Koszul Algebras, Castelnuovo-Mumford Regularity, and Generic Initial Ideals

by

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Abstract

KOSZUL ALGEBRAS, CASTELNUOVO-MUMFORD REGULARITY, AND GENERIC INITIAL IDEALS

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The central topics of this dissertation are: Koszul Algebras, bounds for the Castelnuovo Mumford regularity, and methods involving the use of generic changes of coordinates and generic hyperplane restrictions. We give an introduction to Koszul algebras and prove some criteria to show that an algebra is Koszul. We use these methods to show that the Pinched Veronese, i.e. the toric ring defined as $K[X^3,X^2Y,XY^2,Y^3,X^2Z,Y^2Z,XZ^2,YZ^2,Z^3]$, is Koszul.

The middle chapters are devoted to the Castelnuovo-Mumford regularity. We give a collection of techniques and formulas to compute the regularity by using hyperplane sections. For example we obtain some variations of a criterion due to Bayer and Stillman and a formula for the regularity that involves the postulation numbers.

We study the combinatorial properties of a special kind of monomial ideal that we call weakly stable. We employ them to give a uniform bound, depending on the degree of the generators, for the regularity of all the homogeneous ideals in a polynomial ring. We also provide bounds for the regularity of the tensor product and Hom of two modules.

In chapter seven we study some inequalities on the dimension of graded components of Tor's, and in the last chapter we present a modification of Green's Hyperplane Restriction Theorem. By using this restriction theorem we obtain a general strategy to derive variations of the Eakin-Sathaye Theorem on reductions.

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Contents

Introduction

In this dissertation we mainly study the three following topics: Koszul Algebras, bounds for the Castelnuovo Mumford regularity, and methods involving the use of generic changes of coordinates and generic hyperplane restrictions. The general approach used in this thesis that unifies the three subjects is trying to reduce the problems we consider to the study of initial ideals. More generally, the whole thesis is tied together by the constant effort to compare homological invariants of a special fiber of a flat family to the ones of a generic fiber.

The *first Chapter* gives an introduction to the study of Koszul algebra. All the results presented there are quite standard except for an extension of the notion of *Koszul filtration* which is needed for the second chapter.

In the *second Chapter* we prove that a certain toric ring:

$$R = K[X^3, X^2Y, XY^2, Y^3, X^2Z, Y^2Z, XZ^2, YZ^2, Z^3],$$

called the Pinched Veronese, is Koszul. For about ten years this ring has been the most famous example for which the Koszulness was unknown . The problem about the Koszulness of the pinched Veronese was raised by B.Sturmfels in 1993 in a conversation with Irena Peeva. Ever since it has been circulating as a concrete example to test the efficiency of the new theorems and techniques concerning Koszul algebras. As far as we know, the main approach employed to attack this problem was the use of techniques particularly suited for studying semigroup rings and their associated polytopes. Following a different strategy we achieved our goal by using a combination of flat deformations and Koszul filtrations.

The Chapters three, four, five and six are dedicated to the study of the Castelnuovo-Mumford regularity.

In *Chapter three* we give a collection of techniques and formulas to compute the regularity by using hyperplane sections. Some results presented here include a formula for the regularity that involves the postulation numbers, and several variations of a criterion due to Bayer and Stillman.

Chapter four is focused on the study of a special kind of monomial ideal that we call weakly stable. These ideals, which also include Borel-fixed ideals, have several combinatorial properties: for example, their regularity and projective dimension can be described in a combinatorial way and do not depend on the characteristic of the base field.

The importance of weakly stable ideals becomes more clear in *Chapter five*, where we employ them to give a different proof of a result of E. Sbarra and the author. This result gives a uniform bound, depending on the degree of the generators, for the Castelnuovo-Mumford regularity of all the homogeneous ideals in a polynomial ring. In particular, it answers a question raised by D.Bayer and D.Mumford whether the known bound in characteristic zero holds also in positive characteristic.

Chapter six provides bounds for the regularity of the tensor product and Hom of two modules. In particular, we extend some results due to J.Sidman and Herzog-

Conca. Using our result on the regularity of tensor product, D.Giaimo recently proved the Eisenbud-Goto conjecture for the case of connected curves.

In *Chapter seven* we study some inequalities on the dimension of graded components of Tor's. In particular, we answer a related question asked by A.Conca in a recent paper. Then we analyze the method of K.Pardue of polarizing a monomial ideal and then specializing it generically. We give a slightly different proof of his result on the extremality of lex-segment ideals. At the same time we obtain a different proof of a well-known result of Macaulay on the Hilbert function of a standard graded algebra.

Chapter eight is devoted to a modification of Green's Hyperplane Restriction Theorem. Using this result we obtain a general strategy to derive variations of a theorem due to Eakin and Sathaye. We recover and extend some recent results on the Eakin-Sathaye Theorem obtained by O'Carroll.

Chapter 1

Koszul algebras

The aim of this chapter is to recall the notion of a *Koszul algebra* and prove some properties about them. A standard graded K-algebra R is said to be *Koszul* if its residue field K has a linear free resolution as an R-module. This notion was introduced, in a topological setting, in 1970 by Priddy and later studied in both commutative and non-commutative cases by several authors. In particular R.Fröberg and his collaborators have done an intensive study of the Koszulness and it is not surprising that, for a while, Koszul Algebras were referred as Fröberg Algebras. A survey containing many results on Koszul algebras can be found in [Fr].

There are important relations between the Koszulness (and more generally the study of the free resolution of a residue field) and the structure of the non commutative algebra $\operatorname{Ext}^*_R(k,k)$, i.e the Yoneda-Hopf algebra of K. Moreover the study of the Koszulness for toric varieties and the connection with the corresponding combinatorial objects have given the motivation for the development of interesting results (see for example [BGT], [HHR] and [St2]).

A well known fact, that we will prove later, is that a Koszul algebra R has to be quadratic in the sense that there exists a presentation $R \cong k[X_1,\dots,X_n]/I$ where I is generated by homogeneous forms of degree two. The converse does not hold in general and the first counterexample was found by C.Lech.

Among other things, Koszul algebras are also important because they give, as we will see, an interesting class of quadratic algebras with rational Poincaré series. Two main classes of algebras which are Koszul are coordinate rings of complete intersections of quadrics and algebras with relations given by a Gröbner basis of quadrics with respect to some term order. Indeed many classical varieties, like Grassmannians, Schubert varieties, flag varieties, canonical curves are not only Koszul but they are presented by a quadratic Gröbner basis in their natural embedding. For example Kempf [Ke] proved that the coordinate ring of at most $2n$ points of $\mathbb{P}^n$ in general position is Koszul and later A.Conca, N.V.Trung, G.Valla and M.Rossi [CTRV] showed that it also admits a quadratic Gröbner basis. Also the Veronese subalgebra $R^{(d)}$, of a given commutative graded K-algebra R, has a quadratic Gröbner basis for $d \gg 0$ as shown in [ERT]. This result was later generalized to a larger class of algebras not necessarily generated in degree one, see [BGT]. Sturmfels in [St2] has shown that the subring of $S = k[x_1,\dots,x_n]$ generated by the monomials $\{x_1^{i_1}\cdots x_n^{i_n} | i_1+\cdots+i_n = d, 0 \le i_1 \le s_1,\dots, 0 \le i_n \le s_n\}$ has a Gröbner basis, in a certain ordering, which is not only quadratic but also squarefree. Note that when $s_1 = \cdots = s_n = d$ we get $S^{(d)}$. He called these algebras of *Veronese type*. Further generalizations have been done by S.Blum in [Bl].

In the following we denote by $R = S/I$ a standard graded K-algebra where $S = K[X_1,\dots,X_n]$ is a polynomial ring over a field K and I a homogeneous ideal. By possibly considering a different presentation we can assume that I does not contain any linear forms. Let M be a finitely generated graded R-module, and let $\mathbb{F}$ be a free graded resolution of M over R. We have

$$\mathbb{F}: \dots \longrightarrow F_i \xrightarrow{d_{i-1}} F_i \longrightarrow \dots \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

with $F_i = \bigoplus_{j \in \mathbb{Z}} R(-j)^{b_{ij}}$, $b_{ij} = \dim_K \operatorname{Tor}_i^R(M,K)_j$ and $b_i = \dim_K \operatorname{Tor}_i^R(M,K)$.

When it will be clear from the context, we will try to avoid the more precise notation $b_{ij}(M)$ and $b_i(M)$. Note that since R is not regular, the above resolution is in general not finite.

Definition 1.0.1. The algebra R is said to be *Koszul* if the residue field K has a linear free resolution over R, i.e $\operatorname{Tor}_i^R(K,K)_j = 0$ when $i \neq j$.

The next example is a very easy example of an algebra which is Koszul. In this case the structure of R is so simple that it is possible to describe every map of the resolution of K and deduce the Koszulness directly from them.

Example 1.0.2. Let R be algebra $K[X,Y]/(XY)$ and set x and y the class of X and Y in R. It's easy to check that the following complex $\mathbb{F}$ gives a linear minimal free resolution of K over R.

$$\mathbb{F} : \cdots \longrightarrow R(-i)^2 \xrightarrow{d_i} R(-i+1)^2 \longrightarrow \ldots \longrightarrow R(-1)^2 \xrightarrow{\begin{pmatrix} x, y \end{pmatrix}} R$$

where the map between $R(-i)^2$ and $R(-i+1)^2$ is given by

$$d_i = \begin{cases} \begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix} & \text{when } i \text{ is odd} \\ \\ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} & \text{when } i \text{ is even and positive.} \end{cases}$$

Therefore R is Koszul.

As we have mentioned above any Koszul algebra is given by quadratic relations, this fact can be deduced simply by the linearity of the second syzygy of

K, see for example [BH] Theorem 2.3.2. We state the result with a sketch of the proof.

Theorem 1.0.3. *Let $R = S/I$ be a Koszul algebra. Then I is generated by quadrics.*

Proof. Since R is Koszul, $\mathrm{Tor}_i^R(K,K)_j = 0$ whenever $i \neq j$. In particular we have that $\mathrm{Tor}_2^R(K,K)_j = 0$ for $j \neq 2$. It is then enough to show the following claim:

Claim 1. *The ideal I is generated by quadrics if and only if $\mathrm{Tor}_2^R(K,K)_j = 0$ for any $j \neq 2$.*

Proof of the claim Consider the following exact sequence

$$R(-1)^n \xrightarrow[d_1]{(x_1,\dots,x_n)} R \longrightarrow K \longrightarrow 0.$$

We have to show that $\ker d_1$ is generated by linear elements if and only if I is quadratic. Let $a_1 \dots, a_m$ be a minimal system of homogeneous generators for I. Since I does not contains any linear forms, we can write $a_i = \sum a_{ij} X_j$ with a_{ij} belonging to the homogeneous maximal ideal of S. We denote by $\bar{a}_{ij}$ the class of a_{ij} in R. In order to conclude the proof of the claim it is enough to show that the $\sum_j \bar{a}_{ij}\bar{e}_j$, together with the Koszul relations $x_i\bar{e}_j - x_j\bar{e}_i$, form a minimal system of generators for $\ker d_1$. The fact that they belong to the kernel of d_1 is obvious. On the other hand let $(b_1, \dots, b_n)$ be an element of S^n such that its class is annihilated by d_i. We get that $\sum b_j X_j \in I$, and in particular we can write $\sum_j b_j X_j = \sum_i c_i a_i = \sum_i c_i (\sum_j a_{ij} X_j)$ for some $c_i \in S$. Therefore $(\sum_j b_j e_j) - \sum_i c_i (\sum_j a_{ij} e_j)$ is annihilated by the first map of the Koszul complex (i.e substituting e_j for X_j) and, by the exactness of such a complex, it belongs to the submodule generated by the Koszul relations $X_i e_j - X_j e_i$. Reading this fact in the quotient we get that $\sum_j \bar{a}_{ij}\bar{e}_j$ and $x_i\bar{e}_j - x_j\bar{e}_i$ generate $\ker d_1$. We still have to show the minimality.

Assume that $\sum_i \bar{\alpha}_i(\sum_j \bar{a}_{ij}\bar{e}_j) + \sum_{i<j} \bar{\beta}_{ij}(x_ie_j - x_je_i) = 0$, where $\bar{\alpha}_i$ and $\bar{\beta}_{ij}$ are homogeneous element of R. We want to prove that $\bar{\alpha}_i$ and $\bar{\beta}_{ij}$ belong to $(x_1, \dots, x_n)$. Lifting the previous relation to S^n we get:

$$\sum_i \alpha_i(\sum_j a_{ij}e_j) + \sum_{i<j} \beta_{ij}(X_ie_j - X_je_i) \in IS^n. \tag{1.0.1}$$

Applying the first map of the Koszul complex of $X_1, \dots, X_n$ to (1.0.1) we deduce that $\sum_i \alpha_i(\sum_j a_{ij}X_j) = \sum_i \alpha_i a_i \in I$ and by the minimality of the a_i's we obtain that $\alpha_i \in (X_1, \dots, X_n)$ for every i. Now, because I does not contain any linear forms, $\sum_{i<j} \beta_{ij}(X_ie_j - X_je_i)$ is zero in degree 0 or 1 and therefore the β_{ij}'s are in $(X_1, \dots, X_n)$. Thus the claim is proved and so is the theorem. □

Note that, as we said previously, Theorem 1.0.3 gives only a sufficient condition for the Koszulness and, in general, the implication cannot be reversed. The first counterexample is due to C.Lech and consists of an algebra given by five generic quadratic forms in $K[X_1, \dots, X_4]$. Before analyzing Lech's example we have to recall another characterization of Koszul algebras, first observed by Froberg.

In the next we will denote by $H_M(Z) = \sum \dim_K(M_i)Z^i$ the Hilbert Series of a finitely generated graded R-module M and by $P_R(Z) = \sum b_i(K)Z^i$ and $Q_R(T,U) = \sum b_{ij}(K)T^iU^j$ respectively the Poincaré and the bigraded Poincaré series of R.

Theorem 1.0.4. *The algebra R is Koszul if and only if $H_R(Z)P_R(-Z) = 1$.*

Proof. Let $\mathbb{F} = (F_i, d_i)$ be a minimal free resolution of K over R. It is clear that $1 = H_K(Z) = \sum_{i \geq 0} H_{F_i}(Z)(-1)^i$. On the other hand, since $F_i = \bigoplus R(-j)^{b_{ij}}$, we see that $H_{F_i}(Z) = \sum_j b_{ij}H_R(Z)Z^j$. We can write then

$$1 = \sum_{i \geq 0} \sum_j b_{ij}H_R(Z)Z^j(-1)^i = H_R(Z)Q_R(-1, Z).$$

By the unicity of the inverse in $K[\![Z]\!]$ it is now enough to show that R is Koszul if and only if $Q_R(-1,Z)$ agrees with $P_R(-Z)$, i.e.

$$\sum_{i\geq 0}\sum_{j} b_{ij}(-1)^i Z^j = \sum_{i\geq 0} b_i Z^i (-1)^i. \tag{1.0.2}$$

If R is Koszul we have $b_i = b_{ii}$ and $b_{ij} = 0$ when $i \neq j$, therefore the equation (1.0.2) holds true. Assume now that R is not Koszul. Let a be the smallest index for which $b_{aa} \neq b_a$. Note that, because the b_{ij}'s can be computed from a minimal resolution of K, we have $b_{ij} = 0$ for $j < i$. Then subtracting the two terms in (1.0.2) we obtain

$$\sum_{i\geq 0}\sum_{j\geq i} b_{ij}(-1)^i Z^j - \sum_{i\geq 0} b_i Z^i (-1)^i,$$

which is a formal series with $(b_{aa} - b_a)(-1)^a Z^a$ as lowest non zero term. Therefore (1.0.2) does not hold. $\square$

Note that, substituting Z with $-Z$, Theorem 1.0.4 gives that R is Koszul if and only if $H_R(-Z)P_R(Z) = 1$. Since the Hilbert series of R can be expressed in a rational form, Theorem 1.0.4 shows in particular that a Koszul algebra has a rational Poincaré series. In this way we have obtained also a criteria to check if an algebra could be Koszul: indeed the coefficients in the Poincaré series are dimensions of vector spaces and therefore non-negative. We collect those two facts in the next corollary.

Corollary 1.0.5. *Let R be a Koszul algebra. Then the Poincaré series of R is a rational functions and $1/H_R(-Z)$ has non-negative coefficients.*

We are now ready to discuss Lech's example.

Example 1.0.6 (Lech)**.** Let I be the ideal generated by five generic quadrics in $S = K[X_1, \ldots, X_4]$. Then $R = S/I$ is not Koszul.

Proof. In this example the genericity of the quadrics is only needed to force the Hilbert function of R to have the smallest possible coefficients. Consider for example the ideal J defined by $(X_1^2, X_2^2, X_3^2, X_4^2, X_1X_2 - X_3X_4)$. The Hilbert series of S/J is $H_{S/J}(Z) = 1 + 4Z + (\binom{4+2-1}{2} - 5)Z^2$, which has the smallest possible coefficients and, therefore, it agrees with the Hilbert series of R. We have

$$\frac{1}{H_R(-Z)} = \frac{1}{1-4Z+5Z^2} = 1 + 4Z + 11Z^2 + 24Z^3 + 41Z^4 - 29Z^5 + \dots \quad (1.0.3)$$

Since the negative term $-29Z^5$ appears in (1.0.3), by Corollary 1.0.5 a Koszul algebra cannot have $1+4Z+5Z^2$ as Hilbert series and therefore R and S/J are not Koszul. □

Another consequence of Theorem 1.0.4 is that it allows to prove that the coordinate ring of a complete intersection of quadrics is Koszul. The proof relies on the fact that, by the Tate resolution, it's possible to compute the Poincaré series of such a ring.

Theorem 1.0.7 (Tate resolution)**.** *Let $I = (Q_1, \dots, Q_r) \subset S$ be an ideal generated by a regular sequence of quadratic forms. Then $R = S/I$ is Koszul.*

Proof. Using the Tate resolution [Ta], we know that Poincaré series of R is $(1+Z)^n/(1-Z^2)^r$. On the other hand, by an easy induction, the Hilbert series of R is $(1-Z^2)^r/(1-Z)^n$ and therefore $1/H_R(-Z) = P_R(Z)$. By Theorem 1.0.4 we obtain that R is Koszul. □

Theorem 1.0.7 can be deduced easily, also from one of the next stronger results. We omits the proofs because they require some knowledge of spectral sequences.

Theorem 1.0.8 (Backelin-Froberg). *Let f be a homogeneous regular element of R of degree one or two. Then R is Koszul if and only if $R/(f)$ is Koszul.*

The following result appears as Lemma 6.4 in [CHTV].

Theorem 1.0.9. *Let R be a Koszul algebra and J a homogeneous ideal of R having a linear free resolution over R. Then R/J is Koszul.*

Since the polynomial ring S is Koszul (the Koszul complex is a liner resolution of K), an easy induction shows that either Theorem 1.0.8 or Theorem 1.0.9 implies Theorem 1.0.7. We will give, later in the chapter, another proof which involves lifting and initial ideals.

