \beta_{0,d_t+1}(\mathrm{syz}_S(\mathrm{in}_\omega(I))) \leqslant \beta_{0,d_t+1}(\mathrm{syz}_S(\Gamma))$. $\qquad\square$

2.2. General revlex preorders and complete intersections.
We recall how to define total preorders on monomials starting from a matrix. Let Ω be an $m \times n$ matrix with non-negative integer entries, and let ω_i denote its i-th row. Then Ω induces a total preorder on monomials: we declare that $\mathbf{X^u} \leqslant_\Omega \mathbf{X^v}$ if and only if either $\omega_i \cdot \mathbf{u} = \omega_i \cdot \mathbf{v}$ for all $i = 1, \ldots, m$, or there is $1 \leqslant j < m$ such that $\omega_i \cdot \mathbf{u} = \omega_i \cdot \mathbf{v}$ for all $1 \leqslant i \leqslant j$, and $\omega_{j+1} \cdot \mathbf{u} < \omega_{j+1} \cdot \mathbf{v}$. As a consequence, we can talk about initial forms, and the initial ideal with respect to $\leqslant_\Omega$, which we denote by $\mathrm{in}_\Omega(-)$.

Given a matrix preorder $\leqslant_\Omega$ and a finite set $\mathcal{M}$ of monomials, one can always find a weight ω (depending on the set $\mathcal{M}$) such that for any $m_1, m_2 \in \mathcal{M}$ one has $m_1 \leqslant_\Omega m_2$ if and only if $m_1 \leqslant_\omega m_2$. Thus, when computing $\mathrm{in}_\Omega(I)$ of a given homogeneous ideal I, by Noetherianity one can always reduce to computing $\mathrm{in}_\omega(I)$ for some weight ω. In particular, one can use Algorithm 2.1 and Lemma 2.2 in order to produce a minimal Gröbner basis of a given ideal I with respect to $\leqslant_\Omega$.

We now introduce a special matrix preorder, which is extensively used in this article. Given integers $1 \leqslant h < n$, consider the $(n-h) \times n$ matrix

$$\Omega_h = \begin{bmatrix} 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & 1 & 0 \\ 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \ldots & 1 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \end{bmatrix}.$$

We denote the matrix order induced by Ω_h simply by $\leqslant_h$. Observe that $\leqslant_1$ is nothing but the standard degree-revlex order on monomials of S. Given a homogeneous ideal I, we will denote by $\mathrm{in}_h(I)$ the initial ideal of I with respect to $\leqslant_h$, that is, the ideal generated by the initial forms $\mathrm{in}_h(f)$ of all polynomials $f \in I$.

Lemma 2.5. Let $f_1, \ldots, f_h \in S = \mathbb{k}[x_1, \ldots, x_n]$ be homogeneous polynomials, and assume that $f_1, \ldots, f_h, x_{h+1}, \ldots, x_n$ is a full regular sequence in S. Then $\mathrm{in}_h(f_1, \ldots, f_h, x_{h+1}, \ldots, x_n) = (\mathrm{in}_h(f_1), \ldots, \mathrm{in}_h(f_h), x_{h+1}, \ldots, x_n)$. In particular, $f_1, \ldots, f_h$ are a Gröbner basis with respect to $\leqslant_h$, and $\mathrm{in}_h(f_1), \ldots, \mathrm{in}_h(f_h)$ are still a homogeneous regular sequence in $\overline{S} = \mathbb{k}[x_1, \ldots, x_h]$.

Proof. It suffices to prove the first statement, which is just a consequence of well-known commutation properties of revlex-type preorders with modding out the last variables (for instance, see [Eis95, 15.7]). $\qquad\square$

3. A non-standard grading and the Λ-construction

We introduce a grading on $S = \mathbb{k}[x_1, \ldots, x_n]$ which lies in between the standard grading and the monomial $\mathbb{N}^n$-grading. We view S as $\overline{S}[x_{h+1}, \ldots, x_n]$, and for $i = 0, \ldots, n-h$ we let $\eta_i \in \mathbb{N}^{n-h+1}$ be the vector with 1 in position $i+1$ and 0 elsewhere. We let an element of $\overline{S}_j$ have degree $j \cdot \eta_0$ in S, while for $h+1 \leqslant i \leqslant n$ we let x_i have degree η_{i-h}. Extending this accordingly defines the desired $\mathbb{N} \times \mathbb{N}^{n-h}$-grading on S for any $1 \leqslant h \leqslant n$. Observe that the choice $h = n$ gives the standard grading on S, while $h = 1$ corresponds to the monomial $\mathbb{N}^n$-grading.

Given an element $g \in S$ which is homogeneous with respect to this new grading, we denote by $\deg_{(1,1)}(g) \in \mathbb{N}^{n-h+1}$ its $\mathbb{N} \times \mathbb{N}^{n-h}$-degree. Observe that we can write $\deg_{(1,1)}(g) = (e, \epsilon)$ for some $e \in \mathbb{N}$ and $\epsilon \in \mathbb{N}^{n-h}$. We then let $\deg_{(1,0)}(g) = e$ and $\deg_{(0,1)}(g) = \epsilon$. We still denote by $\deg(g) = e + |\epsilon|$, where $|\epsilon|$ denotes the sum of the entries of $\epsilon \in \mathbb{N}^{n-h}$, the degree of g with respect to the standard grading in S, and we sometimes refer to it as the total degree of g to distinguish it from the previous ones.

Remarks 3.1. It follows directly from the given definitions that:

(i) A polynomial of S is homogeneous with respect to the $\mathbb{N} \times \mathbb{N}^{n-h}$-grading if and only if it is the product of a homogeneous element in $\overline{S}$ with a monomial in the last $n-h$ variables.

(ii) If $\Gamma = \{g_1, \ldots, g_t\}$ is a list of are $\mathbb{N} \times \mathbb{N}^{n-h}$-graded polynomials in S, say $\deg_{(1,1)}(g_i) = \eta_i$, then $\mathrm{syz}_S(\Gamma)$ is an $\mathbb{N} \times \mathbb{N}^{n-h}$-graded submodule of $\bigoplus_{i=1}^{t} S(-\eta_i)$.

(iii) If I is an ideal in S which is homogeneous with respect to the standard grading, then $\mathrm{in}_h(I)$ is $\mathbb{N} \times \mathbb{N}^{n-h}$-graded.

Given a list of $\mathbb{N} \times \mathbb{N}^{n-h}$-graded polynomials $\Gamma = \{\gamma_1, \ldots, \gamma_t\}$. For $1 \leqslant r \leqslant t$ we write $\gamma_r = \overline{\gamma_r} m_r$ for some homogeneous element $\overline{\gamma}_r \in \overline{S}$ and a monic monomial $m_r \in \Bbbk[x_{h+1}, \ldots, x_n]$. Let $L_r = \{\mathrm{lcm}(m_{i_1}, \ldots, m_{i_s}, m_r) \mid \{i_1, \ldots, i_s\} \subseteq \{1, \ldots, r-1\}\}$. We consider the following free $\overline{S}$-module

$$\Lambda(\Gamma) = \bigoplus_{r=1}^{t} \bigoplus_{m \in L_r} \overline{S} \cdot e_m^r.$$

We give to an element $e_m^r \in L_r$ of the basis degree equal to $\deg(\gamma_r) + \deg(m) - \deg(m_r) = \deg(m) + \deg(\overline{\gamma}_r)$. Observe that the j-th graded component of $\Lambda(\Gamma)$ is

$$\Lambda(\Gamma)_j = \bigoplus_{r=1}^{t} \left(\bigoplus_{m \in L_r} \overline{S}_{j-\deg(m)-\deg(\overline{\gamma}_r)} \cdot e_m^r \right).$$

Set $\lambda_j(\Gamma) = \dim_{\Bbbk}(\Lambda(\Gamma)_j)$. We start with a dimension count, which inductively allows to control $\lambda_j(\Gamma)$ as one appends elements to the list Γ.

Lemma 3.2. Let Γ_1 be a list of $\mathbb{N} \times \mathbb{N}^{n-h}$-graded polynomials of S of degree at most $j-1$, and Γ_2 be a list of $\mathbb{N} \times \mathbb{N}^{n-h}$-graded polynomials of degree j such that $|\Gamma_2| \leqslant \lambda_j(\Gamma_1)$. Let Γ be the list obtained by appending Γ_2 to Γ_1. Then

$$\lambda_{j+1}(\Gamma) \leqslant 2\lambda_j(\Gamma_1)^2 + (2h-1)\lambda_j(\Gamma_1).$$

Proof. Let $\Gamma_1 = \{\gamma_1, \ldots, \gamma_t\}$ and $\Gamma_2 = \{\gamma_{t+1}, \ldots, \gamma_{t+s}\}$. Observe that, by construction of $\Lambda(\Gamma_1)$, we have that $|\Gamma_1| = t \leqslant \lambda_j(\Gamma_1)$. Write $\gamma_i = \overline{\gamma_i} m_i$ for some homogeneous $\overline{\gamma_i} \in \overline{S} = \Bbbk[x_1, \ldots, x_h]$ and monic monomials $m_i \in \Bbbk[x_{h+1}, \ldots, x_n]$. We can write

$$\Lambda(\Gamma)_{j+1} = \left(\bigoplus_{r=1}^{t} \bigoplus_{m \in L_r} \overline{S}_{j+1-\deg(m)-\deg(\overline{\gamma_r})} \cdot e_m^r \right) \oplus \left(\bigoplus_{r=1}^{s} \bigoplus_{m \in L_{t+r}} \overline{S}_{j+1-\deg(m)-\deg(\overline{\gamma_{t+r}})} \cdot e_m^r \right) =: V \oplus W.$$