1.1 Koszul filtrations

It is useful, in the study of the Koszulness to give the definition of a Koszul filtration. A possible way of proving that a certain algebra is Koszul is to show that it admits such a filtration, in particular this notion gives an easy proof, as we see later, that an algebra with monomial relation is Koszul. In this section we introduce also an extension of this definition which, for instance, plays an important role (as we see in the next chapter) in proving that the pinched Veronese is Koszul.

We start by recalling the definition of a *Koszul filtration* introduced by A.Conca, N.V.Trung and G.Valla in [CTV], see also [HHR] for related results.

Definition 1.1.1. Let R be a standard graded K-algebra. A family $\mathbf{F}$ of ideals of R is said to be a *Koszul filtration* of R if:

1) Every ideal $I \in \mathbf{F}$ is generated by linear forms.

2) The ideal (0) and the maximal homogeneous ideal $\mathcal{M}$ of R belong to $\mathbf{F}$.

3) For every $I \in \mathbf{F}$ different from (0), there exists $J \in \mathbf{F}$ such that $J \subset I$, I/J is cyclic and $J : I \in \mathbf{F}$.

In [CTV] it is proved that all the ideals belonging to such a filtration have a linear free resolution over R and in particular, since the homogeneous maximal ideal is in $\mathbf{F}$, R will be a Koszul algebra.

As we said before, for many purposes, it's useful to have an extension of this definition to the case of graded modules. In particular we want to substitute $\mathbf{F}$ by a collection of finitely generated graded modules.

Definition 1.1.2. Let R be a standard graded K-algebra. A family $\mathbf{F}$ of finitely generated graded R-modules is said to be a *Koszul filtration* for modules if the following three properties hold:
1) Every module $M \in \mathbf{F}$ is generated by its nonzero component of lowest degree, say s_M.
2) The zero module belongs to $\mathbf{F}$.
3) For every $M \in \mathbf{F}$ different from the zero module there exists $N \subsetneq M$, $N \in \mathbf{F}$ with $N = 0$ or $s_M = s_N$, such that either M/N has a linear free resolution (i.e $\mathrm{Tor}_i^R(M/N,k)_j = 0$ for all $j \neq i + s_M$) or the module of first syzygies $\Omega_1^R(M/N)$ of M/N is generated in degree $s_M + 1$ and $\Omega_1^R(M/N)(1) \in \mathbf{F}$.

The next proposition shows that all the elements in $\mathbf{F}$ have a linear free resolution over R. In particular, because of this fact, the problem of proving that some module M has a linear free resolution over R can be approached by trying to construct a *Koszul filtration* containing M.

Proposition 1.1.3. *Let R be a standard graded K-algebra and* $\mathbf{F}$ *a Koszul filtration as defined in* (1.1.2). *Then every $M \in \mathbf{F}$ has a linear free resolution over R.*

The proof of this result is essentially the same as the one for the case of a Koszul filtration (see [CTV] Prop. 1.2).

Proof. We need to show that for every $M \in \mathbf{F}$ we have $\mathrm{Tor}_i^R(M,K)_j = 0$ for all $j \neq i + s_M$. We argue by induction on the index i. If $i = 0$ the assertion is clearly true; fix an integer $i > 0$. If M is the zero module it has obviously a linear resolution, therefore we can assume that M has a positive minimum number of generators $\mu(M)$. Inducting on $\mu(M)$ we can assume that $\mathrm{Tor}_i^R(N,K)_j = 0$ whenever $j \neq i + s_N$, $N \in \mathbf{F}$ and $\mu(N) < \mu(M)$. From the third property in Definition 1.1.2 we know that M has a submodule $N \subsetneq M$ with $s_M = s_N$ and in particular with $\mu(N) < \mu(M)$. The short exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

gives the exact sequence

$$\mathrm{Tor}_i^R(N,K)_j \longrightarrow \mathrm{Tor}_i^R(M,K)_j \longrightarrow \mathrm{Tor}_i^R(M/N,K)_j. \qquad (1.1.1)$$

From the third property in Definition 1.1.2 either M/N has a linear resolution, so in particular $\mathrm{Tor}_i^R(M/N,K)_j = 0$ for $j \neq i + s_M$, or $\Omega_1^R(M/N)(1) \in \mathbf{F}$ and is generated in degree s_M. The last term in (1.1.1) is $\mathrm{Tor}_{i-1}^R(\Omega_1^R(M/N)(1),K)_{j-1}$. Since the inductive hypothesis on the index of the Tor applies we deduce that $\mathrm{Tor}_i^R(M/N,K)_j = \mathrm{Tor}_{i-1}^R(\Omega_1^R(M/N)(1),K)_{j-1} = 0$ when $j \neq i + s_M$. On the other hand the induction on the minimum number of generators yields $\mathrm{Tor}_i^R(N,K)_j = 0$ for $j \neq s_M + i$. Therefore the middle term in (1.1.1) vanishes when $j \neq s_M + i$. □

Remark 1.1.4. We consider, in the third property in Definition 1.1.2, the fact of having a linear free resolution over R as a possible condition for an element M in $\mathbf{F}$. This is not really essential: indeed if we already know that M has a linear free resolution over R we could add to $\mathbf{F}$ all the modules of syzygies of M filtering them, trivially, with 0. We decide, for the sake of convenience, to try to keep the family $\mathbf{F}$ as small as possible. On the other hand, if we only leave the second part of condition number 3), it reasonable to ask, given a module M with a linear free resolution over R, if there always exists a finite family $\mathbf{F}$ containing M.

Remark 1.1.5. It's easy to see that the Koszul filtration is included in our definition of Koszul filtration for modules. In fact if $J \subset I$ are ideals generated by linear forms (as in Definition 1.1.1) with I/J cyclic, then $J : I \cong \Omega_1^R(I/J)(1)$.

Remark 1.1.6. Our definition of Koszul filtration covers also the definition of *module with linear quotients* recently introduced by Conca and Herzog in [CH] in order to study the linearity of the free resolution of certain modules over a polynomial ring.

The Koszul filtration, already in the original form of [CTV], gives a simple proof the following

Corollary 1.1.7. *Let I be a monomial ideal generated by quadrics. Then $R = S/I$ is Koszul.*

Proof. Define $\mathbf{F}$ to be the set of all ideals in R generated by variables and let $M_1, \dots, M_r$ be a minimal system of monomial generators for I. Note that for any ideal $J \subseteq R$ generated by variables and any $x_i \notin J$, the colon ideal $J : x_i$ is equal to $J + (x_j$ such that X_j divides some $M_l)$. The ideal $J : x_i$ is therefore generated by variables and it belongs to $\mathbf{F}$. The family $\mathbf{F}$ is a Koszul filtration because any ideal in $\mathbf{F}$ can be filtered simply by dropping one variable by its minimal generators. By Proposition 1.1.3 the maximal ideal $(x_1, \dots, x_n) \in \mathbf{F}$ has a linear free resolution and consequently R is Koszul. □

1.2 Quadratic Gröbner bases and weight functions

In this section we prove the well known fact, showed first by Froberg, that an algebra $R = K[X_1 \dots, X_n]/I$ is Koszul if, for a certain term order, it can be generated by a Gröbner basis of quadratic forms. This is the natural generalization of Corollary 1.1.7, indeed the proof, after a flat deformation of the algebra, reduces to the case of an algebra with quadratic monomial relations. More precisely one can say that if, for a certain term order, $\text{in}(I)$ defines a Koszul algebra (that for a monomial ideal simply means being quadratic) so does I.

It is known that the same result is still true considering, instead of a term order, a weight function w given by a vector $(w_1, \dots, w_n)$ of positive integers and replacing $\text{in}(I)$ by the initial ideal $\text{in}_w(I)$ (not necessary monomial) of I with respect to w. We present the standard result of this section under this point of view.

The next Lemma 1.2.1 requires the knowledge of some basic properties about flat families, in particular the ones obtained using weight functions. We refer for notations and generalities concerning flat families to [Ei] Section 15.8.

Given a weight function $w = (w_1, \dots, w_n)$ from $\mathbb{Z}^n$ to $\mathbb{Z}$ we can think about it as a function defined on monomials of S; moreover given $f \in S$ we use $\text{in}_w(f)$ for the sum of all the terms of f that are maximal with respect to w. Given an ideal I we write $\text{in}_w(I)$ for the ideal generated by $\text{in}_w(f)$ for all $f \in I$. Let $A = S[T]$ be a polynomial ring in one variable over S; for any $f \in S$ we define $\tilde{f}$ in the following way: we can write $f = \sum u_i m_i$ where m_i are distinct monomials and $0 \neq u_i \in K$. Let $a = \max w(m_i)$ and set

$$\tilde{f} = T^a f(T^{-w_1} X_1, \dots, T^{-w_n} X_n).$$

Note that $\tilde{f}$ can be written as $\mathrm{in}_\omega(f) + gT$ where g belongs to A. For any ideal I of S define $\tilde{I}$ to be the ideal of A generated by the elements $\tilde{f}$ for all $f \in I$. Setting $\deg X_i = (1, w_i)$ and $\deg T = (0,1)$, the algebra S is bigraded and in particular if $I \subset S$ is an homogeneous ideal then $\tilde{I}$ is bihomogeneous. From the definition it follows that $A/((T)+\tilde{I}) \cong S/\mathrm{in}(I)$ and $A/((T-1)+\tilde{I}) \cong S/I$. Note that T is a non-zerodivisor on $A/\tilde{I}$: let $Tf \in \tilde{I}$ for some $f \in A$. Without loss of generality we can assume f bihomogeneous and moreover specializing at $T = 1$ we have $h = f(X_1, \dots, X_n, 1) \in I$, but f is bihomogeneous therefore it holds that $f = \tilde{h} \in \tilde{I}$. From this fact it follows that also $T-1$ is a non-zerodivisor for $A/\tilde{I}$ since it is sum of two non-zerodivisors of different degrees.

Lemma 1.2.1. *Let $S = K[X_1, \dots, X_n]$. Consider a weight given by a vector of positive integers $w = (w_1, \dots, w_n)$ and homogeneous ideals I, J, H such that $I \subseteq J$ and $I \subseteq H$. Then*

$$\dim_K \mathrm{Tor}_i^{S/I}(S/J, S/H)_j \leq \dim_K \mathrm{Tor}_i^{S/\mathrm{in}_w I}(S/\mathrm{in}_w J, S/\mathrm{in}_w H)_j.$$

Proof. Consider the ideals $\tilde{I}$, $\tilde{J}$, $\tilde{H}$ of $A = S[T]$ defined as above. Define M_i to be $\mathrm{Tor}_i^{A/\tilde{I}}(A/\tilde{J}, A/\tilde{H})$. Note that M_i is bigraded and we can make it a $\mathbb{Z}$-graded module setting $(M_i)_j = \bigoplus_{h \in \mathbb{Z}} (M_i)_{(j,h)}$. Since $(M_i)_j$ is a finitely generated module over $K[T]$, the structure theorem for modules over a PID applies and we obtain the isomorphism $(M_i)_j \cong k[T]^{a_{ij}} \bigoplus B_{ij}$ where B_{ij} is the torsion submodule. Moreover since B_{ij} has to be homogeneous the structure theorem gives $B_{ij} \cong \bigoplus_{h=1}^{b_{ij}} K[T]/(T^{d_h})$. Set $l_1 = T$ and $l_2 = T-1$ and consider the following exact sequence

$$0 \longrightarrow A/\tilde{H} \xrightarrow{\cdot l_r} A/\tilde{H} \longrightarrow A/((l_r)+\tilde{H}) \longrightarrow 0, \qquad (1.2.1)$$

for $r = 1,2$. All the modules appearing in (1.2.1) are over $A/\tilde{I}$ and the multiplication by l_i is a zero degree map with respect to the $\mathbb{Z}-$grading. Tensoring with $A/\tilde{J}$ and passing to the long exact sequence of homologies we have

$$0 \to M_i/l_rM_i \to \operatorname{Tor}_i^{A/\tilde{I}}(A/\tilde{J},A/((l_r)+\tilde{H})) \to \ker(M_{i-1} \xrightarrow{\cdot l_r} M_{i-1}) \to 0. \quad (1.2.2)$$

Since l_r is a regular element for $A/\tilde{I}$ and $A/\tilde{J}$, the middle term is isomorphic to $\operatorname{Tor}_i^{A/((l_r)+\tilde{I})}(A/((l_r)+\tilde{J}),A/((l_r)+\tilde{H}))$ (see [Mat] Lemma 2 page 140) which is $\operatorname{Tor}_i^{S/\operatorname{in}_w I}(S/\operatorname{in}_w J, S/\operatorname{in}_w H))$ when $r = 1$ and is $\operatorname{Tor}_i^{S/I}(S/J,S/H))$ for $r = 2$.

Therefore taking the graded component of degree j in (1.2.2) we obtain:

$$\begin{aligned} \dim_k \operatorname{Tor}_i^{S/\operatorname{in}_w I}(S/\operatorname{in}_w J, S/\operatorname{in}_w H))_j &= a_{ij} + b_{ij} + b_{(i-1)j}, \quad (1.2.3) \\ \dim_k \operatorname{Tor}_i^{S/I}(S/J,S/H))_j &= a_{ij}. \quad (1.2.4) \end{aligned}$$

The Lemma follows by comparing (1.2.3) and (1.2.4). □

Corollary 1.2.2. *Let w be a weight function and I be a homogeneous ideal of $S = K[X_1,\dots,X_n]$ such that $S/\operatorname{in}_w(I)$ is Koszul. Then S/I is Koszul.*

Proof. Since $S/\operatorname{in}_w I$ is Koszul, $\dim_k \operatorname{Tor}_i^{S/\operatorname{in}_w I}(K,K)_j = 0$ for any $i \neq j$. Applying Lemma 1.2.1 with $J = H = (X_1,\dots,X_n)$, we see that $\operatorname{Tor}_i^{S/I}(K,K)_j = 0$ for any $i \neq j$. □

From Corollary 1.2.2 follows:

Theorem 1.2.3 (Froberg)**.** *Let I be an ideal an ideal generated by a Gröbner basis of quadrics with respect to some term order. Then $R = S/I$ is Koszul.*

Remark 1.2.4. Note that, in general, the above implication cannot be reversed. The following example, that can be found in [ERT], has been showed to me by

Conca. Let $S = K[X_1,\dots,X_n]$ and let I be the ideal generated by $(X_1^2 + X_2X_3, X_2^2 + X_1X_3, X_3^2 + X_1X_2)$. By Theorem 1.0.7 the algebra S/I is Koszul since the generators of I are a regular sequence of quadratic forms, on the other hand I does not have a quadratic initial ideal with respect to any term order (even after a change of coordinate).

The obstruction presented by the above example can somehow be overcome if one is allowed to take a lifting of the algebra. We present, in this way, another proof that a regular sequence of quadratic forms defines a Koszul algebra.

Remark 1.2.5. Let $I \subseteq S = K[X_1,\dots,X_n]$ be an ideal generated by a regular sequence, say $Q_1,\dots,Q_r$, of quadratic forms. Set $A = K[X_1,\dots,X_n,Y_1,\dots,Y_r]$ and define $H = (Q_1 + Y_1^2,\dots,Q_r^2 + Y_r^2) \subseteq A$. Note that H is clearly generated by a regular sequence of quadrics and $y_1,\dots,y_r$ is a regular sequence in A/H, because it specializes a complete intersection of r quadrics to a complete intersection of r quadrics. On the other hand any term order for which the Y's are greater than the X's gives $\mathrm{in}_H = (Y_1^2,\dots,Y_r^2)$. The ideal H is therefore generated by a Gröbner basis of quadrics. Theorem 1.0.8 says, in particular, that the Koszulness is preserved taking a quotient by a regular sequence and therefore $A/(H + (Y_1,\dots,Y_r)) = S/I$ is Koszul.

Question 1.2.6. Let $R = K[X_1,\dots,X_n]/I$ be a Koszul algebra. Is it always possible to find a polynomial ring $A = K[X_1,\dots,X_n,Y_1,\dots,Y_s]$ and an ideal $J \subseteq A$ such that J is generated by a Gröbner basis of quadrics and there exists a regular sequence of linear forms $l_1,\dots,l_s$ of A with $A/(J + (l_1,\dots,l_s)) \cong R$?

Chapter 2

The Pinched Veronese is Koszul

An important question, regarding the Koszulness of toric variety, which, as far as we know, is still open is the following: *"Is it true that any quadratic toric varieties with an isolated singularity is Koszul?"* The pinched Veronese, i.e. the K-algebra, where K is a field, defined as $R = K[X^3, X^2Y, XY^2, Y^3, X^2Z, Y^2Z, XZ^2, YZ^2, Z^3]$, has been for a long time the first and the most simple case of the previous question where the answer was unknown. The problem about the Koszulness of the pinched Veronese was raised by B.Sturmfels in the 1993 in a conversation with Irena Peeva, and after that has been circulating as a concrete example to test the efficiency of the new theorems and techniques concerning Koszul algebras. The main goal of the chapeter is to show the following:

Theorem 2.0.7. *The algebra* $R = K[X^3, X^2Y, XY^2, Y^3, X^2Z, Y^2Z, XZ^2, YZ^2, Z^3]$, *where* K *is a field, is Koszul.*

The proof is structured in three different steps. First of all we can consider a presentation for R and write it as $R = S/I$ where I is a homogeneous ideal generated by quadrics and S is a polynomial ring.

The first step consists in taking the initial ideal of I with respect to a carefully chosen weight ω. By Corollary 1.2.2, it's then sufficient to show that $S/\operatorname{in}_\omega(I)$ is Koszul. The use of ω is important because it allows us to study instead of a binomial ideal, an ideal generated by several quadratic monomials and only five quadratic binomials. For this purpose the choice of ω needs to be done very carefully: taking, for example, the trivial weight $\omega = (1,\dots,1)$ we get $\operatorname{in}_\omega(I) = I$ and we do not make any simplification. On the other hand a generic weight will play the role of a term order, bringing in the initial ideal some minimal generator of degree higher than two, and so the ring defined by the initial ideal with respect to a generic weight cannot be Koszul.

The second reduction consists in writing $\operatorname{in}_\omega(I)$ as the sum two ideals: U generated by the monomial part of $\operatorname{in}_\omega(I)$ plus a distinguished binomial of $\operatorname{in}_\omega(I)$ and the ideal $(Q_1,\dots Q_4)$ given by the remaining four binomials of $\operatorname{in}_\omega(I)$. The ideal U is generated by a Gröbner basis of quadrics, so S/U is Koszul. We need the following fact, which is part of Lemma 6.6 of [CHTV]:

Fact 1. *Let T be a Koszul algebra let $Q \subset T$ be a quadratic ideal with a linear free resolution over T. Then T/Q is Koszul.*

Using this fact it is enough to show that the class $(q_1,\dots,q_4)$ of $(Q_1,\dots,Q_4)$ in S/U has a linear free resolution over S/U. Note that among all the possible five binomials, the distinguished one we pick is the only one giving at the same time the Koszulness of S/U and the linearity of the ideal given by the other four. It's maybe possible to show that the whole binomial part has a linear resolution over S modulo the monomial one, but for this purpose the amount of calculations required seems much higher.