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Observe that

$$V = \left(\bigoplus_{r=1}^{t} \bigoplus_{m \in L_r, \deg(e_m^r) \leqslant j} \overline{S}_{j+1-\deg(m)-\deg(\overline{\gamma}_r)} \cdot e_m^r \right) \oplus \left(\bigoplus_{r=1}^{t} \bigoplus_{m \in L_r, \deg(e_m^r)=j+1} \Bbbk \cdot e_m^r \right).$$

The dimension of the first summand of V is

$$\sum_{r=1}^{t} \sum_{m \in L_r, \deg(e_m^r) \leqslant j} \dim_{\Bbbk} \overline{S}_{j+1-\deg(m)-\deg(\overline{\gamma}_r)}$$

$$\leqslant \sum_{r=1}^{t} \sum_{m \in L_r, \deg(e_m^r) \leqslant j} h \dim_{\Bbbk} \overline{S}_{j-\deg(m)-\deg(\overline{\gamma}_r)} = h\lambda_j(\Gamma_1).$$

The dimension of the second summand of V is bounded above by all the possible least common multiples between an element $m' \in L_r$ for some $1 \leqslant r \leqslant t$ such that $\deg(e_{m'}^r) \leqslant j$, together with a monomial $m_i \in \{m_1, \ldots, m_r\}$. This number is bounded above by the possible choices of unordered pairs of distinct elements $\{m, m'\} \subseteq L_t$ with $\max\{\deg(e_m^r), \deg(e_{m'}^r)\} \leqslant j$. It is immediate to see from the description of $\Lambda(\Gamma_1)$ that the number of elements $m \in L_t$ such that $\deg(e_m^r) \leqslant j$ is at most $\lambda_j(\Gamma_1)$, and therefore the dimension of the second summand of V is at most $\binom{\lambda_j(\Gamma_1)}{2}$.

The dimension of W is bounded above by three contributions. The first comes from the summand

$$\bigoplus_{r=1}^{s} \left(\bigoplus_{m \in L_{t+r}, \deg(e_m^{t+r})=j} \overline{S}_1 e_m^{t+r} \right) = \bigoplus_{r=1}^{s} \overline{S}_1 e_{m_{t+r}}^{t+r},$$

whose dimension is $\sum_{r=1}^{s} \dim_{\Bbbk}(\overline{S}_1) = h|\Gamma_2| \leqslant h\lambda_j(\Gamma_1)$.

Since $\deg(e_{m_{t+r}}^{t+r}) = j$ for every $1 \leqslant r \leqslant s$, the second and the third contributions to $\dim_{\Bbbk}(W)$ come from least common multiples m performed between a monomial m_{t+r} for some $1 \leqslant r \leqslant s$ and another monomial which either belongs to $\{m_1, \ldots, m_t\}$ or to $\{m_{t+1}, \ldots, m_{t+r-1}\}$, in such a way that $\deg(e_m^{t+r}) = j+1$. Note that there are at most $|\Gamma_2| \cdot |\Gamma_1| \leqslant \lambda_j(\Gamma_1)^2$ many possible least common multiples coming from the first case scenario, while there are at most $\binom{|\Gamma_2|}{2} \leqslant \binom{\lambda_j(\Gamma_1)}{2}$ possible ones coming from the second. Putting all estimates together, we conclude that

$$\lambda_{j+1}(\Gamma) \leqslant 2\lambda_j(\Gamma_1)^2 + (2h-1)\lambda_j(\Gamma_1). \qquad \square$$

The following proposition justifies the introduction of $\Lambda(\Gamma)$.

Proposition 3.3. Let $\varphi_1, \ldots, \varphi_h \in \overline{S}$ be homogeneous elements, and let $D > \max\{j \in \mathbb{N} \mid \beta_{1,j}(\varphi_1, \ldots, \varphi_h) \neq 0\}$. Let $\gamma_1, \ldots, \gamma_t$ be $\mathbb{N} \times \mathbb{N}^{n-h}$ homogeneous elements of degrees at least D and at most $D+j$ for some $j \geqslant 0$. Set $\Gamma = \{\gamma_1, \ldots, \gamma_t\}$ and let $J = (\varphi_1, \ldots, \varphi_h, \gamma_1, \ldots, \gamma_t)$. Then $\beta_{1,D+j+1}(J) \leqslant \lambda_{D+j+1}(\Gamma)$.