The last part of the proof consists in showing the linearity of the free resolution of $(q_1,\dots,q_4)$ over S/U via the construction of a Koszul filtration containing

$(q_1,\ldots,q_4)$.

Theorem 2.0.7. *The algebra* $R = K[X^3, X^2Y, XY^2, Y^3, X^2Z, Y^2Z, XZ^2, YZ^2, Z^3]$ *is Koszul.*

Proof. Since R contains all monomials in X,Y,Z of degree 9 its Hilbert function is $H_R(0) = 1$, $H_R(1) = 9$ and $H_R(n) = \binom{3n+2}{2}$ for $n \geq 2$. The Hilbert polynomial of R is given by $\binom{3n+2}{2}$ and the Krull dimension of R is 3. One computes that the Hilbert series of R is given by:

$$H_R(Z) = \frac{Z^4 - 3Z^3 + 4Z^2 + 6Z + 1}{(1-Z)^3}.$$

Consider a presentation $S/\ker\phi \cong R$ where $S = K[X_1,\ldots,X_9]$ and and ϕ is the homomorphism from S to R defined by sending X_i to the i^{th} monomial of R in $(X^3, X^2Y, XY^2, Y^3, X^2Z, Y^2Z, XZ^2, YZ^2, Z^3)$. Let I be the ideal defined as

$$\begin{aligned} I = \quad & (X_8^2 - X_6X_9, X_6X_8 - X_4X_9, X_5X_8 - X_2X_9, X_7^2 - X_5X_9, \\ & X_6X_7 - X_3X_9, X_5X_7 - X_1X_9, X_4X_7 - X_3X_8, X_3X_7 - X_2X_8, \\ & X_2X_7 - X_1X_8, X_6^2 - X_4X_8, X_5X_6 - X_2X_8, X_5^2 - X_1X_7, X_4X_5 - X_2X_6, \\ & X_3X_5 - X_1X_6, X_3^2 - X_2X_4, X_2X_3 - X_1X_4, X_2^2 - X_1X_3). \end{aligned}$$

It is immediate to see that $I \subseteq \ker\phi$. On the other hand also the opposite inclusion holds, in fact it is sufficient to check that R and S/I have the same Hilbert function. We will prove this below.

Consider the weight function ω from $\mathbb{Z}^9$ to $\mathbb{Z}$ given by $(3,3,1,3,3,3,2,3,3)$ and take its natural extension to the monomials of S. Let J be the ideal generated by the initial forms with respect to ω of the generators of I given previously. We have

$$
\begin{aligned}
J &= (X_8^2 - X_6X_9, X_6X_8 - X_4X_9, X_5X_8 - X_2X_9, X_5X_9, X_6X_7,\\
&\quad X_1X_9, X_4X_7, X_2X_8, X_1X_8, X_6^2 - X_4X_8, X_5X_6, X_5^2,\\
&\quad X_4X_5 - X_2X_6, X_1X_6, X_2X_4, X_1X_4, X_2^2).
\end{aligned}
$$

We claim that $J = \mathrm{in}_\omega I$: one inclusion is clear and to prove the other is enough to show, as stated previously, that R/J and R/I have the same Hilbert function. Consider the degrevlex order σ on the monomials of S. Note first that $X_2X_9^2$ and $X_2X_6X_9$ belong to J since $X_2X_9^2 = (X_5X_9)X_8 - (X_5X_8 - X_2X_9)X_9$ and $X_2X_6X_9 = (X_2X_8)X_8 - (X_8^2 - X_6X_9)X_2$, therefore the following ideal

$$
\begin{aligned}
H &= (X_5X_9, X_1X_9, X_8^2, X_6X_8, X_5X_8, X_2X_8, X_1X_8, X_6X_7, X_4X_7, X_6^2,\\
&\quad X_5X_6, X_1X_6, X_5^2, X_4X_5, X_2X_4, X_1X_4, X_2^2, X_2X_9^2, X_2X_6X_9)
\end{aligned}
$$

is contained in $\mathrm{in}_\sigma J$. The Hilbert series of S/H is easy to compute and it is $H_{S/H}(Z) = \frac{Z^4-3Z^3+4Z^2+6Z+1}{(1-Z)^3}$. Coefficient-wise we have:

$$
H_{S/\ker\phi}(Z) \le H_{S/I}(Z) = H_{S/\mathrm{in}_\omega I}(Z) \le H_{S/J}(Z) = H_{S/\mathrm{in}_\sigma J}(Z) \le H_{S/H}(Z).
$$

The first and the last term agree, thus all the previous inequalities are in fact equalities, and in particular it follows that $\ker\phi = I$, $\mathrm{in}_\omega I = J$ and $\mathrm{in}_\sigma J = H$.

Applying Corollary 1.2.2 to S/I and ω, in order to finish the proof of the theorem it's enough to show the following

Claim 2. *The K-algebra S/J is Koszul.*

Proof of the claim. We can write J as a sum of two ideals: one generated by all the quadratic monomials of J together with the quadratic binomial $X_6X_8 - X_4X_9$, namely

$$\begin{aligned} U &= (X_5X_9, X_1X_9, X_2X_8, X_1X_8, X_6X_7, X_4X_7, X_5X_6, X_1X_6, X_5^2, X_2X_4, \\ &\quad X_1X_4, X_2^2, X_6X_8 - X_4X_9), \end{aligned}$$

and the other one generated by the remaining binomials $Q_1 = X_6^2 - X_4X_8$, $Q_2 = X_4X_5 - X_2X_6$, $Q_3 = X_8^2 - X_6X_9$ and $Q_4 = X_5X_8 - X_2X_9$. Note first that U is generated by a Gröbner basis of quadrics with respect to the degrevlex order σ, in fact all the S-pairs we need to check are:

$$\begin{aligned} (X_6X_8 - X_4X_9)X_2 - (X_2X_8)X_6 &= -(X_2X_4)X_9, \\ (X_6X_8 - X_4X_9)X_1 - (X_1X_8)X_6 &= -(X_1X_9)X_4, \\ (X_6X_8 - X_4X_9)X_7 - (X_6X_7)X_8 &= -(X_4X_7)X_9, \qquad (2.0.1) \\ (X_6X_8 - X_4X_9)X_5 - (X_5X_6)X_8 &= -(X_5X_9)X_4, \\ (X_6X_8 - X_4X_9)X_1 - (X_1X_8)X_6 &= -(X_1X_9)X_4. \end{aligned}$$

One can observe that for any ideal $L = (X_{i_1}, \dots, X_{i_r})$ generated by variables, the ideal $U + L$ is again generated by a Gröbner basis of at most quadrics. Indeed there are no more S-pairs to check than the ones in (2.0.1). Moreover if L is chosen in a such a way that $X_6X_8 - X_4X_9 \in L$ or $X_6X_8 \notin L$, we obtain that $\text{in}_\sigma(U) + L = \text{in}_\sigma(U + L)$, in fact the only case in which this very last equality doesn't hold is when X_4X_9 appears in the sum without being in L. By Theorem 2.2 of [BHV] if $\text{in}_\sigma(U) + L = \text{in}_\sigma(U + L)$ then not only S/U is Koszul but also the ideal $(L+U)/U$ has a linear free resolution over S/U.

Now set $S/U = T$. In the following we will denote by x_i the class of X_i and by q_j the class of Q_j in T. Since T is Koszul we can use Fact 1 to conclude the proof of Claim 2 if one shows, and we do, that $(q_1,\dots,q_4)$ has a linear free resolution over T. We prove this by constructing a *Koszul filtration* $\mathbf{F}$ over T containing $(q_1,\dots,q_4)(1)$ because this implies, by Proposition 1.1.3, that $(q_1,\dots,q_4)(1)$ has a linear free resolution over T and so $(q_1,\dots,q_4)$ does.

It will be useful to add to $\mathbf{F}$ a set of ideals $\mathbf{G}$ for which we already know they have a linear free resolution over T. Setting

$$\begin{aligned}\mathbf{G} \;=\; & \{\text{Ideals } (x_{i_1},\dots,x_{i_r}) \text{ of } T \text{ such that } X_6X_8 - X_4X_9 \in (X_{i_1},\dots,X_{i_r}) \\ & \text{or } X_6X_8 \notin (X_{i_1},\dots,X_{i_r})\},\end{aligned}$$

from what we have seen above any ideal in $\mathbf{G}$ has a linear resolution over T.

We define $\mathbf{F}$ to be

$$\begin{aligned}\mathbf{F} = \mathbf{G} \cup \{&(q_1,q_2)(1),(q_1,\dots,q_4)(1),M_1^{1,\dots,8},M_1^{1,2,4,6,7,8},\\ &M_2^{1,\dots,8},M_2^{1,2,4,5,7,8},M_3^{1,\dots,10},M_3^{1,2,3,5,6,7,8,9,10}\}\cup\{0\}.\end{aligned}$$

The modules $M_1^{1,\dots,8}$, $M_1^{1,2,4,6,7,8}$, $M_2^{1,\dots,8}$,$M_2^{1,2,4,5,7,8}$,$M_3^{1,\dots,10}$,$M_3^{1,2,3,5,6,7,8,9,10}$ are constructed as follows. We consider T-homomorphisms defined by matrices:

$$M_1 = \begin{pmatrix} x_7 & 0 & x_5 & x_1 & x_2 & 0 & 0 & 0 \\ 0 & x_7 & x_8 & 0 & x_6 & x_5 & x_2 & x_1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} x_7 & 0 & -x_8 & 0 & x_2 & -x_5 & x_1 & 0 \\ 0 & x_7 & x_6 & x_5 & 0 & x_2 & 0 & x_1 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} x_6 & 0 & x_5 & -x_8 & x_4 & 0 & x_2 & 0 & x_1 & 0 \\ 0 & x_6 & 0 & x_5 & 0 & x_4 & 0 & x_2 & 0 & x_1 \end{pmatrix}$$

$$M_1 : T(-1)^8 \to T^2, \quad M_2 : T(-1)^8 \to T^2, \quad M_3 : T(-1)^{10} \to T^2.$$

We use now an upper index notation on the matrices to indicate the module generated by the images of the elements of the standard basis corresponding to those indeces: for instance $M_1^{1,4}$ is the module generated by the images under M_1 of $(1,0,\dots,0)$ and $(0,0,0,1,0,\dots,0)$.

We prove that $\mathbf{F}$ is a *Koszul filtration* for T. For what concerns the elements in $\mathbf{G}$ there is nothing to check since $0 \in \mathbf{F}$ and they have a linear free resolution over T. For all the other modules $M \in \mathbf{F}$ we exhibit a submodule $N \in \mathbf{F}$, $N \subsetneq M$, such that M/N has a linear free resolution or $\Omega_1(M/N)(1)$ belongs to $\mathbf{F}$. We have the following isomorphisms

$$\Omega_1((q_1,q_2)(1))(1) \cong M_1^{1,\dots,8} \in \mathbf{F} \tag{2.0.2}$$

$$\Omega_1((q_1,\dots,q_4)/(q_1,q_2)(1))(1) \cong M_3^{1,\dots,10} \in \mathbf{F} \tag{2.0.3}$$

$$\Omega_1(M_1^{1,\dots,8}/M_1^{1,2,4,6,7,8})(1) \cong M_2^{1,\dots,8} \in \mathbf{F} \tag{2.0.4}$$

$$M_1^{1,2,4,6,7,8}/M_1^{1,4} \cong (x_7,x_5,x_2,x_1) \in \mathbf{G} \subseteq \mathbf{F} \tag{2.0.5}$$

$$\Omega_1(M_2^{1,\dots,8}/M_2^{1,2,4,5,7,8})(1) \cong M_1^{1,\dots,8} \in \mathbf{F} \tag{2.0.6}$$

$$M_2^{1,2,4,5,7,8}/M_2^{1,5,7} \cong (x_7,x_5,x_1) \in \mathbf{G} \subseteq \mathbf{F} \tag{2.0.7}$$

$$\Omega_1(M_3^{1,\dots,10}/M_3^{1,2,3,5,6,7,8,9,10})(1) \cong (x_6,x_5,x_4,x_2,x_1) \in \mathbf{G} \subseteq \mathbf{F} \tag{2.0.8}$$

$$M_3^{1,2,3,5,6,7,8,9,10}/M_3^{1,3,5,7,9} \cong (x_6,x_4,x_2,x_1) \in \mathbf{G} \subseteq \mathbf{F} \tag{2.0.9}$$

where (2.0.2), (2.0.3), (2.0.4) and (2.0.6) have been checked with the help of the computer algebra system MACAULAY2 [M2] over the field of rational numbers. In particular by flat extension these isomorphisms work over any field of characteristic zero. On the other hand we performed by hand exactly the same Gröbner basis based computation, suggested by the calculations over $\mathbb{Q}$. Since integer coefficients different from 1 or -1 never appear, those calculations are enough to prove the previous isomorphisms also over any field of positive characteristic.

In (2.0.5) and (2.0.7) the modules $M_1^{1,4}$ and $M_2^{1,5,7}$ are clearly isomorphic to $(x_7,x_4) \in \mathbf{G} \subseteq \mathbf{F}$ and to $(x_7,x_2,x_1) \in \mathbf{G} \subseteq \mathbf{F}$ respectively. Similarly in (2.0.9) the module $M_3^{1,3,5,7,9}$ is isomorphic to (x_6,x_5,x_4,x_2,x_1) which belongs to $\mathbf{G} \subseteq \mathbf{F}$. This show that $\mathbf{F}$ is a *Koszul filtration* and, as we said before, by Proposition 1.1.3 the ideal $(q_1 \ldots, q_4)(1)$ has a linear free resolution over T. Thus the claim is proved and so is the theorem. $\square$

Chapter 3

Castelnuovo-Mumford regularity and Hyperplane sections

This chapter gives an introduction to the methods of computing the Castelnuovo-Mumford regularity using hyperplane sections. First, we treat the definitions and some basic properties of the regularity. Second, we explore some equivalent definitions of regularity obtained by the use of generic hyperplane sections. Our focus is on a well-known criterion of Bayer and Stillman (see [BS]) for detecting regularity: we will show how to use a single approach to derive this one and other similar criteria. More precisely, we deduce from a formula of Serre that the Castelnuovo-Mumford regularity can be described in terms of the postulation numbers of filter regular hyperplane restrictions, where the postulation number $\alpha(M)$ of a module M is defined as the largest nonnegative integer for which the Hilbert function of M is not equal to the corresponding Hilbert polynomial.

Finally, we draw a parallel comparison between Bayer-Stillman and our criterion. In particular, we obtain, for both of them, a result that is very closely related to the *Crystallization Principle* for generic initial ideals.

3.1 Castelnuovo-Mumford regularity

We recall the definition of the Castelnuovo-Mumford regularity, and we refer the reader to [EG], [Ei] and [BS] for further details on the subject.

Definition 3.1.1. Let M be a finitely generated graded R-module and let $\beta_{ij}(M)$ denote the graded Betti numbers of M (i.e. the numbers $\dim_K \operatorname{Tor}_i(M,K)_j$). The *Castelnuovo-Mumford regularity* $\operatorname{reg}(M)$ of M is

$$\max_{i,j}\{j-i|\beta_{ij}(M)\neq 0\}.$$

Remember also this equivalent definition of regularity in terms of the local cohomology modules of M, which we shall use later. Since the graded local cohomology modules $H^i_{\mathbf{m}}(M)$ with support in the maximal graded ideal $\mathbf{m}$ of R are Artinian, one defines $\operatorname{Max}(H^i_{\mathbf{m}}(M))$ as the maximum integer k such that $H^i_{\mathbf{m}}(M)_k \neq 0$. Then

$$\operatorname{reg}(M) = \max_{i}\{\operatorname{Max}(H^i_{\mathbf{m}}(M))+i\}.$$

Finally, a finitely generated R-module M is said to be *m-regular* for some integer m if and only if $\operatorname{reg}(M)\leq m$.

Example 3.1.2. Let $R = K[X_1,\dots,X_n]$ be a polynomial ring and $I = (f_1,\dots,f_r)$ a homogeneous ideal generated by a regular sequence of forms of degree $d_1,\dots,d_r$. Looking at the Koszul complex given by $f_1,\dots,f_r$ we note that the maximum of $\{j-i|\beta_{ij}(R/I)\neq 0\}$ is obtained at $\beta_{ra}(R/I)$ where $a=\sum_1^r d_i$. Therefore $\operatorname{reg}(R/I)$ is $(\sum_1^r d_i)-r=\sum_1^r(d_i-1)$.

3.1.1 Partial Regularities and short exact sequences

It is useful to recall the behavior of the regularity with respect to short exact sequences. In order to get some more precise statements, that we will need in the next sections, we want to introduce some partial Castelnuovo-Mumford regularity.

We define partial Castelnuovo-Mumford regularities for M with respect to a set of indices $\mathcal{X} \subseteq \{0,\dots,n\}$ as following:

Definition 3.1.3. Given a set of indices $\mathcal{X} \subseteq \{0,\dots,n\}$ and a finitely generated graded R-module M we set $\operatorname{reg}^{\mathcal{X}}(M)$ to be:

$$\operatorname{reg}^{\mathcal{X}}(M) = \max_{i \in \mathcal{X}} \{\operatorname{Max}(H^i_{\mathbf{m}}(M)) + i\}.$$

We say that M is *m-regular with respect to* $\mathcal{X}$ ($i.e$ m-reg$^{\mathcal{X}}$) if $\operatorname{reg}^{\mathcal{X}}(M) \le m$. Similarly we set $\operatorname{reg}_{\mathcal{X}}(M)$ to be:

$$\operatorname{reg}_{\mathcal{X}}(M) = \max_{i \in \mathcal{X}} \{\operatorname{Max}(\operatorname{Tor}_i(M,K)) - i\},$$

and we say that M is m-reg$_{\mathcal{X}}$ if $\operatorname{reg}_{\mathcal{X}}(M) \le m$.

Remark 3.1.4. Note that when $\mathcal{X} = \{0,\dots,n\}$ the m-reg$^{\mathcal{X}}$ agrees with m-regularity in the sense of Castelnuovo-Mumford. We notice that, from the Grothendieck vanishing theorem, all the local cohomology modules are zero for indexes bigger than n. On the other hand since the projective dimension of M is always bounded by n also the modules $\operatorname{Tor}_i(M,K)$ are zero for indexes bigger than n, therefore it makes sense to use the following notation: given $a \in \mathbb{Z}$ we set $\mathcal{X} + a$ to be $\{i+a | i \in \mathcal{X}\} \bigcap \{0,\dots,n\}$.

The next lemma describes the behavior of the regularity with respect to $\mathcal{X}$ for exact sequences.

Lemma 3.1.5. *Given an exact sequence of finitely generated graded R-module,*

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0,$$

we have:

(1) *If M and P are* m-reg^X *so is N.*

(2) *If N is* m-reg^X *and P is* $(m-1)$-reg^{X-1} *then M is* m-reg^X.

(3) *If M is* $(m+1)$-reg^{X+1} *and N is* m-reg^X *then P is* m-reg^X.

Similarly:

(a) If M and P are m-reg$_X$ *so is N.*

(b) If N is m-reg$_X$ *and P is* $(m+1)$-reg$_{X+1}$ *then M is* m-reg$_X$.