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Proof. Write $\gamma_i = \overline{\gamma_i} m_i$ for some monic monomials $m_i \in \Bbbk[x_{h+1}, \ldots, x_n]$, and homogeneous elements $\overline{\gamma_i} \in \overline{S}$. Given an $\mathbb{N} \times \mathbb{N}^{n-h}$-graded syzygy σ of $\varphi_1, \ldots, \varphi_h, \gamma_1, \ldots, \gamma_t$, we can represent it as a $(h+t)$-uple

$$\sigma = (s_1, \ldots, s_{h+t}) \in \left(\bigoplus_{i=1}^{h} S\left(-(\deg(\varphi_i), 0)\right) \right) \oplus \left(\bigoplus_{i=1}^{t} S\left(-(\deg(\overline{\gamma_i}), \epsilon_i)\right) \right),$$

where ϵ_i represents the exponent vector of the monomial m_i with respect to the variables $x_{h+1}, \ldots, x_n$. If $\sigma \neq 0$, we let $W(\sigma) = \max\{i \mid s_i \neq 0\}$. Given a finite set Σ of $\mathbb{N} \times \mathbb{N}^{n-h}$-graded syzygies of Γ, we let $W(\Sigma) = \sum_{\sigma \in \Sigma} W(\sigma)$. Among all sets Σ of $\mathbb{N} \times \mathbb{N}^{n-h}$-graded elements which minimally generate $\mathrm{syz}_S(J)$ up to degree $D + j + 1$, we pick one which minimizes $W(\Sigma)$.

We claim that there is an injective map of $\Bbbk$-vector spaces $\psi : (\Bbbk\Sigma)_{D+j+1} \hookrightarrow \Lambda(\Gamma)_{D+j+1}$, defined on a $\Bbbk$-basis as follows. First of all, we note that any element $\sigma \in \Sigma$ must satisfy $W(\sigma) > h$, otherwise it would correspond to a minimal syzygy of $\varphi_1, \ldots, \varphi_h$ of degree $D + j + 1$, contradicting our choice of D. For every $1 \leqslant r \leqslant t$, we let $\Sigma_r = \{\sigma \in \Sigma \mid W(\sigma) = h + r\}$. For $\sigma = (s_1, \ldots, s_h, \sigma_1, \ldots, \sigma_t) \in \Sigma_r$ we can write $\sigma_r = \overline{\sigma_r} m_r'$ for some monic monomial $m_r' \in \Bbbk[x_{h+1}, \ldots, x_n]$ and some element $\overline{\sigma_r} \in \overline{S}_{D+j+1-\deg(\gamma_r)-\deg(m_r')}$. Since $\sigma \in \Sigma_r$ is part of a minimal graded generating set of $\mathrm{syz}_S(J)$, the monomial $m_r m_r'$ must be obtained as the least common multiple of some of the monomials $m_1, \ldots, m_{r-1}$, together with m_r. In other words, $m := m_r m_r' \in L_r$, and we can set $\psi(\sigma) = (\overline{\sigma_r} m) e_m^r \in \Lambda(\Gamma)_{D+j+1}$. To show that this map is injective, let $\sigma^{(1)}, \ldots, \sigma^{(s)}$ be a $\Bbbk$-basis of $(\Bbbk\Sigma)_{D+j+1}$, and assume by way of contradiction that $\psi(\sigma^{(1)}), \ldots, \psi(\sigma^{(s)})$ are $\Bbbk$-linearly dependent. Without loss of generality, we may assume that we have non-zero elements $\alpha_1, \ldots, \alpha_s \in \Bbbk$ and a $\Bbbk$-linear combination $\alpha_1 \psi(\sigma^{(1)}) + \ldots + \alpha_s \psi(\sigma^{(s)}) = 0$, where all the elements $\sigma^{(i)}$ belong to Σ_r for some $1 \leqslant r \leqslant t$, and $\sigma_r^{(i)} = \overline{\sigma_r}^{(i)} m_r'$ for all i. We let $m = m_r m_r'$. We have

$$0 = \sum_{i=1}^{s} \alpha_i \psi(\sigma^{(i)}) = \left(\sum_{i=1}^{s} \alpha_i \overline{\sigma_r}^{(i)} \right) e_m^r.$$

It follows that $\sigma' = \sum_{i=1}^{s} \alpha_1 \sigma^{(i)}$ is an $\mathbb{N} \times \mathbb{N}^{n-h}$ which, together with $\sigma^{(2)}, \ldots, \sigma^{(s)}$, constitutes a minimal $\mathbb{N} \times \mathbb{N}^{n-h}$-graded generating set Σ' of $\mathrm{syz}_S(J)$ in degree up to $D + j + 1$ such that $W(\Sigma') < W(\Sigma)$. This contradicts our choice of Σ. Therefore ψ is injective, and the proof is complete. $\qquad\square$

4. Bounds on Betti numbers of prime ideals

We start this main section of the article with an observation that, even if rather elementary, will come very handy in the study of generators of prime ideals. We say that an ideal of S is unmixed if all its associated primes are minimal, and they all have the same height. In the next lemma we show that, to compute the graded Betti numbers of an unmixed homogeneous radical ideal of height $h > 0$, one can always reduce to studying the Betti numbers of a suitable almost complete intersection, that is, an ideal of height h generated by $h + 1$ elements.