(c) If M is $(m-1)$-reg$_{X-1}$ *and N is* m-reg$_X$ *then P is* m-reg$_X$.

Proof. The proof of the first three facts follows from the long exact sequence of local cohomology modules. The remaining three statement can be proved by looking at the long exact sequence of Tor's. □

3.1.2 Regularity of a filter regular hyperplane section

Using the definition of Castelnuovo-Mumford regularity that involves the local cohomology modules, it is easy to see that the regularity behaves quite well under certain hyperplane section. These sections, called filter regular, are the ones that avoid all the associated primes different from the homogenous maximal ideal. More precisely:

Definition 3.1.6. A homogeneous element $l \in R$ of degree D is *filter regular* on a graded R-module M if the multiplication map $l : M_{i-D} \to M_i$ is injective for all $i \gg 0$. A sequence $l_1, \dots, l_r$ of homogeneous elements of R is called a *filter regular sequence* on M if l_i is filter regular on $M/(l_1, \dots, l_i - 1)M$ for $i = 1, \dots, m$.

Remark 3.1.7. Since $H^0_{\mathbf{m}}(M) = \{u \in M \mid \mathbf{m}^k u = 0 \text{ for some } k\}$, then l is filter regular on M if an only if l is a non-zerodivisor on $M/H^0_{\mathbf{m}}(M)$.

Remark 3.1.8. The regularity of a module does not change by extending the field K, therefore we can assume K to be infinite. This will ensure the existence of filter regular elements (for example any generic element is filter regular).

Proposition 3.1.9. *Let M be a finitely generated graded R-module and $l \in R$ a filter regular element on M of degree D. Then for any set of indices $\mathcal{X} \subseteq \{0,\ldots,n\}$ we have:*

(1) $\operatorname{reg}^{\mathcal{X}+1}(M) \leq \operatorname{reg}^{\mathcal{X}\cup(\mathcal{X}+1)}(M/lM) - D + 1$

(2) $\operatorname{reg}^{\mathcal{X}}(M/lM) - D + 1 \leq \operatorname{reg}^{\mathcal{X}\cup(\mathcal{X}+1)}(M)$.

Proof. Consider the short exact sequence

$$0 \longrightarrow (M/0:_M l)(-d) \xrightarrow{\cdot l} M \longrightarrow M/lM \longrightarrow 0.$$

Note that, since l is filter regular on M, $H^i_{\mathbf{m}}((M/0:_M l)(-D)) \cong H^i_{\mathbf{m}}(M)(-D)$ for all $i > 0$. Looking at the long exact sequence of local cohomology modules, we have

$$\begin{array}{l} \ldots \longrightarrow H^i_{\mathbf{m}}(M) \longrightarrow H^i_{\mathbf{m}}(M/lM) \longrightarrow H^{i+1}_{\mathbf{m}}(M)(-D) \longrightarrow \\ \longrightarrow H^{i+1}_{\mathbf{m}}(M) \longrightarrow H^{i+1}_{\mathbf{m}}(M/lM) \longrightarrow \ldots \end{array}$$

for all $i \geq 0$.

Let $j > \operatorname{reg}^{\mathcal{X}\cup(\mathcal{X}+1)}(M/lM) - D + 1$ and $i \in \mathcal{X}$. Consider the exact sequence of K-vector spaces given by the graded pieces of degree $j - i + D - 1$ of the previous sequence. Because of the choice of j, we have

$$H^i_{\mathbf{m}}(M/lM)_{j-i+D-1} = H^{i+1}_{\mathbf{m}}(M/lM)_{j-i+D-1} = 0.$$

Therefore, $H^{i+1}_{\mathbf{m}}(M)(-D)_{j-i+D-1} \cong H^{i+1}_{\mathbf{m}}(M)_{j-i+D-1}$, that is $H^{i+1}_{\mathbf{m}}(M)_{j-i-1} \cong H^{i+1}_{\mathbf{m}}(M)_{j-i+D-1}$. An induction shows that $H^{i+1}_{\mathbf{m}}(M)_{j-i-1} \cong H^{i+1}_{\mathbf{m}}(M)_{j-i+sD-1}$ for any $s > 0$. Since $H^{i+1}_{\mathbf{m}}(M)$ is Artinian, we obtain that $H^{i+1}_{\mathbf{m}}(M)_{j-i-1} = 0$ for all $i \in \mathcal{X}$, which implies part (1).

We prove now part (2). Take $j > \operatorname{reg}^{\mathcal{X}\cup(\mathcal{X}+1)}(M) + D - 1$ and $i \in \mathcal{X}$. From the choice of j, we have $H^{i}_{\mathbf{m}}(M)_{j-i} = H^{i+1}_{\mathbf{m}}(M)(-D)_{j-i} = 0$ for any $i \in \mathcal{X}$. In particular looking at the $(j-i)^{\text{th}}$ graded component of the long exact sequence of local cohomology modules we get $H^{i}_{\mathbf{m}}(M/lM)_{j-i} = 0$ for all $i \in \mathcal{X}$, which implies part (2). □

Proposition 3.1.9 has the following corollary:

Corollary 3.1.10. *Given a finitely generated graded R-module M and a filter regular element l of degree D we have:*

$$\operatorname{reg}(M/H^0_{\mathbf{m}}(M)) \leq \operatorname{reg}(M/lM) - D + 1.$$

Proof. Set $\mathcal{X} = \{0, \dots, n\}$ and note that $\operatorname{reg}(M/H^0_{\mathbf{m}}(M)) = \operatorname{reg}^{\mathcal{X}+1}(M)$. The conclusion follows from Proposition 3.1.9 (1). □

3.2 Equivalent definitions of regularity using hyperplane sections

As we said in the introduction of this chapter, our main goal is to obtain results relating regularity and invariants of hyperplane sections. The first example of such a result is another corollary of Proposition 3.1.9.

Corollary 3.2.1 ([CH] Proposition 1.2)**.** *Given a finitely generated graded R-module M and a filter regular element l of degree D we have:*

$$\operatorname{reg}(M) = \max\{\operatorname{Max} H^0_{\mathbf{m}}(M), \operatorname{reg}(M/lM) - D + 1\}. \tag{3.2.1}$$

Proof. Let $\mathcal{X} = \{0,\dots,n\}$, and note that $\mathrm{reg}(M) = \max\{\mathrm{reg}^{\{0\}}(M), \mathrm{reg}^{\mathcal{X}+1}(M)\}$. Clearly $\mathrm{reg}^{\{0\}}(M) = \mathrm{Max}\, H^0_{\mathbf{m}}(M)$.
From Proposition 3.1.9 (1) we have $\mathrm{reg}^{\mathcal{X}+1}(M) \leq \mathrm{reg}^{\mathcal{X}\cup(\mathcal{X}+1)}(M/lM) - D + 1 = \mathrm{reg}(M/lM) - D + 1$. Thus we get $\mathrm{reg}(M) \leq \max\{\mathrm{Max}\, H^0_{\mathbf{m}}(M), \mathrm{reg}(M/lM) - D + 1\}$. On the other hand, $\mathrm{Max}\, H^0_{\mathbf{m}}(M) \leq \mathrm{reg}(M)$ and, by Proposition 3.1.9 (1), we have $\mathrm{reg}^{\mathcal{X}}(M/lM) - D + 1 \leq \mathrm{reg}^{\mathcal{X}\cup(\mathcal{X}+1)}(M) = \mathrm{reg}(M)$. □

Note that for a finitely generated graded module N of dimension zero $H^0_{\mathbf{m}}(N) = N$, therefore $\mathrm{reg}(N) = \mathrm{Max}\, H^0_{\mathbf{m}}(N)$. An easy induction on the formula 3.2.1 shows the known fact:

Theorem 3.2.2 ([CH] Proposition 1.2)**.** *Let M be a finitely generated graded R-module of dimension d. Then* $\mathrm{reg}(M) = \max_{i\in\{0,\dots,d\}}\{\mathrm{Max}\, H^0_{\mathbf{m}}(M/(l_1,\dots,l_i)M) - \sum_{j=1}^{i}(D_j - 1))\}$ *where $l_1,\dots,l_d$ is a filter regular sequence of degrees $D_1,\dots,D_d$.*

Theorem 3.2.2 can be found in [Gr1] (see Theorem 2.30 (5),(6)) under the more restricted assumptions that the field K has characteristic zero and the l_i's are generic linear forms.

3.2.1 Regularity and Postulation Numbers

We prove how the Castelnuovo-Mumford regularity of M, with $\dim M = d$, can be obtained as the maximum of all the postulations numbers of d filter regular hyperplane sections. More precisely we want to obtain an analogue of Theorem 3.2.2 where the function $\mathrm{Max}\, H^0_{\mathbf{m}}(N)$ is replaced by the postulation number $\alpha(N)$.

Below we will denote by $H_M(i)$ the value at i of the Hilbert function of M (i.e $H_M(i) = \dim_K M_i$), and with $P_M(i)$ the corresponding Hilbert polynomial. It is well-known that $P_M(i)$ agrees with $H_M(i)$ for $i \gg 0$. We recall also that, by a theorem of Hilbert, the Hilbert series (i.e. the formal series defined as $\sum_{i\in\mathbb{Z}} H_M(i)Z^i$)

has a rational expression $\frac{h(Z)}{(1-Z)^d}$ where $h(Z) \in \mathbb{Z}[Z, 1/Z]$. When a graded R-module M has dimension 0, we will denote by $\max M$ the degree of its highest nonzero graded component.

Definition 3.2.3. Let M be a finitely generated graded R-module with Hilbert series $\frac{h(Z)}{(1-Z)^d}$. Let $h(Z) = \sum_{i=a}^{b} c_i Z^i$ with $c_b \neq 0$. We set the *postulation number* of M to be $\alpha(M) = b - d$.

Remark 3.2.4. It is a well-known fact that the postulation number of M is equal to the highest degree i for which the Hilbert function differs from the Hilbert polynomial (i.e $H_M(i) - P_M(i) \neq 0$). For a proof see for example Proposition 4.1.12 in [BH]. The following formula of Serre (see [BH] Theorem 4.4.3 for a proof)

$$H_M(i) - P_M(i) = \sum_{j=0}^{d} (-1)^j \dim_K H^j_{\mathbf{m}}(M)_i \text{ for all } i \in \mathbb{Z}, \tag{3.2.2}$$

shows how the postulation number of M can be defined in terms of the local cohomology modules $H^i_{\mathbf{m}}(M)$.

Theorem 3.2.5. *Let M be a finitely generated graded R-module with* $\dim(M) = d$. *Then*

$$\operatorname{reg}(M) = \max_{i \in \{0,\dots,d\}} \{\alpha(M/(l_1,\dots,l_i)M) - \sum_{j=1}^{i} (D_j - 1))\}$$

where $l_1,\dots,l_d$ is a filter regular sequence of degrees $D_1,\dots,D_d$.

Proof. By definition, given any finitely generated graded R-module N and any $i > \operatorname{reg}(N)$, we have $H^j_{\mathbf{m}}(N)_{i-j} = 0$. In particular $H^j_{\mathbf{m}}(N)_i = 0$, hence from (3.2.2) it is clear that $\operatorname{reg}(N) \geq \alpha(N)$ for every N.

By Corollary 3.2.1, $\operatorname{reg}(M) \geq \operatorname{reg}(M/lM) - \deg l + 1$ for any filter regular element l, so we have:

$$\operatorname{reg}(M) \geq \max_{i \in \{0,\dots,d\}} \{\alpha(M/(l_1,\dots,l_i)M) - \sum_{j=1}^{i} (D_j - 1))\}.$$

We need to prove the reverse inequality. We do an induction on the dimension of M. If $\dim M = 0$, then $\operatorname{reg}(M) = \operatorname{Max} H^0_{\mathbf{m}}(M)$ which equals to $\alpha(M)$, by (3.2.2). Assume $d > 0$. By induction hypothesis we get:

$$\operatorname{reg}(M/l_1M) = \max_{i\in\{1,\dots,d\}}\{\alpha(M/(l_1,l_2,\dots,l_i)M) - \sum_{j=2}^{i}(D_j-1))\}.$$

Consequently, setting $a = \max_{i\in\{0,\dots,d\}}\{\alpha(M/(l_1,\dots,l_i)M) - \sum_{j=1}^{i}(D_j-1))\}$ we have:

$$\operatorname{reg}(M/l_1M) - D_1 + 1 \leq a.$$

Because of Corollary 3.2.1 we still need to prove that $\operatorname{Max} H^0_{\mathbf{m}}(M) \leq a$. By Corollary 3.1.10, since $H^j_{\mathbf{m}}(M) \cong H^j_{\mathbf{m}}(M/H^0_{\mathbf{m}}(M))$ for all $j > 0$, we have $H^j_{\mathbf{m}}(M)_{>a-j} = 0$ for all $j > 0$. In particular, for any $b > a$, $H^j_{\mathbf{m}}(M)_b = 0$ for all $j > 0$. Hence, by (3.2.2) we deduce $H_M(b) - P_M(b) = \dim_K H^0_{\mathbf{m}}(M)_b$. But $a \geq \alpha(M)$ so $H_M(b) - P_M(b) = 0$ for all $b > a \geq \alpha(M)$. Therefore, $\max H^0_{\mathbf{m}}(M) \leq a$. □

An interesting corollary of the Theorem 3.2.5 is the following:

Corollary 3.2.6. *Let M be a finitely generated graded R-module. Let* $\dim M = d$, *and let $l_1,\dots,l_d$ be a filter regular sequence on M of degree $D_1,\dots,D_d$. Then the number*

$$\max_{i\in\{0,\dots,d\}}\{\alpha(M/(l_1,\dots,l_i)M) - \sum_{j=1}^{i}(D_j-1))\}$$

is independent of the choice of the filter regular sequence and of its degrees.

3.2.2 Regularity and hyperplane sections: a general approach

We want to study the general properties of the functions $\operatorname{Max}(H^0_{\mathbf{m}}(\_))$ and $\alpha(\_)$ on which Theorem 3.2.2, Theorem 3.2.5 and Corollary 3.2.6 rely.

From Remark 3.2.4, the number $\alpha(M)$ is the highest integer i for which the function ϕ defined as

$$\phi(i, M_0, M_1, M_2, \ldots, M_n) := \sum_{j=0}^{n} (-1)^j \dim_K(M_j)_i$$

is not zero at $(i, H^0_{\mathbf{m}}(M), H^1_{\mathbf{m}}(M), \ldots, H^n_{\mathbf{m}}(M))$.

On the other hand $\mathrm{Max}(H^0_{\mathbf{m}}(M))$ is trivially the highest integer i for which the function θ defined as

$$\theta(i, M_0, M_1, M_2, \ldots, M_n) := dim_K(M_0)_i$$

is not zero at $(i, H^0_{\mathbf{m}}(M), H^1_{\mathbf{m}}(M), \ldots, H^n_{\mathbf{m}}(M))$.

It is possible to replace for ϕ and θ any other function ψ such that, whenever $(M_j)_{>i-j} = 0$ for all $j > 0$, we have:

$$\psi(i, M_0, M_1, M_2, \ldots, M_n) \neq 0 \text{ if and only if } (M_0)_i \neq 0. \tag{3.2.3}$$

For example, instead of $\alpha(\_)$ or $\mathrm{Max}\, H^0_{\mathbf{m}}(\_)$ we could use the function $\beta(\_)$ defined as:

$$\beta(M) = \sup\{i \mid \psi(i, H^0_{\mathbf{m}}(M), H^1_{\mathbf{m}}(M), \ldots, H^n_{\mathbf{m}}(M)) \neq 0\}.$$

The following result holds:

Theorem 3.2.7. *For such a β defined as above we have:*

$$\mathrm{reg}(M) = \max_{i \in \{0, \ldots, d\}} \{\beta(M/(l_1, \ldots, l_i)M) - \sum_{j=1}^{i} (D_j - 1))\}$$

where M is a finitely generated graded module of dimension d and $l_1, \ldots, l_d$ is a filter regular sequence of degrees $D_1, \ldots, D_d$.

Remark 3.2.8. If two functions ψ_1 and ψ_2 satisfying the above property (3.2.3) then also $\min\{\psi_1, \psi_2\}$ and $\max\{\psi_1, \psi_2\}$ satisfy (3.2.3). Moreover if we call β_1 and β_2 the corresponding functions associated with ψ_1, ψ_2 then $\min\{\psi_1, \psi_2\}$ and $\max\{\psi_1, \psi_2\}$ are associated with $\min\{\beta_1, \beta_2\}$ and $\max\{\beta_1, \beta_2\}$. This observation allows us to obtain the following result of independence.

Theorem 3.2.9. *Let $\beta_1, \ldots, \beta_d$ be defined as above and let $l_1, \ldots, l_d$ be a filter regular sequence of forms of degrees $D_1, \ldots, D_d$ over a module M of dimension d. Then the number*

$$\max_{i \in \{0,\ldots,d\}} \{\beta_i(M/(l_1, \ldots, l_i)M) - \sum_{j=1}^{i} (D_j - 1))\} \tag{3.2.4}$$

is equal to the regularity of M and therefore does not depend on the filter regular sequence chosen not on the functions β_i.

Proof. Define the function γ_1 to be $\min\{\beta_i\}$ and γ_2 to be $\max\{\beta_i\}$. Thanks to Remark 3.2.8 we can apply Theorem 3.2.7 and get:

$$\begin{aligned} \operatorname{reg}(M) = \max_{i \in \{0,\ldots,d\}} \{\gamma_1(M/(l_1, \ldots, l_i)M) - \sum_{j=1}^{i} (D_j - 1))\} \leq \\ \max_{i \in \{0,\ldots,d\}} \{\beta_i(M/(l_1, \ldots, l_i)M) - \sum_{j=1}^{i} (D_j - 1))\} \leq \\ \max_{i \in \{0,\ldots,d\}} \{\gamma_2(M/(l_1, \ldots, l_i)M) - \sum_{j=1}^{i} (D_j - 1))\} = \operatorname{reg}(M). \end{aligned}$$

Therefore the middle term is equal to $\operatorname{reg}(M)$. □

3.2.3 Bayer and Stillman criterion for detecting regularity and some similar further criteria

In this section we discus the Bayer and Stillman criterion for detecting regularity. Below we will focus on modules as our main object of study: working with ideals does not give a significant simplification to the treatment. The reader can refer to Bayer and Stillman's original paper [BS] for the ideal-theoretic discussion. This criterion, as outlined in [BS], is a key point for the introduction and the study of generic initial ideals. Similarly, we will explore consequences of criteria for regularity in the next chapters: bounds for regularity and the structure of Gins rely significantly on those criteria.

Consider a finitely generated module M with a minimal presentation as $M = F/N$, where F is a free module (maybe with some shifts: i.e. $F = \oplus R(-i)^{b_i}$) and N is a nonzero submodule. The basic question behind these criteria is the following: *Assuming the knowledge of the highest degree of a minimal homogeneous generator of N (i.e.* $\max \operatorname{Tor}_1(M,K)$ *or* $\operatorname{reg}_{\{1\}}(M)+1$ *), how can one improve the formulas in Theorem 3.2.1, 3.2.5, 3.2.7, and 3.2.9?*

Concerning Theorem 3.2.1 an answer is given by the following criterion of Bayer and Stillman:

Theorem 3.2.10 (Bayer and Stillman criterion)**.** *Let M be a finitely generated graded module. Let f be a homogenous polynomial such that $(0:_M f)_{a+1} = 0$, for some $a \geq \max\{\operatorname{reg}(M/fM) - (\deg(f)-1), \operatorname{reg}_{\{1\}}(M)\}$. Then $(0:_M f)_{\geq a+1}$ is zero (if M has positive dimension f is therefore filter regular) and moreover $\operatorname{reg}(M) \leq a$.*

Proof. Write M minimally as F/N where F is a free module. First we want to show that the degree of the minimal generators of N and $N:_F f$ are bounded by $a+1$. This is equivalent to showing $\operatorname{reg}_{\{1\}}(M) \leq a$ and $\operatorname{reg}_{\{1\}}(M/(0:_M f) \leq a$. While the first inequality is by assumption, the second follows from the short exact sequence

$$0 \longrightarrow (M/0:_M f)(-deg(f)) \xrightarrow{\cdot f} M \longrightarrow M/fM \longrightarrow 0.$$

In fact, using Lemma 3.1.5 we have:

$$\mathrm{reg}_{\{1\}}(M/0:_M f) \leq \max\{\mathrm{reg}_{\{1\}}(M), \mathrm{reg}_{\{2\}}(M/fM)+1\} - \deg(f)$$

which is bounded by a. Now, because $(0:_M f) = (N:_F f)/N$, the fact that this module is zero in a degree $a+1$, greater or equal than the degree of the minimal generators of $N:_F f$ and N, implies $((N:_F f)/N)_{\geq a} = 0$. In particular $(0:_M f)$ has finite length and, therefore, if the dimension of M is positive, f is filter regular. We still have to show that $\mathrm{reg}(M) \leq a$. Note that since $(0:_M f)_{\geq a+1} = 0$ then $(0:_M f^\infty)_{\geq a+1} = 0$. This implies $(0:_M f^\infty) = H^0_{\mathbf{m}}(M)$ and, in particular, it gives $\max H^0_{\mathbf{m}}(M) \leq a$, that is enough for the dimension zero case. If the dimension of M is positive we know that f is filter regular and by Corollary 3.2.1 we have

$$\mathrm{reg}(M) \leq \{\max H^0_{\mathbf{m}}(M), \mathrm{reg}(M/fM) - \deg(f) + 1\},$$

which is less than or equal to a. □

Remark 3.2.11. Note that in the proof of Theorem 3.2.10, in order to obtain $(0:_M f)_{\geq a+1} = H^0_{\mathbf{m}}(M)_{\geq a+1} = 0$ it was enough to have $a \geq \max\{\mathrm{reg}_{\{2\}}(M/fM) - (\deg(f)-1), \mathrm{reg}_{\{1\}}(M)\}$.

Corollary 3.2.12. *Let M be a finitely generated graded module and let f be a filter regular element. Set $c = \max\{\mathrm{reg}(M/fM) - (\deg(f)-1), \mathrm{reg}_{\{1\}} M\}$. Then*

$$\begin{aligned}\mathrm{reg}(M) &= \min\{a \mid (0:_M f)_{a+1} = 0 \text{ and } a \geq c\} \\ &= \min\{a-1 \mid H^0_{\mathbf{m}}(M)_{a+1} = 0 \text{ and } a \geq c\}.\end{aligned}$$

Proof. By the previous Theorem the first term is smaller than or equal to the second. On the other hand, $(0:_M f) \subseteq (0:_M f^\infty) = H^0_{\mathbf{m}}(M)$, therefore, the second term is smaller than or equal to the third. Corollary 3.2.1 gives $\mathrm{reg}(M) \geq$

$\operatorname{reg}(M/fM) - (D-1)$ and, in particular, $\operatorname{reg}(M) \geq c$. Since $H^0_{\mathbf{m}}(M)_{\operatorname{reg}(M)+1}$ is zero, we have that

$$\operatorname{reg}(M) \in \{a | (0 :_M f)_{a+1} = 0 \text{ and } a \geq c\}$$

which proves that the third term is smaller than or equal to the first. □

Remark 3.2.13. Using the notation of the previous section we will consider now a function ψ satisfying condition (3.2.3). With an abuse of notation we will denote the function $\psi(i, H^0_{\mathbf{m}}(M), H^1_{\mathbf{m}}(M), \ldots, H^n_{\mathbf{m}}(M))$ by $\psi(i,M)$. Recall that the difference between the Hilbert polynomial and the Hilbert function of a module is one of such a ψ.

We can state then two variations of Theorem 3.2.10 and Corollary 3.2.12.

Proposition 3.2.14. *Let M be a finitely generated graded module and let ψ be a function defined as above. Let f be a homogenous filter regular polynomial such that* $\psi(a+1,M) = 0$, *for some* $a \geq \max\{\operatorname{reg}(M/fM) - (\deg(f) - 1), \operatorname{reg}_{\{1\}}(M)\}$. *Then* $\psi(i,M) = 0$ *for all* $i \geq a+1$ *and moreover* $\operatorname{reg}(M) \leq a$.

Proof. In order to prove that $\operatorname{reg}(M) \leq a$ it is enough to show that the hypotheses of Theorem 3.2.10 are satisfied. By part (1) of Proposition 3.1.9 we know that $\operatorname{reg}^{\{1,\ldots,n\}}(M) \leq \operatorname{reg}(M/fM) - (\deg(f) - 1)$ which is bounded by a. Therefore, $H^i_{\mathbf{m}}(M)_{a+1-i} = 0$ and by the properties of ψ we have that $\psi(a+1,M) = 0$. This implies $H^0_{\mathbf{m}}(M)_{a+1} = 0$, which gives $(0 :_M f)_{a+1} = 0$.

To prove that $\psi(i,M) = 0$ for all $i \geq a+1$ it is enough to observe that $\psi(i,M) = 0$ if and only if $H^0_{\mathbf{m}}(M)_i = 0$. This condition is satisfied because we know that $\operatorname{reg}(M) \leq a < i$, and we can use Corollary 3.2.12. □

Corollary 3.2.15. *Under the same assumptions of Corollary 3.2.12 we have:*

$$\mathrm{reg}(M) = \min\{a|\psi(a+1,M) = 0 \text{ and } a \geq c\}. \tag{3.2.5}$$

Proof. We know that for $a \geq c$ the function $\psi(a+1,M)$ is equal to zero if and only if $H^0_{\mathbf{m}}(M)_{a+1} = 0$. Therefore the result follows directly from Corollary 3.2.12. □

3.2.4 Crystallization principle

In this section we want to underline one immediate consequences of Corollary 3.2.15. The choice of the title will be clarified later when we will study some applications of the result of this section. In particular we will give a proof of the crystallization principle for generic initial ideals in characteristic zero, by using this result. Below ψ will denote a function defined in Remark 3.2.13.

The following Lemma is an immediate and direct consequence of Corollary 3.2.12 and Corollary 3.2.15.

Lemma 3.2.16. *Let M be a finitely generated graded module and let f be a filter regular form. Let $c \geq \max\{\mathrm{reg}_{\{1\}} M, \mathrm{reg}(M/fM) - (\deg(f) - 1)\}$ Then the following sets of indexes are the same:*

(1) $S_1 = \{j|(0_M : f)_j \neq 0, \text{ and } j \geq c\}$

(2) $S_2 = \{j|H^0_{\mathbf{m}}(M)_j \neq 0 \text{ and } j \geq c\}$

(3) $S_3 = \{j|\psi(j,M) \neq 0 \text{ and } j \geq c\}$

(4) $S_4 = \{j|c \leq j \leq \mathrm{reg}(M)\}$.

Proof. As we said above the proof follows from Corollary 3.2.12 which shows $S_1 = S_2 = S_4$, and from Corollary 3.2.15 which gives $S_3 = S_4$. □

Proposition 3.2.17 (Crystallization Principle)**.** *Let M be a finitely generated graded module over $K[x_1,\ldots,x_n]$ and let $l_1,\ldots,l_n$ be a filter regular sequence of linear linearly independent over K. Let $N_0 = M$, $N_i = M/(l_1,\ldots,l_i)M$ and for $i > 0$ define $c_i = \max\{\operatorname{reg}_{\{1\}}(M), \operatorname{reg}(N_i)\}$. Then the following sets of indexes are the same:*

(1) $S_1 = \cup_{i=0}^{n-1}\{j|(0_{N_i} : l_{i+1})_j \neq 0, \text{ and } j \geq c_{i+1}\}$

(2) $S_2 = \cup_{i=0}^{n-1}\{j|H^0_{\mathbf{m}}(N_i)_j \neq 0 \text{ and } j \geq c_{i+1}\}$

(3) $S_3 = \cup_{i=0}^{n-1}\{j|\psi(j,N_i) \neq 0 \text{ and } j \geq c_{i+1}\}$

(4) $S_4 = \{j|\operatorname{reg}_{\{1\}}(M) \leq j \leq \operatorname{reg}(M)\}$.

Proof. First note that $\operatorname{reg}_{\{1\}}(M) \geq \operatorname{reg}_{\{1\}}(N_1) \geq \ldots, \geq \operatorname{reg}_{\{1\}}(N_n) = 0$ moreover each N_i is a module over a polynomial ring in $n-i$ variables.
Define $S_{1,i} = \{j|(0_{N_i} : l_{i+1})_j \neq 0, \text{ and } j \geq c_{i+1}\}$ for $i = 0,\ldots,n-1$, and similarly define $S_{2,i}$ and $S_{3,i}$. Set $S_{4,i}$ to be $\{j|c_{i+1} \leq j \leq \operatorname{reg}(N_i)\}$. To conclude the proof, it is enough to show the following claim:

Claim 3. *For any $i = 0,\ldots,d-1$ we have $S_{1,i} = S_{2,i} = S_{3,i} = S_{4,i}$.*

Which follows from Lemma 3.2.16 applied to N_i and c_{i+1}. □

Remark 3.2.18. Following the same idea as in the proof of Theorem 3.2.9, we could substitute the third set above for a more general one:

$$\cup_{i=0}^{n-1}\{j|\psi_i(j,N_i) \neq 0 \text{ and } j \geq c_{i+1},\}$$

where ψ_i - exactly as ψ - are functions defined in Remark 3.2.13.

Chapter 4

Weakly Stable Ideals and Castelnuovo-Mumford Regularity

This chapter is devoted to the study of a special kind of monomial ideals called *weakly stable ideals* (see Definition 4.1.3). The notion has been introduced by Enrico Sbarra and the author in [CS] in order to have a combinatorial property satisfied both by strongly stable ideals and p-Borel ideals. In particular in [CS] we use weakly stable ideals to reproduce an argument of Giusti to bound uniformly, in characteristic zero, the Castelnuovo-Mumford regularity of all the ideals generated at most in degree d. We refer to the next chapter for the proofs of the bounds in [CS], in particular we show how it is possible to use weakly stable ideals to give a different proof of these bounds.

It is well known that the regularity of a stable ideal I is equal to the highest degree of a minimal generator of I. This fact can be deduced, for example, by looking at the Eliahou-Kervaire resolution of I, see [EK]. On the other hand in the literature there is no equivalent formula for p-Borel ideals, and the only known result, which was conjectured by Pardue and recently proved by J.Herzog and D.Popescu [HP], is a quite complicated formula for the special case of p-Borel principal ideals.

Later we show how to extend the formula for the regularity of stable ideals to weakly stable ideals. We will prove:

Theorem 4.1.10. *Let $I \subset K[x_1 \ldots, x_n]$ be a weakly stable ideal minimally generated by the monomials $u_1, \ldots, u_r$. Assume that $u_1 > u_2 > \cdots > u_r$ with respect to the reverse lexicographic order (note that it is not the degree revlex). Then*

$$\operatorname{reg}(I) = \max\{\deg u_i + C(u_i)\} \qquad \text{(A)}$$

where $C(u_i)$ is set to be the highest degree of a monomial v in $K[X_1, \ldots, X_j]$ such that $vu_i \notin (u_1, \ldots, u_{i-1})$ and X_{j+1} is the last variable dividing u_i.

It will follow easily that when I is strongly stable, the correction term $C(u_i)$ is zero for all i.

In the first section we give the definition of weakly stable ideals and we show that this combinatorial notion is equivalent to saying that all the primes associated to such ideals are generated by lex-segments. This property allows us to make use of the Bayer and Stillman criterion for detecting regularity, and prove that their regularity does not depend on the characteristic of the base field. On the other hand, we give an example of a weakly stable ideal for which the Betti numbers depend on $\operatorname{char}(K)$.

In the second section we prove the formula (A) for the Castelnuovo-Mumford regularity that was mentioned above.

4.1 General properties of Weakly Stable ideals

Strongly stable ideals, stable ideals and p-Borel ideals play an important role in those areas of Commutative Algebra and Algebraic Geometry where certain homological invariants, for example projective dimension, Castelnuovo-Mumford regularity and extremal Betti numbers, can be computed by combinatorial properties of the generic initial ideal. Generic initial ideals ideals are strongly stable (and in particular stable) when $\operatorname{char} K = 0$ and they are p-Borel if $\operatorname{char} K = p > 0$. We recall briefly those two notions (see [Pa1] for further details).

Notation 4.1.1. Given a monomial ideal I we define $G(I)$ to be the set of its minimal monomial generators. Given a monomial u we denote $\max\{i$ such that $X_i \mid u\}$ by $m(u)$ and the value $\max\{j$ such that $X_i^j \mid u\}$ by $|u|_i$. These notion can be naturally extended to a monomial ideal by setting $m(I) = \max\{m(u)$ with $u \in G(I)\}$ and $|I|_i = \max\{|u|_i$ with $u \in G(I)\}$.

A monomial ideal I is *strongly stable* if for all $u \in I$, whenever $X_i \mid u$ then $\frac{X_j u}{X_i} \in I$, for every $j < i$.

The wider class of *stable* ideals is defined by the following weaker exchange condition on the variables of the monomials: an ideal I is stable if for every monomial $u \in I$, $\frac{X_j u}{X_{m(u)}} \in I$, for every $j < m(u)$.

Example 4.1.2. In $K[X,Y,Z]$ the smallest stable ideal containing XYZ is $I = (X^3, X^2Y, XY^2, XYZ)$, which is not strongly stable since $X^2Z \notin I$.

Let p be a prime number. Given two integers a and b, we write their *p-adic expansion* as $a = \sum_i a_i p^i$ and $b = \sum_i b_i p^i$ respectively. One defines a partial order $\leq_p$ by saying that $a \leq_p b$ if and only if $a_i \leq b_i$ for all i.

An ideal I is said to be *p-Borel* if for every monomial $u \in I$, if b is the maximum integer such that $X_i^b | u$, then $\frac{X_j^a u}{X_i^a} \in I$, for every $i < j$ and $a \leq_p b$.

Notation. Given two monomial u and v, we will denote the monomial generator of the ideal $(u) : v^\infty$ simply by $u : v^\infty$.

Definition 4.1.3. A monomial ideal I is called *weakly stable* if the following property holds. For all $u \in I$ and for all $j < m(u)$ there exists a positive integer a such that $(u : X_{m(u)}^{\infty})X_j^a \in I$.

Remark 4.1.4. First of all note, as we said in the introductory section, that strongly stable, stable and p-Borel ideals are weakly stable.

From the definition we can also deduce the following:

(1) Let I and J be weakly stable ideals. Then $I+J$, IJ and $I \cap J$ are also weakly stable.

(2) Let I be a weakly stable ideal and J be a monomial ideal. Then $I : J$ is weakly stable.

(3) If I is weakly stable and $x_i^a \in I$, then there exists positive integer $a_1, \ldots, a_{i-1}$ such that $X_j^{a_j} \in I$ for all $0 < j < i$. Which can be rephrased as $(X_1, \ldots, X_i) \subseteq \mathrm{rad}(I)$.

(4) Ideals defining Artinian algebras are weakly stable.

Proof of (1). The fact that $I+J$ and IJ are weakly stable follows directly from the definition. For what concerns $I \cap J$ we note that if $u \in I \cap J$ then there exist a_1 and a_2 such that $(u : X_{m(u)}^{\infty})X_j^{a_1} \in I$ and $(u : X_{m(u)}^{\infty})X_j^{a_2} \in J$. Therefore taking $b = \max\{a_1, a_2\}$ we get $(u : X_{m(u)}^{\infty})X_j^b \in I \cap J$. The proof of (3) and (4) is straightforward, while (2) needs some explanations.

Proof of (2). Let $(u_1, \ldots, u_r)$ be a system of monomial generators for J. Since $I : J = \cap_i (I : u_i)$, without loss of generality we can assume that J is a principal ideal generated by u_1. Let $u \in I : u_1$. Note that the only nontrivial case is when $m(u)$ is greater than one. We can write $u = qX_{m(u)}^a$ and $u_1 = st$ where

$q,s \in K[X_1,\dots,X_{m(u)-1}]$ and $t \in K[X_{m(u)},\dots,X_n]$. Since $qsX_{m(u)}^a t \in I$, by applying several times the property of the weak stability, we obtain $qsX_{m(u)}^b \in I$ for some b. On the other hand $m(qsX_{m(u)}^b) = m(u)$ and therefore, by weak stability, for any positive integer $j < m(u)$ there exists a c such that $qsX_j^c \in I$. In particular $u : X_{m(u)}^{\infty} X_j^c = qX_j^c$ belongs to $I : s \subseteq I : u_1$. □

We show now how the weak stability is in fact equivalent to some other, less combinatorial, properties.

Set $R = K[X_1,\dots,X_n]$, and let I to be a homogeneous ideal. We recall that a homogenous element l of degree d is said to be *filter regular* for $R/I = S$ if the multiplication by l from S_a to S_{d+a} is an injective map for $a >> 0$. This is equivalent to say that l does not belong to any of the associated primes of I different from the homogeneous maximal ideal. The elements $l_1,\dots,l_i$ form an filter regular sequence if, for any $j = 1,\dots,i$, the form l_j is regular for $R/(l_1,\dots,l_{j-1})$.

Proposition 4.1.5. *Let $I \subseteq R$ be a monomial ideal. Then the following properties are equivalent.*

$i)$ *The ideal I is weakly stable.*

$ii)$ *Any $P \in \operatorname{Ass}(I)$ is a lex-segment ideal.*

$iii)$ *The variables $X_n, X_{n-1},\dots,X_1$ are an filter regular sequence for R/I.*

Proof. The fact that weak stability implies $ii)$ is an immediate consequence of Remark 4.1.4. Let P be an associated prime of I, we can then write it as $P = I : m$ for some monomial m. By Remark 4.1.4 part (2) P is weakly stable. Let X_i be the variable belonging to P having the greatest index, in particular, by Remark 4.1.4 part (3), $(X_1,\dots,X_i) \subseteq P$. The other inclusion holds by the choice of X_i.

Assume now that *ii*) holds true. It is clear that X_n is a filter regular element. In order to prove *iii*), by a decreasing induction, it is enough to show that X_{n-1} is filter regular for $K[X_1,\dots,X_{n-1}]/(I \cap K[X_1,\dots,X_{n-1}])$. The result follows because property *ii*) is satisfied by $I \cap K[X_1,\dots,X_{n-1}]$.