Lemma 4.1. Assume that $\Bbbk$ is infinite. Let $I \subsetneq S = \Bbbk[x_1, \ldots, x_n]$ be a homogeneous unmixed radical ideal of height $h < n$, and $\mathfrak{f} \subseteq I$ be an ideal generated by a regular sequence with

$\mathrm{ht}(\mathfrak{f}) = h$. If $I \neq \mathfrak{f}$, there exists an integer D_0 with the following property: for any $D > D_0$, there exists a homogeneous element $g \in S_D$ such that $\beta_{0,j}(I) = \beta_{1,D+j}(\mathfrak{f}+(g))$ for all $j \geqslant 0$.

Proof. Let $D' = \max\{j \in \mathbb{Z} \mid \beta_{0,j}(\mathfrak{f}) \neq 0 \text{ or } \beta_{1,j}(\mathfrak{f}) \neq 0\}$. Observe that the minimal primes of I are contained in the minimal primes of $\mathfrak{f}$. Since I is radical, unmixed and homogeneous, we can find a homogeneous element $g \in S$ such that $I = \mathfrak{f} : g$. Let $D_0 = \max\{D', \deg(g)\}$, and fix an integer $D > D_0$. Since $\mathrm{depth}(S/I) > 0$ and $\Bbbk$ is infinite, we can find a linear form ℓ which is regular modulo I. In particular, $I = \mathfrak{f} : g\ell^N$ for any $N \geqslant 0$. After replacing g with $g\ell^{D-\deg(g)}$, we may assume without loss of generality that $D = \deg(g)$. Let $\mathfrak{a} = \mathfrak{f}+(g)$. After applying the functor $- \otimes_S \Bbbk$ to the graded exact sequence $0 \to S/I(-D) \to S/\mathfrak{f} \to S/\mathfrak{a} \to 0$, and looking at the component of degree m in the long exact sequence of $\mathrm{Tor}_\bullet^S(-, \Bbbk)$ modules, we get an exact sequence of $\Bbbk$-vector spaces

$$\mathrm{Tor}_2^S(S/\mathfrak{f}, \Bbbk)_m \to \mathrm{Tor}_2^S(S/\mathfrak{a}, \Bbbk)_m \to \mathrm{Tor}_1^S(S/I, \Bbbk)_{m-D} \to \mathrm{Tor}_1^S(S/\mathfrak{f}, \Bbbk)_m.$$

For $j \geqslant 0$ and $m = j + D > D_0$ we have that $\mathrm{Tor}_2^S(S/\mathfrak{f}, \Bbbk)_m = \mathrm{Tor}_1^S(S/\mathfrak{f}, \Bbbk)_m = 0$. It follows that $\beta_{0,j}(I) = \dim_\Bbbk(\mathrm{Tor}_1^S(S/I, \Bbbk))_j = \dim_\Bbbk(\mathrm{Tor}_2^S(S/\mathfrak{a}, \Bbbk))_{j+D} = \beta_{1,j+D}(\mathfrak{a})$, as claimed. $\qquad\square$

We are finally ready to prove our main theorem.

Theorem 4.2. Let $I \subsetneq S$ be a homogeneous unmixed radical ideal of height h. For all $j \geqslant 0$ we have that

$$\beta_{0,j}(I) \leqslant h^{2^{j+1}-3}.$$

Proof. The cases $h \leqslant 1$ or $j = 0$ are trivially satisfied, therefore we will assume that $h \geqslant 2$ and $j \geqslant 1$. The case in which $h = n = \dim(S)$ is also trivial, since in this case I is forced to be equal to $(x_1, \ldots, x_n)$ and the claimed bound is satisfied. Assume that $h < n$. A suitable extension of the base field does not affect our assumptions and the desired conclusion, therefore we may assume that $\Bbbk$ is infinite. We may find an ideal $\mathfrak{f} \subseteq I$, with $\mathrm{ht}(\mathfrak{f}) = h$, generated by a homogeneous regular sequence of degrees $d_1 \leqslant \ldots \leqslant d_h$. If $I = \mathfrak{f}$, then $\beta_{0,j}(I) \leqslant h$ for every $j \geqslant 1$, and the claimed bound is satisfied. Assume that $I \neq \mathfrak{f}$. By Lemma 4.1, for $D \gg 0$ we can find a homogeneous element $g \notin \mathfrak{f}$ of degree D such that $\beta_{0,j}(I) = \beta_{1,D+j}(\mathfrak{a})$, where $\mathfrak{a} = \mathfrak{f} + (g)$ is an almost complete intersection of height h. In particular, as seen in the proof of Lemma 4.1, we assume that $D > \max\{j \in \mathbb{Z} \mid \beta_{1,j}(\mathfrak{f}) \neq 0\}$.