Assume now that *iii*) holds. Let u be monomial in I and set J to be $I \cap K[X_1,\dots,X_{m(u)}]$. Since $X_{m(u)}$ is filter regular in $K[X_1,\dots,X_{m(u)}]/J$, the module $(J : X_{m(u)}^{\infty})/J$ has finite length and therefore for any positive i less than $m(u)$ there exists an a_i for which $(u : X_{m(u)}^{\infty})X_i^{a_i} \in J \subseteq I$. □

A consequence of Proposition 4.1.5 is outlined by the next proposition, which shows that the regularity and the projective dimension of a weakly stable ideal are conbinatorial invariants. They are, therefore, independent of the characteristic of the base field.

Proposition 4.1.6. *Let $J \subset \mathbb{Z}[X_1,\dots,X_n]$ be a weakly stable monomial ideal and let $R = K[X_1,\dots,X_n]$. Define the ideal I of R as $I = JR$. Then* $\mathrm{reg}(I)$ *and the projective dimension* $\mathrm{pd}(I)$ *depend only on J. Moreover* $\mathrm{pd}(I) = m(I) - 1$.

Proof. Since I is weakly stable then $X_n,\dots,X_1$ is a filter regular sequence. Clearly $X_n,\dots,X_{m(I)+1}$ is a regular sequence for R/I and $X_{m(I)}$ is a zerodivisor. Then by Proposition 4.1.5 *ii*), we know that $(X_1,\dots,X_{m(I)})$ is associated to R/I and, therefore, $\mathrm{depth}(R/I) = n - m(I)$ and $\mathrm{pd}(I) = m(I) - 1 = m(J) - 1$. To complete the proof we show that $\mathrm{reg}(R/I)$, which is $\mathrm{reg}(I) - 1$, depends only on J. If $n = 1$ the result is clear, moreover $\mathrm{reg}_1 R/I$ is also clearly independent of R since it is the highest degree of a minimal monomial generator of J. We can then do an induction on the number of variables. By using Corollary 3.2.12 we know that $\mathrm{reg}(R/I) = \min\{a | (0 :_{R/I} X_n)_{a+1} = 0 \text{ and } a \geq \max\{reg_1(R/I), \mathrm{reg}(R/(I + (X_n))\}\}$. Whether

$(0 :_{R/I} X_n)$ is zero in a certain degree is independent of R and, therefore, by induction, the above set is independent too. □

Remark 4.1.7. In general the regularity and the projective dimension of a monomial ideal depend on $\mathrm{char}(K)$. The well-known example to show this fact, obtained from the triangulation of $\mathbb{P}^2$, is the following: Let $R = K[X_1, X_2, \ldots, X_6]$ and let $I = (X_1X_2X_3, X_1X_2X_4, X_1X_3X_5, X_2X_4X_5, X_3X_4X_5, X_2X_3X_6, X_1X_4X_6, X_3X_4X_6, X_1X_5X_6, X_2X_5X_6)$. If $\mathrm{char}(K) = 0$ then R/I has the following resolution:

$$0 \longrightarrow R^6(-5) \longrightarrow R^{15}(-4) \longrightarrow R^{10}(-3) \longrightarrow R/I \longrightarrow 0.$$

Which gives $\mathrm{reg}(I) = 3$ and $\mathrm{pd}(I) = 2$. On the other hand, if $\mathrm{char}(K) = 2$ we have the following resolution:

$$0 \longrightarrow R(-6) \longrightarrow R^6(-5) \bigoplus R(-6) \longrightarrow R^{15}(-4) \longrightarrow R^{10}(-3) \longrightarrow R/I \longrightarrow 0,$$

which provides $\mathrm{reg}(I) = 4$ and $\mathrm{pd}(I) = 4$.
Note that I is not a weakly stable ideal since it does not contain a pure power of the variable X_1.

Proposition 4.1.6 is refined by Theorem 4.1.10 which gives a precise value for $\mathrm{reg} I$. In particular, Theorem 4.1.10 gives a different proof that the regularity of a weakly stable ideal does not depend on $\mathrm{char}(K)$. Notice that, on the other hand, it is possible to construct examples of weakly stable ideals whose graded Betti numbers depends of $\mathrm{char}(K)$. Take an ideal I, for example the one of the triangulation of $\mathbb{P}^2$, whose Betti numbers depend on $\mathrm{char}(K)$ and then add a power of the homogeneous maximal ideal greater than $\mathrm{reg}(I)$. This new ideal, say J, is weakly stable because it gives and Artinian Algebra and has graded Betti numebers depending on $\mathrm{char}(K)$.

Before proving Theorem 4.1.10 it is useful to give the following two Lemmas.

Lemma 4.1.8. *Let I be a monomial ideal and let $u_1 > u_2 > \cdots > u_r$ be the minimal monomial generators of I ordered revlex. Then $m((u_1\, u_2, \ldots, u_{i-1}) : u_i) < m(u_i)$.*

Proof. Set $J = (u_1\, u_2, \ldots, u_{i-1}) : u_i$. It is clear that $m(J) \leq m(u_1, u_2, \ldots, u_{i-1}) \leq m(u_i)$. Let v be a minimal generator of J for which $m(v) = m(J)$. We know that there exists a monomial w such that $vu_i = wu_j$ for some $1 \leq j \leq i-1$; this forces $v > w$. On the other hand, since v is a minimal generator of J, v and w have no common nontrivial factor, and in particular we get that $m(v) < m(w) \leq m(wu_j) = m(vu_i)$. This gives $m(v) < m(u_i)$. □

The next Lemma explains the real meaning of the correction terms $C(u_i)$ appearing in Theorem 4.1.10.

Lemma 4.1.9. *Let $I \subset K[X_1 \ldots, X_n]$ be a weakly stable ideal, and let $u_1 > u_2 > \cdots > u_r$ be the minimal monomial generators of I, ordered in revlex. Let j be $m(u_i) - 1$ and define $C(u_i)$ as the highest degree of a monomial v in $K[X_1, \ldots, X_j]$ such that $vu_i \notin (u_1, \ldots, u_{i-1})$. Then $C(u_i) = \operatorname{reg} R/J$ where $J = ((u_1, \ldots, u_{i-1}) : u_i)$.*

Proof. By Lemma 4.1.8 we know that $m(J) \leq m(u_i) - 1 = j$. Hence $X_n, \ldots, X_{j+1}$ is a regular sequence for R/J, so $\operatorname{reg} R/J = \operatorname{reg} R/(J + (X_n, \ldots, X_{j+1}))$. On the other hand, since the ideal $(u_1, \ldots, u_i)$ is weakly stable, there exist $a_1, \ldots, a_j$ such that $(u_i : X_{j+1}^{\infty})X_l^{a_l} \in I$, for $l = 1 \ldots, j$. More precisely, since $(u_i : X_{j+1}^{\infty})X_l^{a_l} < u_i$, we get that $(u_i : X_{j+1}^{\infty})X_l^{a_l} \in (u_1, \ldots, u_{i-1})$. In particular, $(X_1^{a_1}, \ldots, X_j^{a_j}) \subseteq (J + (X_n, \ldots, X_{j+1}))$, and, therefore, $R/(J + (X_n, \ldots, X_{j+1})$ Artinian. The Castelnuovo-Mumford regularity of $R/(J + (X_n, \ldots, X_{j+1})$ is equal to the degree of its highest nonzero graded component, which is exactly the highest degree of a monomial v in $K[X_1, \ldots, X_j]$ such that $vu_i \notin (u_1, \ldots, u_{i-1})K[X_1, \ldots, X_{j+1}]$. □

We can now prove the following:

Theorem 4.1.10. *Let $I \subset K[X_1 \ldots, X_n]$ be a weakly stable ideal minimally generated by the monomials $u_1, \ldots, u_r$. Assume that $u_1 > u_2 > \cdots > u_r$ with respect to the reverse lexicographic order and let j_i be $m(u_i) - 1$. Then*

$$\operatorname{reg}(I) = \max\{\deg u_i + C(u_i)\}, \tag{A}$$

where $C(u_i)$ is set to be the highest degree of a monomial v in $K[X_1, \ldots, X_j]$ such that $vu_i \notin (u_1, \ldots, u_{i-1})$.

Proof. Set $\alpha(I) = \max\{\deg u_i + C(u_i)\}$. If $r = 1$ the formula is clear. We can, therefore, do an induction on the number of generators of I. From the following short exact sequence:

$$0 \to (u_1, \ldots, u_{r-1}) \to I \to ((u_1, \ldots, u_{r-1}) : u_r)(-\deg u_r) \to 0,$$

we deduce $\operatorname{reg}(I) \leq \max\{\operatorname{reg}((u_1, \ldots, u_{r-1}), \operatorname{reg}((u_1, \ldots, u_{r-1}) : u_r) + \deg u_r\}$. By Lemma 4.1.9 $\operatorname{reg}((u_1, \ldots, u_{r-1}) : u_r) + \deg u_r$ is precisely $\deg u_r + C(u_r)$. By induction we know that $\operatorname{reg}(u_1, \ldots, u_{r-1}) = \alpha((u_1, \ldots, u_{r-1}))$, and therefore

$$\operatorname{reg}(I) \leq \max\{\alpha((u_1, \ldots, u_{r-1})), \deg u_r + C(u_r)\} = \alpha(I).$$

Assume, by contradiction, that $\operatorname{reg} I < \alpha(I)$. Let i be the lowest index for which J, defined as $J = I \cap K[X_1, \ldots, X_i]$, satisfies $\alpha(J) = \alpha(I)$. Since $X_n \ldots, X_{i+1}$ is a filter regular sequence for R/I we have that $\operatorname{reg}(J) = \operatorname{reg}(I + (X_n \ldots, X_{i+1})) \leq \operatorname{reg}(I) < \alpha(I) = \alpha(J)$. By substituting I for J, without loss of generality, we can assume that $\operatorname{reg}(I) < \alpha(I)$ and $\alpha((I \cap K[X_1, \ldots X_{n-1}])) < \alpha(I)$, moreover this inequality implies that $\alpha(I)$ is obtained at some u_l with $m(u_l) = n$. Since $\operatorname{reg}(I) \leq \alpha(I) - 1$ we know that $H^0_{\mathbf{m}}(R/I) = (I : \mathbf{m}^\infty)/I = (I : X_n{}^\infty)/I$ is zero in degree $\alpha(I) - 1$.

In particular, $((I : X_n)/I)_{\alpha(I)-1} = 0$. On the other hand, there exists a monomial $v \in K[X_1, \ldots, X_{n-1}]$ such that $v \notin (u_1, \ldots, u_{l-1}) : u_l$ and $\deg(vu_l) = \alpha(I)$. Note that since the u_i's are ordered reverse lexicographically $m(u_l) = n$ implies $m(u_j) = n$ for all $j \geq l$, therefore $v \notin I$. The element vu_l/X_n shows that $((I : X_n)/I)_{\alpha(I)-1} \neq 0$ which is a contradiction. □

As a corollary we get the well-known fact that the regularity of a strongly stable ideal is equal to the highest degree of a minimal monomial generator.

Corollary 4.1.11. *Let $I \subset K[X_1 \ldots, X_n]$ be a strongly stable ideal minimally generated by the monomials $u_1, \ldots, u_r$. Then* $\operatorname{reg}(I) = \max\{\deg u_i\}$.

Proof. We can assume that $u_1 > u_2 > \cdots > u_r$ with respect to the reverse lexicographic order. By using Theorem 4.1.10 it is enough to show that the correction terms $C(u_i)$ are zero. Since $(u_i/X_{m(u_i)})X_j \in I$ for all $j < i$ we know that $(X_1, \ldots, X_{m(u_i)-1}) \subseteq (u_1, \ldots, u_{i-1}) : u_i$. On the other hand, Lemma 4.1.8 gives the other inclusion, so $(u_1, \ldots, u_{i-1}) : u_i = (X_1, \ldots, X_{m(u_i)-1})$. This implies that $C(u_i) = 0$. □

The use of weakly stable ideals for investigating the regularity of homogeneous ideals is made more clear by the following results.

Lemma 4.1.12. *Let $I \subset K[X_1 \ldots, X_n]$ be a homogeneous ideal. Then $X_n, \ldots, X_i$ is a filter regular sequence for R/I if and only if it is a filter regular sequence for $R/\operatorname{in}_{rlex}(I)$.*

Proof. For all s, $i \leq s \leq n$, define J_s as $I + (X_n, \ldots X_s)$ and H_s as $(\operatorname{in}_{rlex}(I) + (X_n, \ldots X_s))$. Since the term-order we are using is the reverse lexicographic order we have $\operatorname{in}_{rlex}(J_s) = H_s$. The sequence $X_n, \ldots, X_i$ is a filter regular sequence for R/I (resp. for $R/\mathrm{in}_{rlex} I$) if and only if $(J_{s-1}:X_s)/J_{s-1}$ (resp. $(H_{s-1}:X_s)/H_{s-1}$) has finite length for all $i \leq s \leq n$.

From the short exact sequences

$$0 \to (J_{s-1}:X_s)/J_{s-1} \to R/J_{s-1} \xrightarrow{\cdot X_s} R/J_{s-1} \to R/J_s \to 0$$

and

$$0 \to (H_{s-1}:X_s)/H_{s-1} \to R/H_{s-1} \xrightarrow{\cdot X_s} R/H_{s-1} \to R/H_s \to 0$$

we deduce that $(J_{s-1}:X_s)/J_{s-1}$ and $(H_{s-1}:X_s)/H_{s-1}$ have the same length since for any index i the Hilbert function of R/J_i and R/H_i are the same. □

Remark 4.1.13. From the above proof we also deduce that $(I:X_n)/I$ and $(\mathrm{in}_{\mathrm{rlex}}(I):X_n)/\mathrm{in}_{\mathrm{rlex}}(I)$ have the same Hilbert function.

Corollary 4.1.14. *Let $I \subset K[X_1 \ldots, X_n]$ be a homogeneous ideal. Then $X_n, \ldots, X_1$ is a filter regular sequence for R/I if and only if* $\mathrm{in}_{\mathrm{rlex}}(I)$ *is weakly stable.*

Proof. The result follows immediately from Lemma 4.1.12 and the equivalence given by Proposition 4.1.5. □

Remark 4.1.15. In the above Corollary, since any nonzero homogeneous form is filter regular over an Artinian ring, we can substitute $X_n, \ldots X_d$ for $X_n, \ldots X_1$, with $d = \dim R/I$.

The next Proposition is a slight extension of the well-know fact that the regularity and the projective dimension of an ideal are preserved by taking generic initial ideal with respect to the reverse lexicographic order.

Proposition 4.1.16. *Let $I \subset K[X_1 \ldots, X_n]$ be a homogeneous ideal. If* $\mathrm{in}_{rlex}(I)$ *is weakly stable, then* $\mathrm{reg}(I) = \mathrm{reg}(\mathrm{in}_{rlex}(I))$ *and* $\mathrm{pd}(I) = \mathrm{pd}(\mathrm{in}_{rlex}(I))$.

Proof. The fact that the projective dimension is preserved is equivalent to say that the depth is preserved. Assume that $\operatorname{depth}(R/I) = a$, then since $X_n, \dots, X_1$ is a filter regular sequence for R/I we get that $X_n, \dots, X_a$ is a regular sequence for R/I. On the other hand the Hilbert functions of $R/(I + (X_n, \dots, X_a))$ and $R/(\operatorname{in}_{rlex}(I) + (X_n, \dots, X_a)) = R/\operatorname{in}_{rlex}((I) + (X_n, \dots, X_a))$ are the same, therefore $X_1, \dots, X_a$ is a regular sequence also for $R/\operatorname{in}_{rlex}(I)$. If $\operatorname{depth}(R/\operatorname{in}_{rlex}(I)) = b$, we deduce that $X_n, \dots, X_b$ is regular for $R/\operatorname{in}_{rlex}(I)$, and in the same way, we get $\operatorname{depth}(R/I) \geq b$. For what concerns the regularity we recall that for any ideal N and for any filter regular sequence of linear forms $l_n \dots, l_1$ for R/N we have

$$\operatorname{reg}(N) = \max\{\operatorname{Max}(N + (l_n, \dots, l_{i-1}) : l_i)/(N + (l_n, \dots, l_{i-1})\}$$

where $\operatorname{Max}(M)$ stands for the highest nonzero graded component of a finite length module M. Keeping the same notation of Lemma 4.1.14, we see that its proof shows that the Hilbert function of $(J_{s-1} : X_s)/J_{s-1}$ and $(H_{s-1} : X_s)/H_{s-1}$ are the same. Therefore, by using the above formula, we get $\operatorname{reg}(I) = \operatorname{reg}(\operatorname{in}_{rlex}(I))$. □

In the next section we present an example of a family of ideals with high regularity. This gives us the opportunity to show how to combine the use of Theorem 4.1.10 and Proposition 4.1.16.

4.2 Ideals with high Castelnuovo-Mumford regularity

Given an ideal $I \subseteq K[X_1, \dots, X_n]$ generated by polynomials of degrees $d_1, \dots, d_r$ we know that $\max\{d_i\} \leq \operatorname{reg}(I)$. In general, without any assumption on I, the difference between the degree of the generators and the regularity of I can be very large. The most famous example in the literature, showing this bad behavior, was given by Mayr and Meyer in [MM]. This important example consists of a family of ideals of quartics (a variation, due to Jee Koh, gives an ideal of quadrics), depending on the number of variables of the ring. It is quite natural to ask whether it is possible to construct examples of ideals with large regularity (with respect to the degree of the generators) for a fixed polynomial ring. In this setting there is a uniform bound for the regularity of all the ideals generated in a fixed degree, so it makes more sense to ask for a family of ideals depending on the degree of the generators.

The example we give in this section is an extremely simple one: it is given by two monomial and a binomial of degree d in four variable. The computation of the regularity for these ideals is obtained quite easily by the use of results concerning weakly stable ideals.

Example 4.2.1. Let $I = (X_1^d, X_2^d, X_1X_3^{d-1} - X_2X_4^{d-1})$ be a homogeneous ideal of $K[X_1, X_2, X_3, X_4]$ with $d \geq 2$. Then $\operatorname{reg}(I) = d^2 - 1$.

Proof. First we show that:

$$\operatorname{in}_{rlex}(I) = (X_1^d, X_2^d, X_1X_3^{d-1}, X_1^{d-i}(X_2X_4^{d-1})^i \text{ for } i = 1, \dots, d-1).$$

More generally we construct a Gröbner basis for I as follow.
Set $g_1 = X_1X_3^{d-1} - X_2X_4^{d-1}$, $g_2 = X_2^d, g_3 = X_1^d$ and then define recursively

$$g_{3+i} = g_{2+i}X_3^{n-1} - g_1X_1^{d-i}(X_2X_4^{d-1})^{i-1}.$$

Note that $g_{3+i} = X_1^{d-i}(X_2X_4^{d-1})^i$. By construction we have that all the g_j's belong to I, moreover all the S-pairs are reducible. Therefore the g_j's are a Gröbner basis for I. It is immediate to check, using the definition involving the exchange property, that $\operatorname{in}_{rlex}(I)$ is weakly stable. Therefore by Proposition 4.1.16 we know that $\operatorname{reg}(I) = \operatorname{reg}(\operatorname{in}_{rlex}(I))$. By using Theorem 4.1.10 and by ordering the generators as $X_1^d < X_2^d < X_1X_3^d - 1 < X_1^{d-1}(X_2X_4^{d-1}) \cdots < X_1(X_2X_4^{d-1})^{d-1}$ we have to find the highest value for $\deg(u_j) + C(u_j)$ where u_j is the j^{th} minimal generator. This maximum is obtained at the very last generator, i.e $X_1(X_2X_4^{d-1})^{d-1}$. We have that $\deg(X_1(X_2X_4^{d-1})^{d-1}) = 1 + d(d-1)$ and that $C(X_1(X_2X_4^{d-1})^{d-1})$ is given, for example, by X_3^{d-2}. We obtain that $\operatorname{reg}(I) = 1 + d(d-1) + d - 2 = d^2 - 1$. □

This example shows that even with a fixed number of variable and a fixed number of generators for an ideal I (even with a fixed cardinality of the monomial supports for the generator of the ideal) the Castelnuovo-Mumford regularity of I could be much larger than the one corresponding to the complete intersection case.

Remark 4.2.2. A possible generalization of the above example is given by the following ideals. Let $R = K[X_1, \ldots, X_r, Y_1, \ldots, Y_r]$ and set $I_d = (X_1^d, Y_1^d) + (X_iX_{i+1}^{d-1} - Y_iY_{i+1}^{d-1}) | 1 \leq i \leq n-1)$. Computational experiments seem to indicate that $\operatorname{reg}(I_d)$, as a function of d, is given by a polynomial $P(d)$ of degree r.

Chapter 5

Uniform bounds for the Castelnuovo-Mumford regularity

In the literature we frequently find attempts bound the Castelnuovo-Mumford regularity and, in general, the expected results range quite widely, from the well-behaved examples coming from the algebraic geometry, as suggested by the Eisenbud-Goto Conjecture [EG], to the worst case provided by the example of Mayr and Meyer [MM].

In general, under quite unrestrictive assumptions, the regularity can be very large. If one works with a homogeneous ideal I in a polynomial ring with n variables over a field K, a very natural question to ask is whether the regularity can be bounded just by knowing the highest degree, say d, of a minimal homogeneous generator.

If $\operatorname{char} K = 0$, as observed in [BM] (Proposition 3.8), from the work of Giusti [Gi] and Galligo [Ga1], [Ga2] one can derive

$$\operatorname{reg}(I) \leq (2d)^{2^{n-2}}. \tag{A}$$

On the other hand, in any characteristic, it has been proven by Bayer and Mumford [BM], using cohomological methods, that

$$\operatorname{reg}(I) \leq (2d)^{(n-1)!}, \tag{B}$$

but in the same paper it is asked whether (A) holds in general independently of the characteristic.

This question was answered positively by the author and E.Sbarra in [CS]. The main effort in extending the result to positive characteristic is that Giusti's proof utilizes the combinatorial structure of the *generic initial ideal*, in characteristic zero, with respect to the reverse lexicographic order. More precisely a key point in that proof is the following fact known as Crystallization Principle.

CP: Let I be a homogeneous ideal generated in degrees $\leq d$. Assume also that $\mathrm{Gin}_{\mathrm{rlex}}(I)$ has no generator in degree $d+1$. Then there are no generators of $\mathrm{Gin}_{\mathrm{rlex}}(I)$ of degree higher than d ([Gr1], Proposition 2.28).

Note that the Crystallization Principle, as stated above, only holds in characteristic zero. Consider, for instance, the ideal (X^{2p}, Y^{2p}) in $K[X,Y]$ with $\operatorname{char} K = p \neq 2$. In this case $\mathrm{Gin}_{\mathrm{rlex}}(I)$ can be computed by observing that the ideal (X^{2p}, Y^{2p}) is the ideal generated by the images of X^2 and Y^2 under the Frobenius map $R \to R$, $X \to X^p$. In fact from the next result it follows that $\mathrm{Gin}_{\mathrm{rlex}}(I) = (X^{2p}, X^pY^p, Y^{3p})$.

Proposition 5.0.3. *Let I be a homogeneous ideal of $R = K[X_1, \ldots, X_n]$. Assume* $\operatorname{char} K = p$ *and let F be the Frobenius map. Then, for any term order τ one has*

$$\mathrm{Gin}_\tau(F(I)) = F(\mathrm{Gin}_\tau(I)).$$

Proof. Note that the computation of the initial ideal of $F(I)$ can be performed in $K[X_1^p, \ldots, X_n^p]$, i.e. the S-pairs of $F(I)$ are just the p-th power of the S-pairs of I, so that $F(\mathrm{in}_\tau(I)) = \mathrm{in}_\tau(F(I))$. This suffices, since by definition $\mathrm{Gin}_\tau(F(I)) = \mathrm{in}_\tau(g(F(I))) = \mathrm{in}_\tau(F(g(I)))$, where g is a generic change of coordinates. □

In [CS] we replaced the use of the crystallization principle by using the Bayer and Stillman criterion for detecting regularity. A posteriori this substitution seems quite natural, especially in the light of a result such as Proposition 3.2.17. This Proposition is a generalization of the Crystallization principle and a direct consequence of Bayer and Stillman criterion.

In this chapter we present a proof of the main result of [CS], (i.e. that (A) holds independently of the characteristic of the base field) quite different from the one in [CS]. The approach adopted in this chapter makes a more significant use of weakly stable ideals.

5.1 A Bound for the Castelnuovo-Mumford regularity in term of filter regular sections

Our goal in this section is to prove a bound for the regularity of an ideal by using the regularity of its hyperplane sections. This theorem appears first in [CS]. The proof we give in this chapter is simpler in nature and we belive that a deeper analysis of the combinatorics involved in this proof could give some improvements of the bounds themselves.

First of all, we need to state and prove a lemma which follows from Bayer and Stillman Criterion for detecting regularity.

Lemma 5.1.1. *Let $I \subseteq K[X_1, \ldots, X_n]$ be an ideal generated by homogeneous polynomials of degree less than or equal to d. Let l be a filter regular element for R/I. Then*

$$\operatorname{reg}(I) \leq \max\{\operatorname{reg}(I+l), d\} + \lambda((I:l)/I),$$

where λ denotes the length.

Proof. Set $\max\{\operatorname{reg}(I+l),d\}$ to be a and $\lambda((I:l)/I)$ to be b. Note that $((I:l)/I)_j = 0$ for some $j \in \{a, a+1, \ldots, a+b\}$ since the set of indexes has a cardinality bigger than b. We can use Bayer and Stillman criterion, Theorem 3.2.10, to get $\operatorname{reg}(R/I) \leq j-1 \leq a+b-1$. Therefore $\operatorname{reg}(I) \leq a+b$. □

Remark 5.1.2. The use of $\lambda((I:l)/I)$ to get the above bound does not seem the best possible choice. In fact it would be much better to consider the number of nonzero graded components of $(I:l)/I$ (maybe of degree higher than of equal to $\operatorname{reg}(I+l),d\}$). The trouble is that, in the technical part of our proof and especially in the one appearing in [CS], a bound for these numbers comes only from a bound for the length. We believe that the advantage of a combinatorial proof is that it leaves some hope to overcome these problems.

Before proving the next theorem we want to recall the notations we used in Chapter 4 Section 4.1. We denote by $G(I)$ the set of minimal monomial generators of a monomial ideal I, we set $\max\{i \text{ such that } X_i \mid u\}$ to be $m(u)$ and the value $\max\{j \text{ such that } X_i^j \mid u\}$ to be $|u|_i$. More generally we define $m(I)$ and $|I|_i$ to be $\max\{m(u) \text{ such that } u \in G(I)\}$ and $\max\{|u|_i \text{ such that } u \in G(I)\}$ respectively.

We need to prove first a technical lemma (see [CS]). In the following, given a monomial ideal I, we will denote by $I_{[i]}$ the contraction $I \cap K[X_1, \ldots, X_i]$.

Lemma 5.1.3. *Let I be a weakly stable ideal. Then $|I_{[i]}|_i = |I|_i$. Moreover setting c to be the greatest index for which $I_{[c]}$ (as an ideal of $K[X_1, \ldots, X_c]$) gives an Artinian algebra we have $|I|_i \leq \operatorname{reg} I_{[i]} - 1$ for all $n \geq i > c$.*

Proof. It is immediate that $|I_{[i]}|_i \leq |I|_i$. Suppose by contradiction that $|I_{[i]}|_i < |I|_i$. Let s be $|I_{[i]}|_i$. Then, there exists $u \in G(I)$ such that $X_i^{s+1} \mid u$ and $m(u) > i$. Choose such a counterexample in a way that $m(u)$ is the smallest possible. Because I

is weakly stable it follows that there exists a positive integer k such that the monomial $v = \frac{uX_i^k}{X_{m(u)}^{|u|_{m(u)}}}$ is in I. Hence there exists $p \in G(I)$ such that $p \mid v$ and $m(p) < m(u)$. Therefore, $|p|_i \leq s$, so that $p \mid uX_i^k$ and $p \nmid u$. But this implies that $s \geq |p|_i \geq |u|_i + 1$ which is greater than or equal to $s+2$ and this is impossible.

To show that $|I|_i \leq \operatorname{reg} I_{[i]} - 1$ for all $n \geq i > m$ it is enough to prove that $|I_{[i]}|_i$ is smaller than the maximum degree of a minimal generator of $I_{[i]}$. If this is not true it follows that $X_i^{a_i} \in I_{[i]}$ for some a_i. By the weakly stability of $I_{[i]}$ we deduce that $I_{[i]}$ gives an Artinian algebra, contradicting the choice of i. □

Theorem 5.1.4. *Let $I \subseteq K[X_1, \ldots, X_n]$ be an ideal of height c generated by homogeneous polynomials of degree less than or equal to d. Then, if $l_n, \ldots, l_{c+1}$ is a filter-regular sequence of linear forms, one has*

$$\operatorname{reg}(I) \leq \max\{d, \operatorname{reg}(I + (l_n))\} + \lambda(R/(I + (l_n, \ldots, l_{c+1}))) \prod_{i=c+2}^{n} \operatorname{reg}(I + (l_n, \ldots, l_i)).$$

Proof. Performing a change of coordinate we can assume that $l_i = X_i$ for all $n \geq i \geq c+1$. Moreover since the height of I is c we get that $X_n, \ldots, X_1$ is a filter regular sequence. By Lemma 5.1.1 it is enough to show that

$$\lambda((I : X_n)/I) \leq \lambda(R/(I + (X_n, \ldots, X_{c+1}))) \prod_{i=c+2}^{n} \operatorname{reg}(I + (X_n, \ldots, X_i)).$$

Let $J = \operatorname{in}_{\mathrm{rlex}}(I)$. Due to Lemma 4.1.12 we know that J is a weakly stable ideal and, moreover, $\lambda((I : X_n)/I) = \lambda((J : X_n)/J)$. Note that $\operatorname{in}_{\mathrm{rlex}}(I + (X_n, \ldots, X_i)) = J + (X_n, \ldots, X_i)$ for all i, hence by using Proposition 4.1.16 we get

$$\operatorname{reg} J_{[i-1]} = \operatorname{reg}(I + (X_n, \ldots, X_i)).$$

The Theorem is proved if we can show that

$$\lambda((J : X_n)/J) \leq \lambda(R/(J + (X_n, \ldots, X_{c+1}))) \prod_{i=c+1}^{n-1} \operatorname{reg}(J_{[i]}).$$

Let $\mathbf{X} = \{v_1, \ldots, v_s\}$ be the set of all monomials in $(J : X_n) \setminus J$. For all i, write $v_i = w_i X_n^{a_i}$ with $X_n \not| w_i$. Note that if $w_i = w_j$, by setting b to be the greatest exponent for which $w_i X_n^b \notin I$, we deduce $v_i = w_i X_n^b = w_j X_n^b = v_j$ and, therefore, $i = j$. Hence $|\mathbf{X}| = |\{w_1, \ldots, w_s\}|$. For all i, write $w_i = t_i u_i$ with $t_i \in K[X_1, \ldots, X_c]$ and $u_i \in K[X_{c+1}, \ldots, X_n]$. We can immediately observe that $|\{t_1, \ldots, t_s\}| \leq \lambda(R/(J + (X_n, \ldots, X_{c+1}))$. On the other hand, since $v_i X_n \in J$, by the weak stability of J, for all j there exist b_j for which $v_i X_j^{b_j} \in J$. Since $v_i \notin J$ we deduce $|v_j|_i \leq |J|_i$ which is less than $\text{reg}(J_{[i]})$ by Lemma 5.1.3. This shows that $|\{u_1, \ldots, u_s\}| \leq \prod_{i=c+1}^{n-1} \text{reg}(J_{[i]})$. Finally we get

$$|\mathbf{X}| = |\{w_1, \ldots, w_s\}| \leq \lambda(R/(J + (X_n, \ldots, X_{c+1}))) \prod_{i=c+1}^{n-1} \text{reg}(J_{[i]}).$$

□

Remark 5.1.5. Note that since $(R/(I + (X_n, \ldots, X_{c+1}))$ is Artinian and generated in degree less than or equal to d, its length is bounded by d^c.

5.2 Doubly exponential bound for the Castelnuovo-Mumford regularity

We are now ready to use Theorem 5.1.4 to show that the known bound in characteristic zero holds also in any characteristic. This follows as a straightforward recursive application of Theorem 5.1.4. Our approach, for this section, is the one of [CS]. The behavior of the regularity in some special case is outlined by the following remark.

Remark 5.2.1. Let $I \subset K[X_1, \ldots, X_n]$ be a homogeneous ideal generated in degree less than or equal to d. If the height of I is n, then I contains a complete intersection of forms of degree at most d, therefore, $\text{reg}(I) \leq n(d-1)+1$.

Furthermore, if I has height one then there exists a homogeneous polynomial f of degree $0 < a \leq d$ such that $I = (f)J$ and J is an ideal generated in degree $\leq d - a$. Thus, the ideal I is a shifted copy of J and $\operatorname{reg}(I) = \operatorname{reg}(J) + a$.

Theorem 5.2.2. *Let $I \subset K[X_1, \ldots, X_n]$ be an ideal of height $c < n$ and generated in degree $\leq d$. Then*

$$\operatorname{reg}(I) \leq (d^c + (d-1)c + 1)^{2^{n-c-1}}.$$

Proof. Let $l_n, \ldots, l_{c+1}$ be an almost-regular sequence of linear forms. By virtue of Theorem 5.1.4 we are able to compute a bound for the regularity of $I + (l_n, \ldots, l_i)$, $i \geq c+1$, in the following way. First we observe that the regularity of $I + (l_n, \ldots, l_i)$ equals that of its image $\bar{I}$ in $K[X_1, \ldots, X_{i-1}]$ by restriction. Moreover, the quotient algebra $R/(I + (l_n, \ldots, l_{c+1})) \simeq K[X_1, \ldots, X_c]/\bar{I}$ is Artinian and its regularity is bounded by $c(d-1)+1$. We set B_0 to be $(d-1)c+1$. Now we apply Theorem 5.1.4 to the image of $I + (l_n, \ldots, l_{c+2})$ in $K[X_1, \ldots, X_{c+1}]$ and we obtain that $\operatorname{reg}(I + (l_n, \ldots, l_{c+2})) \leq (d-1)c + 1 + d^c$. We set the latter to be B_1. For all $i \geq 2$ we define recursively B_i to be $B_{i-1} + \prod_{j=1}^{i-1} B_j$. It is easy to deduce that $B_i = (B_{i-1} - B_{i-2})B_{i-1} + B_{i-1} \leq (B_{i-1})^2$. Hence $B_i \leq (B_1)^{2^{i-1}}$ for all $i \geq 1$ and

$$\operatorname{reg}(I) \leq B_{n-c} \leq ((d-1)c + 1 + d^c)^{2^{n-c-1}},$$

as desired. □

The next corollary shows that formula (A) holds in general.

Corollary 5.2.3. *Let $I \subset K[X_1, \ldots, X_n]$ be an ideal generated in degree $\leq d$. If $n = 2$ then $\operatorname{reg}(I) \leq 2d - 1$ otherwise, for $n \geq 3$, we have*

$$\operatorname{reg}(I) \leq ((d^2 + 2d - 1)^{2^{n-3}} \leq (2d)^{2^{n-2}}.$$

Proof. The case $n = 2$ is easy. If $n \geq 3$, we have only to verify that the worst possible situation occurs when the height of I is 2. Since the bounds are decreasing as a function of c, this is equivalent to saying that the case height one is not the worst possible, and this follows by the discussion in Remark 5.2.1. □

Example 5.2.4. One could be interested in a slightly better estimate for the regularity and for this purpose could follow step-by-step the proof of Corollary 5.2.2.

Consider for instance the case of $n = 4$. As we said before, the worst possible case is provided by an ideal of height 2. Since $B_2 = (B_1 - B_0)B_1 + B_1$, we have that the regularity of a homogeneous ideal in $K[X_1, \ldots, X_4]$ is bounded by $((d^2 + 2d - 1) - (2d - 1))(d^2 + 2d - 1) + (d^2 + 2d - 1) = d^4 + 2d^3 + 2d - 1$.

Chapter 6

Bounds on the regularity of tensor product and Hom of modules

Let $R = K[X_1,\dots,X_n]$ be a polynomial ring over a field K, M a finitely generated graded R-module and $I \subset R$ and ideal. Recently some work has been done to study when the Castelnuovo-Mumford regularity of I^r can be bounded by r times the regularity of I and more generally when the regularity of IM can be bounded by the sum of the regularity of I and M. This is not always the case, see the papers of Sturmfels [St1], and Conca, Herzog [CH] for counterexamples. On the other hand, under the hypothesis that $\dim(R/I) \leq 1$, Chandler[Ch] and Geramita, Gimigliano and Pitteloud [GGP] showed that $\mathrm{reg}(I^r) \leq r\,\mathrm{reg}(I)$. In a recent paper Conca and Herzog [CH] proved, using similar methods to the one in [Ch] that, under the same assumption (i.e. $\dim(R/I) \leq 1$), $\mathrm{reg}(IM) \leq \mathrm{reg}(I) + \mathrm{reg}(M)$. An extension of the latter was recently done by Sidman [Si] who showed that if two ideals of R, say I and J, define schemes whose intersection is a finite set of points then $\mathrm{reg}(IJ) \leq \mathrm{reg}(I) + \mathrm{reg}(J)$. She deduced this theorem from a result in the same paper [Si] which bounded the regularity of a tensor product of sheaves.

In this chapter we show how the same technique as in [Si] can be applied to prove a stronger statement, i.e that given M and N graded R-modules such that $\dim \mathrm{Tor}_1^R(M,N) \leq 1$, then $\mathrm{reg}(M \otimes N) \leq \mathrm{reg}(M) + \mathrm{reg}(N)$. It easy to see that this result implies all the previous work mentioned above. This theorem has been recently applied by Daniel Giaimo [Gi] to prove the Eisenbud-Goto regularity conjecture for connected absolutely reduced curves. The results of this chapter can be found in [Ca1].