By upper semi-continuity it suffices to prove that $\beta_{1,D+j}(\mathrm{in}_h(\mathfrak{a})) \leqslant h^{2^{j+1}-3}$. After a sufficiently general change of coordinates, if we let $\varphi_1 = \mathrm{in}_h(f_1), \ldots, \varphi_h = \mathrm{in}_h(f_h) \in \overline{S}$, then by Lemma 2.5 we may assume that $\varphi_1, \ldots, \varphi_h, x_{h+1}, \ldots, x_n$ form a regular sequence.

We prove by induction on $j \geqslant 1$ that there exists a list Γ of $\mathbb{N} \times \mathbb{N}^{n-h}$-graded elements that generate $\mathrm{in}_h(\mathfrak{a})$ in degrees between D and $D+j-1$, and such that $\lambda_{D+j}(\Gamma) \leqslant h^{2^{j+1}-3}$. It will then follow from Propositions 2.4 and 3.3 that $\beta_{1,D+j}(\mathrm{in}_h(\mathfrak{a})) \leqslant h^{2^{j+1}-3}$, and this will conclude the proof.

The base case $j = 1$ is satisfied, since we can choose $\Gamma = \{\mathrm{in}_h(g)\}$, and it follows that $\lambda_{D+1}(\Gamma) = h = h^{2^{j+1}-3}$.

Let $j \geqslant 2$, and assume that the claimed inequality is true for $j - 1$. Let Γ_1 be a set of minimal $\mathbb{N} \times \mathbb{N}^{n-h}$-graded generators of $\mathrm{in}_h(\mathfrak{a})$ in degrees between D and $D + j - 2$, and Γ_2 be minimal $\mathbb{N} \times \mathbb{N}^{n-h}$-graded generators of $\mathrm{in}_h(\mathfrak{a})$ of degree precisely equal to $D + j - 1$. Let Γ be the list obtained by appending Γ_2 to Γ_1. It follows from Proposition 3.3 that

$|\Gamma_2| = \beta_{0,D+j-1}(\text{in}_h(\mathfrak{a})) \leqslant \lambda_{D+j-1}(\Gamma_1)$. By Lemma 3.2 we have that $\lambda_{D+j}(\Gamma) \leqslant 2\lambda_{D+j-1}(\Gamma_1)^2 + (2h-1)\lambda_{D+j-1}(\Gamma_1)$. Since $\lambda_{D+j-1}(\Gamma_1) \leqslant h^{2^j-3}$ by induction, and because $2^j - 1 \leqslant 2^{j+1} - 5$ for $j \geqslant 2$, we conclude that

$$\lambda_{D+j}(\Gamma) \leqslant 2\left(h^{2^j-3}\right)^2 + (2h-1)h^{2^j-3} \leqslant 2h^{2^{j+1}-6} + h^2 h^{2^j-3}$$

$$\leqslant h^{2^{j+1}-5} + h^{2^j-1} \leqslant 2h^{2^{j+1}-5} \leqslant h^{2^{j+1}-3}. \qquad \square$$

In the case of quadrics, Theorem 4.2 provides h^5 as an upper bound, which is definitely larger compared to the one of $\binom{h+1}{2}$ given by Castelnuovo's Theorem. The following example shows that one cannot expect the bound of Castelnuovo to hold for unmixed radical ideals.

Example 4.3. For any $h \geqslant 1$, the ideal $(x_1,\ldots,x_h) \cap (y_1,\ldots,y_h) \subseteq S = \Bbbk[x_1,\ldots,x_h,y_1,\ldots,y_h]$ is unmixed and radical, and it is minimally generated by the h^2 monomials $\{x_iy_j \mid 1 \leqslant i,j \leqslant h\}$.

If one does not assume that $\Bbbk$ is algebraically closed, the upper bound of Castelnuovo's theorem does not even hold for prime ideals.

Example 4.4. Let $S = \mathbb{R}[a,b,c,d]$ and $P = (a^2 + c^2, b^2 + d^2, ad - bc, ab + cd)$. Observe that $S/P \cong \mathbb{R}[x,y,ix,iy]$, therefore P is a prime ideal of height two. However, P is minimally generated by $4 > \binom{3}{2}$ quadrics.

The next theorem provides a more refined upper bound for the number of quadratic minimal generators of *any* unmixed radical ideal in a standard graded polynomial ring over *any* field. In order to achieve this, we run a more careful analysis of the first few inductive steps in the proof of Theorem 4.2.

Theorem 4.5. Let $I \subsetneq S$ be an unmixed radical ideal of height h. Then $\beta_{0,2}(I) \leqslant 2h^2 + h$.

Proof. As in the proof of Theorem 4.2, we may assume that $h \geqslant 2$ and that $\Bbbk$ is infinite.

Let $\mathfrak{a} = (f_1,\ldots,f_h,g)$ be an almost complete intersection of degrees $d_1 \leqslant \ldots \leqslant d_h \ll D$ such that $\beta_{0,j}(I) = \beta_{1,D+j}(\mathfrak{a})$ for all j, constructed as in Lemma 4.1. After performing a sufficiently general change of coordinates, our goal is to prove that $\beta_{1,D+2}(\text{in}_h(\mathfrak{a})) \leqslant 2h^2 + h$.

Let $g_1 = g, g_2,\ldots,g_t$ be homogeneous elements of S such that, if $\varphi_i = \text{in}_h(f_i)$ and $\gamma_i = \text{in}_h(g_i)$, then the elements $\varphi_1,\ldots,\varphi_h,\gamma_1,\ldots,\gamma_t$ minimally generate $\text{in}_h(\mathfrak{a})$ in degree up to $D+1$. We let $\Gamma_1 = \{\gamma_1\}$, $\Gamma_2 = \{\gamma_2,\ldots,\gamma_t\}$, and $\Gamma = \{\gamma_1,\gamma_2,\ldots,\gamma_t\}$. Observe that by Proposition 3.3 we have that $t - 1 = |\Gamma_2| \leqslant \lambda_{D+1}(\Gamma_1) = h$.

Write $\gamma_i = \overline{\gamma_i}m_i$ for some monomials $m_i \in \Bbbk[x_{h+1},\ldots,x_n]$ and homogeneous elements $\overline{\gamma_i} \in \overline{S}$.

We have seen in the proof of Lemma 3.2 that $\lambda_{D+2}(\Gamma)$ is bounded above by the contributions of two vector spaces, V and W. In the current notation we have that $V = \overline{S}_2m_1$, and thus $\dim_{\Bbbk}(V) = \binom{h+1}{2}$. On the other hand, we have already seen in the proof of Lemma 3.2 that $\dim_{\Bbbk}(W) \leqslant h\lambda_{D+1}(\Gamma_1) + \binom{\lambda_{D+1}(\Gamma_1)}{2} + |\Gamma_2| \cdot |\Gamma_1| \leqslant h^2 + \binom{h}{2} + h$. We conclude by Proposition 3.3 that $\beta_{1,D+2}(\text{in}_h(\mathfrak{a})) \leqslant \lambda_{D+2}(\Gamma) \leqslant 2h^2 + h$. $\qquad \square$

We conclude the article by providing an upper bound on the Betti numbers $\beta_{i,j}(I)$ of any radical ideal I which only depends only on i,j and its bigheight, i.e., on the largest

height among the minimal primes of I. In particular, if P is a prime ideal, $\beta_{i,j}(P)$ can be bounded above by a function depending only on i, j and the height of P.

We first show that, if we know an upper bound on the number of generators of an ideal I in every degree, then we can accordingly bound the graded Betti numbers $\beta_{i,j}(\text{in}_{\leqslant}(I))$ of its initial ideal with respect to *any* monomial order $\leqslant$. This fact is embedded in the proof of the next result.

Proposition 4.6. Let $I \subsetneq S$ be a homogeneous ideal and integers $a, h \geqslant 1$ be such that $\beta_{0,j}(I) \leqslant h^{2^{j+a}-3}$ for all $j \geqslant 0$. For all $i, j \geqslant 0$ we have that

$$\beta_{i,\leqslant i+j}(I) \leqslant \binom{h^{2^{a+j+1}-3}}{i+1}.$$

Proof. The case $h \leqslant 1$ is trivially satisfied, therefore we may assume that $h \geqslant 2$. Let $\text{in}(-)$ denote the initial ideal with respect to any monomial order $\leqslant$, and let $J = \text{in}(I)$. By upper semi-continuity, it suffices to prove the claimed upper bound for $\beta_{i,\leqslant i+j}(J)$. We first prove the case $i = 0$ by induction on $j \geqslant 0$. The case $j = 0$ is trivial. Assume the claim is proved for some $j \geqslant 0$, and let us show it for $j + 1$. Let Γ be a set of homogeneous elements of I whose initial forms minimally generate $\text{in}(I)$ up to degree j. By inductive hypothesis, we have that $|\Gamma| = \beta_{0,\leqslant j}(J) \leqslant h^{2^{a+j+1}-3}$. Since $\leqslant$ is a monomial order, every S-pair involves only two polynomials at a time. As every minimal generator of degree j of J is either the initial form of the reduction of one such S-pair between elements of Γ, or it is the initial form of a minimal generator of I of degree $j + 1$, we conclude by induction that

$$\beta_{0,j+1}(J) \leqslant \binom{h^{2^{a+j+1}-3}}{2} + h^{2^{a+j+1}-3} \leqslant h^{2^{a+j+3}-6} + h^{2^{a+j+1}-3}.$$

It follows that $\beta_{0,\leqslant j+1}(J) = \beta_{0,\leqslant j}(J) + \beta_{0,j+1}(J) \leqslant h^{2^{a+j+1}-3} + h^{2^{a+j+2}-6} + h^{2^{a+j+1}-3} \leqslant h^{2^{a+j+2}-3}$.

For the claim about Betti numbers, observe first that, by degree considerations, we have that $\beta_{i,\leqslant i+j}(J) = \beta_{i,\leqslant i+j}(J_{\leqslant j})$. Since the Taylor complex $T_{\bullet}$ on a generating set of $J_{\leqslant j}$ is a (possibly non minimal) free resolution $S/J_{\leqslant j}$, it follows that

$$\beta_{i,\leqslant i+j}(J) \leqslant \text{rank}(T_{i+1}) = \binom{\beta_0(J_{\leqslant j})}{i+1} \leqslant \binom{h^{2^{a+j+1}-3}}{i+1}. \qquad \square$$

Ideals of bigheight one are principal, so this case is trivial. The next result gives an upper bound on the graded Betti numbers of *any* radical ideal of bigheight $h \geqslant 2$.

Theorem 4.7. Let $I \subsetneq S$ be a radical ideal of bigheight h. For all $i, j \geqslant 0$ we have that

$$\beta_{i,\leqslant i+j}(I) \leqslant \binom{h^{2^{j+h}-2}}{i+2}.$$

Proof. For any $1 \leqslant \ell \leqslant h$, we let I_ℓ be the intersection of all minimal primes of I of height ℓ, so that $I = \bigcap_{\ell=1}^{h} I_\ell$. If we let f be a generator of I_1, then since I is radical we must have $I = fI'$, with $I' = \bigcap_{\ell=2}^{h} I_\ell$. It follows that $\beta_{i,j}(I) = \beta_{i,j-\deg(f)}(I')$. Since our upper bound is an increasing function in the variable j, it suffices to show the upper bound for the graded Betti numbers of I'. Thus, without loss of generality, we assume that $f = 1$, so that $I = \bigcap_{\ell=2}^{h} I_\ell$.

We proceed by induction on $h \geqslant 2$. If $h = 2$, then Theorem 4.2 yields $\beta_{0,j}(I) \leqslant h^{2^{j+1}-3}$ and Proposition 4.6 gives that

$$\beta_{i,\leqslant i+j}(I) \leqslant \binom{h^{2^{j+2}-3}}{i+1} \leqslant \binom{h^{2^{j+h}-2}-1}{i+1} \cdot \frac{h^{2^{j+h}-2}}{i+2} = \binom{h^{2^{j+h}-2}}{i+2}.$$

Assume $h \geqslant 3$, and let $J = \bigcap_{\ell=2}^{h-1} I_\ell$ and $L = J + I_h$. Consider the graded short exact sequence

$$0 \longrightarrow S/I \longrightarrow S/J \oplus S/I_h \longrightarrow S/L \longrightarrow 0.$$

Applying the functor $- \otimes_S \Bbbk$ and counting dimensions of the graded components of the long exact sequence of $\operatorname{Tor}_\bullet^S(-, \Bbbk)$ we obtain that $\beta_{i,\leqslant i+j}(I) \leqslant \beta_{i,\leqslant i+j}(J) + \beta_{i,\leqslant i+j}(I_h) + \beta_{i+1,\leqslant i+j}(L)$.

By induction, we have that $\beta_{i,\leqslant i+j}(J) \leqslant \binom{(h-1)^{2^{j+h-1}-2}}{i+2}$, and in particular we have that $\beta_{0,j}(J) \leqslant \binom{(h-1)^{2^{j+h-1}-2}}{2} \leqslant h^{2^{j+h}-4}$. By Theorem 4.2 we have that $\beta_{0,j}(I_h) \leqslant h^{2^{j+1}-3}$, and therefore $\beta_{0,j}(L) \leqslant \beta_{0,j}(J) + \beta_{0,j}(I_h) \leqslant h^{2^{j+h}-3}$ for every j. By Proposition 4.6 we get $\beta_{i,\leqslant i+j}(I_h) \leqslant \binom{h^{2^{j+2}-3}}{i+1}$ and $\beta_{i+1,\leqslant i+j}(L) \leqslant \binom{h^{2^{j+h}-3}}{i+2}$. Putting all these estimates together, some easy calculations show that

$$\begin{aligned}\beta_{i,\leqslant i+j}(I) &\leqslant \binom{(h-1)^{2^{j+h-1}-2}}{i+2} + \binom{h^{2^{j+2}-3}}{i+1} + \binom{h^{2^{j+h}-3}}{i+2} \\ &\leqslant \binom{h^{2^{j+h}-3}-1}{i+2} + \binom{h^{2^{j+h}-3}+1}{i+2} \leqslant \binom{h^{2^{j+h}-2}}{i+2}.\end{aligned} \qquad \square$$

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