6.1 Castelnuovo-Mumford regularity and complexes of modules

In the following we will use the notion of partial Castelnuovo mumford regularity of a mofule M, with respect to a set of indeces $\mathcal{X}$, $\mathrm{reg}^{\mathcal{X}}(M)$, as defined in Chapter 3. The following lemma was inspired by Lemma 1.4 in [Si].

Lemma 6.1.1. *Let* $\mathbf{C}$ *be a complex of finitely generated graded R-modules*

$$\mathbf{C}:\ 0 \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \dots \longrightarrow C_0 \longrightarrow 0.$$

If C_i *is* $(m+i)$-$\mathrm{reg}^{\mathcal{X}+i}$ *for all* $i>0$ *and the* i^{th} *homology* $H_i(\mathbf{C})$ *is* $(m+i+1)$-$\mathrm{reg}^{\mathcal{X}+i+1}$ *for all* $i>0$ *then:*

(1) The i^{th} *boundary* B_i *is* $(m+i+1)$-$\mathrm{reg}^{\mathcal{X}+i+1}$ *for all* $i\geq 0$.

(2) If C_0 *is* m-$\mathrm{reg}^{\mathcal{X}}$ *then so is* $H_0(\mathbf{C})$.

If C_{n-i} *is* $(m-i)$-$\mathrm{reg}^{\mathcal{X}-i}$ *for all* $i\geq 0$ *and the* $(n-i)^{th}$ *homology* $H_{n-i}(\mathbf{C})$ *is* $(m-i-1)$-$\mathrm{reg}^{\mathcal{X}-i-1}$ *for all* $i>0$ *then:*

(1') The $(n-i)^{th}$ *cycles* Z_{n-i} *are* $(m-i)$-$\mathrm{reg}^{\mathcal{X}-i}$ *for all* $i\geq 0$.

(2') In particular $H_n(\mathbf{C})$ *is* m-$\mathrm{reg}^{\mathcal{X}}$.

Proof. First we prove (1). Note that when $i=n$ the set $\mathcal{X}+i+1$ is the empty set (see Remark 3.1.4) and in particular the n^{th} boundary B_n is trivially $(m+n+1)$-$\mathrm{reg}^{\mathcal{X}+n+1}$ since there are no conditions to check. We can therefore do a reverse induction on i. Consider the following diagram with exact rows and column:

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & & & & & \\
 & & & & \downarrow & & & & & & \\
0 & \longrightarrow & B_i & \longrightarrow & Z_i & \longrightarrow & H_i(\mathbf{C}) & \longrightarrow & 0 & & \\
 & & & & \downarrow & & & & & & \\
\dots & \longrightarrow & C_{i+1} & \longrightarrow & C_i & \longrightarrow & C_{i-1} & \longrightarrow & \dots & & \\
 & & & & \downarrow & & & & & & \\
 & & 0 & \longrightarrow & B_{i-1} & \longrightarrow & Z_{i-1} & \longrightarrow & H_{i-1}(\mathbf{C}) & \longrightarrow & 0. \\
 & & & & \downarrow & & & & & & \\
 & & & & 0 & & & & & &
\end{array}
$$

By induction we know that B_i is $(m+i+1)$-reg$^{\mathcal{X}+i+1}$ and by assumption $H_i(\mathbf{C})$ is $(m+i+1)$-reg$^{\mathcal{X}+i+1}$ so, applying Lemma 3.1.5 to the top exact row in the diagram above, we deduce that Z_i is $(m+i+1)$-reg$^{\mathcal{X}+i+1}$. Now, since C_i is $(m+i)$-reg$^{\mathcal{X}+i}$, applying Lemma 3.1.5 to the exact column of the diagram we obtain that B_{i-1} is $(m+i)$-reg$^{\mathcal{X}+i}$; this completes the induction and proves (1).

We now prove (2). Consider the exact sequence

$$0 \longrightarrow B_0 \longrightarrow C_0 \longrightarrow H_0 \longrightarrow 0.$$

By part (1) we know that B_0 is $(m+1)$-reg$^{\mathcal{X}+1}$ and by assumption C_0 is m-reg$^{\mathcal{X}}$ therefore from Lemma 3.1.5 follows H_0 is m-reg$^{\mathcal{X}}$.

The proof of (1') and (2') follow similar lines. Note that since $Z_n \cong H_n(\mathbf{C})$ it is sufficient to prove (1'). Moreover $Z_0 = C_0$ is $(m-n)$-reg$^{\mathcal{X}-n}$; we can therefore do a reverse induction on i. Apply Lemma 3.1.5 (2) to the last row in the diagram to get B_{n-i} is $(m-i)$-reg$^{\mathcal{X}-i}$ and then apply Lemma 3.1.5 (2) to the exact column to get Z_{n-i+1} is $(m-i+1)$-reg$^{\mathcal{X}-i+1}$. This complete the induction. □

6.1.1 Bounds on the regularity of the tensor product

In this section we outline some direct consequences of Lemma 6.1.1 (2). The first result is following:

Theorem 6.1.2. *Let M and N be finitely generated graded R-modules such that $\mathcal{X} = \{a,\ldots,n\}$, for $a \geq 0$, M is m-regular (i.e m-$\mathrm{reg}^{\{0,\ldots,n\}}$), N is s-$\mathrm{reg}^{\mathcal{X}}$ and $\mathrm{Tor}_i^R(M,N)$ is $(m+s+i+1)$-$\mathrm{reg}^{\mathcal{X}+i+1}$ for all $i > 0$. Then $M \otimes_R N$ is $(m+s)$-$\mathrm{reg}^{\mathcal{X}}$.*

Proof. Take a minimal graded free resolution $\mathbb{F} : \cdots \to F_i \to \cdots \to F_0$ of M. Note that since M is m-regular the lowest possible shift appearing in F_i is $-m-i$. Hence $F_i \otimes N$ is $(m+s+i)$-$\mathrm{reg}^{\mathcal{X}}$ and so in particular it is $(m+s+i)$-$\mathrm{reg}^{\mathcal{X}+i}$. The homologies of the complex $\mathbf{C} \otimes_R N$ are $\mathrm{Tor}_i^R(M,N)$, and by assumption they are $(m+s+i+1)$-$\mathrm{reg}^{\mathbb{X}+i+1}$, for $i > 0$. The conclusion follows from Lemma 6.1.1 part (2) applied to $\mathbb{F} \otimes N$ after noticing that $H_0(\mathbb{F} \otimes N)$ is $M \otimes N$. □

Remark 6.1.3. Note that the condition, "$\mathrm{Tor}_i^R(M,N)$ is $(m+s+i+1)$-$\mathrm{reg}^{\mathcal{X}+i+1}$", of Theorem 6.1.2 is clearly satisfied when the Krull dimension of $\mathrm{Tor}_i^R(M,N)$ is less than or equal to the minimum of $\mathcal{X}+i$ (since the relevant local cohomology modules are zero for reasons of dimension).

Setting $\mathcal{X} = \{0,\ldots,n\}$ (and noticing that by rigidity of Tor, see [An] Theorem 2.1, $\dim \mathrm{Tor}_1^R(M,N) \leq 1$ is equivalent to $\dim \mathrm{Tor}_i^R(M,N) \leq 1$ for all $i \geq 1$) we have the following corollary:

Corollary 6.1.4. *Let M be an m-regular finitely generated graded R-module and N be an n-regular finitely generated graded R-module such that $\dim \mathrm{Tor}_1^R(M,N) \leq 1$. Then $M \otimes N$ is $(m+n)$-regular.*

From Corollary 6.1.4 we can deduce:

Theorem 6.1.5. *Let $I \subseteq R$ be an homogeneous ideal and M a finitely generated graded R-module such that the dimension of* $\operatorname{Tor}_1^R(M, R/I)$ *is less than or equal to 1. Then* $\operatorname{reg}(IM) \leq \operatorname{reg}(I) + \operatorname{reg}(M)$.

Proof. First note that unless I is the whole ring (in which case the result is obvious), we can assume that $\operatorname{reg}(I) > 0$. From the exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

we get $\operatorname{reg}(R/I) = \operatorname{reg}(I) - 1$. By Corollary 6.1.4 $\operatorname{reg}(M/IM) = \operatorname{reg}(M \otimes_R R/I) \leq \operatorname{reg}(M) + \operatorname{reg}(I) - 1$. On the other hand, applying Lemma 3.1.5 (2) to the following exact sequence:

$$0 \longrightarrow IM \longrightarrow M \longrightarrow M/IM \longrightarrow 0,$$

we obtain $\operatorname{reg}(IM) \leq \max\{\operatorname{reg}(M), \operatorname{reg}(M/IM) + 1\}$ which is less than or equal to

$$\max\{\operatorname{reg}(M), \operatorname{reg}(M) + \operatorname{reg}(I) - 1 + 1\} \leq \operatorname{reg}(M) + \operatorname{reg}(I).$$

□

Theorem 6.1.5 implies the following:

Theorem 6.1.6 (Conca, Herzog Theorem 2.5 [CH])**.** *Let $I \subset R$ be an homogeneous ideal with* $\dim R/I \leq 1$ *and M a finitely generated graded R-module. Then* $\operatorname{reg}(IM) \leq \operatorname{reg}(I) + \operatorname{reg}(M)$.

Theorem 6.1.7 (Sidman Theorem 1.8 [Si])**.** *Let I, J be homogeneous ideals of R such that the dimension of $R/(I+J)$ is less or equal to 1. Then* $\operatorname{reg}(IJ) \leq \operatorname{reg}(I)$+$\operatorname{reg}(J)$.

Remark 6.1.8. The condition on the Krull dimension of $\operatorname{Tor}_1^R(M,N)$ (or, in the above theorem, on $R/(I+J)$) cannot be made weaker. For example one can consider a variation of Example 4.2.1. Set $R=[X_1,\dots,X_5]$, $I=(X_1^d,X_2^d,X_1X_3^{d-1}-X_2X_4^{d-1}+X_5^d)$ and $J=X_5$. Note that I is a complete intersection, hence $\operatorname{reg}(I)=3d-2$. On the other hand, by Example 4.2.1, we know that $\operatorname{reg}(I+J)=d^2-1$, which is bigger than $\operatorname{reg}(I)+\operatorname{reg}(J)=3d-1$ for $d\geq 4$. In this situation $\dim R/(I+J)=\dim \operatorname{Tor}_1^R(R/I,R/J)=2$. Moreover this example shows that for a fixed ring, in this case $K[X_1,\dots,X_5]$, the regularity of a special linear hyperplane section cannot be bounded uniformly by $a\operatorname{reg}(I)+b$ where a and b depend only on the ring and I is any homogeneous ideal. Some possible questions to ask are the following:

(1) Is it possible to find a,b,c depending on a fixed ideal $J\subset R$ such that $\operatorname{reg}(I+J)\leq a(reg(I))^c+b$ uniformly for any ideal I of R. ?

(2) Assume that J is generated by just a linear form. Is it possible to find a,b,c such that $\operatorname{reg}(I+J)\leq a(reg(I))^c+b$ uniformly for any ideal I of any polynomial ring?

6.1.2 Bounds on the regularity of $\operatorname{Hom}_R(M,N)$

Similar reasoning as in Theorem 6.1.2 can be used to prove a bound for the regularity of $\operatorname{Hom}_R(M,N)$ where M and N are finitely generated graded R-modules. In this context the dimensional condition required of $\operatorname{Tor}_1^R(M,N)$ has an analogue in certain conditions on the depth of $\operatorname{Ext}_R^i(M,N)$.

We prove the following:

Theorem 6.1.9. *Let M and N be finitely generated graded R-modules. Let m be the lowest degree of a homogeneous minimal generator for M, and let $X=$*

$\{0,\ldots,a\}$, $a \leq n$ *be a set of indices. If* N *is* s-reg$^{\mathcal{X}}$ *and* $\mathrm{Ext}^i_R(M,N)$ *is* $(s-m-i-1)$-reg$^{\mathcal{X}-i-1}$ *for all* $i>0$*, then* $\mathrm{Hom}_R(M,N)$ *is* $(s-m)$-reg$^{\mathcal{X}}$.

Proof. Take a minimal graded free resolution $\mathbb{F}: \cdots \to F_i \to \cdots \to F_0$ of M. Note that, since the lowest degree of a homogeneous minimal generator for M is m, the biggest possible shift appearing in F_i is less than or equal to $-m-i$. Hence $\mathrm{Hom}_R(F_i,N)$ is $(s-m-i)$-reg$^{\mathcal{X}}$ so in particular it is $(s-m-i)$-reg$^{\mathcal{X}-i}$. The homologies of the complex $\mathrm{Hom}_R(\mathbb{F},N)$ are $\mathrm{Ext}^i_R(M,N)$, and by assumption they are $(s-m-i-1)$-reg$^{\mathbb{X}-i-1}$ for all $i>0$. The conclusion follow from Lemma 6.1.1 part (2') applied to $\mathrm{Hom}_R(\mathbb{F},N)$ after noticing that $H_n(\mathrm{Hom}_R(\mathbb{F},N))$ is $\mathrm{Hom}_R(M,N)$. □

Remark 6.1.10. The condition: "$\mathrm{Ext}^i_R(M,N)$ is $(s-m-i-1)$-reg$^{\mathcal{X}-i-1}$ for all $i>0$" of Theorem 6.1.9 is obtained for example when $\operatorname{depth}\mathrm{Ext}^i_R(M,N)$ is greater than or equal to $n-i$ for all $i>0$, because in this case $H^j_{R_+}(\mathrm{Ext}^i_R(M,N))=0$ for $j<n-i-1$. On the other hand, since for any prime ideal P of $\operatorname{ht}P<i$, $\mathrm{Ext}^i_R(M,N)_P=0$, we have $\dim\mathrm{Ext}^i_R(M,N)\leq n-i$. Therefore $\operatorname{depth}\mathrm{Ext}^i_R(M,N)\geq n-i$ if and only if $\mathrm{Ext}^i_R(M,N)$ is Cohen-Macaulay.

Hence we have the following result analogous to Theorem 6.1.4.

Theorem 6.1.11. *Let* M *be a finitely generated graded* R*-module with* m *the lowest degree of a homogeneous minimal generator of* M*, and let* N *be a finitely generated graded* R*-module such that* $\mathrm{Ext}^i_R(M,N)$ *is Cohen-Macaulay for all* $i>0$. *Then* $\mathrm{reg}(\mathrm{Hom}_R(M,N))\leq \mathrm{reg}(N)-m$.

Chapter 7

Initial ideals, Lex-segments ideals and inequalities on Tor

The starting point for this chapter is an article by Aldo Conca [Co] where he proves, among other things, the next two interesting inequalities on Tor .

Let K be an infinite field. Given a term order τ on $R = K[X_1,\dots,X_n]$ and a homogeneous ideal I we denote by $gin_\tau(I)$ the generic initial ideal with respect to τ and by I^{lex} the only lex-segment ideal of R with the same Hilbert function of I. Conca proved the result below.

Theorem. *(Conca) Let I be a homogeneous ideal of $R = K[X_1,\dots,X_n]$, let $|K| = \infty$ and $r \le n$. Then for any term order τ and any generic linear forms $l_1,\dots,l_r$ we have*

$$\dim_K \operatorname{Tor}_i(R/I, R/(l_1,\dots,l_r))_j \le \dim_K \operatorname{Tor}_i(R/gin_\tau(I), R/(l_1,\dots,l_r))_j. \quad (7.0.1)$$

Moreover, if $\operatorname{char}(K) = 0$*, then*

$$\dim_K \operatorname{Tor}_i(R/I, R/(l_1,\dots,l_r))_j \le \dim_K \operatorname{Tor}_i(R/I^{\text{lex}}, R/(l_1,\dots,l_r))_j. \quad (7.0.2)$$

The first section of this chapter is devoted to the improvement of the formula (7.0.1). More precisely, we substitute $\operatorname{gin}_\tau(I)$ for $\operatorname{in}_\tau(I)$ and we show that the result is still true even if $l_1,\dots,l_r$ are generic forms of some fixed degrees $d_1,\dots,d_r$. Moreover, with this approach r needs not to be smaller than or equal to n.

The second section is motivated by the following conjecture, although the goal of proving this inequality was not reached.

Conjecture 7.0.12. In any characteristic and for any r,

$$\dim_K \operatorname{Tor}_i(R/I, R/(l_1,\dots,l_r))_j \leq \dim_K \operatorname{Tor}_i(R/I^{\text{lex}}, R/(l_1,\dots,l_r))_j$$

where $l_1,\dots,l_r$ are generic forms of certain fixed degrees $d_1,\dots,d_r$.

The strategy that seems reasonable to us is to analyze the proof of a result of Pardue, [Pa1], where he shows the above formula when $(l_1,\dots,l_r)$ is the homogeneous maximal ideal. In other words, Pardue proved that a lex-segment ideal has the biggest graded Betti numbers among all the homogeneous ideals with the same Hilbert function. It is important to mention that this *extremality* of the lex-segment ideal was first proved in characteristic zero, using a different method, by Bigatti and independently by Hulett.

Our hope is that with a better understanding of Pardue's techniques, and by using the methods of the first section of this chapter, one could prove the above conjecture.

The proof of Pardue is based partially on the idea that by performing a certain sequence of operations (polarizations, generic specializations, and taking initial ideal) one can transform a monomial ideal into its corresponding lex-segment ideal. We study this phenomenon very closely, and in particular we give a proof of Pardue's theorem where the polarizations we consider are just partial. Moreover the specializations are done precisely with the aim of making a new specific monomial appear in the ideal. This proof is similar, in a certain way, to a sequence of little surgical operations for improving the ideal.

In our proof, given a homogeneous ideal I, we do not assume the well known existence, according to Macaulay, of I^{lex}. However we obtain such an existence by the same proof. Note also that, given a finite number of ideals and a term order τ, their initial ideals with respect to τ can be obtained as the initial ideals with respect to a weight function ω (see for example [St2] or [Gr1]). Thus, by Lemma 1.2.1 we have:

Lemma 7.0.13. *Let $R = K[X_1,\dots,X_n]$. Consider a term order τ and homogeneous ideals I,J,H such that $I \subseteq J$ and $I \subseteq H$. Then*

$$\dim_K \operatorname{Tor}_i^{R/I}(R/J,R/H)_j \leq \dim_K \operatorname{Tor}_i^{R/\operatorname{in}_\tau I}(R/\operatorname{in}_\tau J,R/\operatorname{in}_\tau H)_j.$$

Using Lemma 7.0.13 we get a different proof of a result due to Pardue and Iyengar (see [IP]). More precisely we obtain:

Theorem 7.2.14. *Let $R = K[X_1,\dots,X_n]$ and let I,J be homogeneous ideals such that $I \subseteq J$. Then*

$$\dim_K \operatorname{Tor}_i^{R/I}(R/J,K)_j \leq \dim_K \operatorname{Tor}_i^{R/I^{\text{lex}}}(R/J^{\text{lex}},K)_j.$$

7.1 Initial ideals and inequalities on Tor's

In this section we will use the notion of weight function discussed in the Section 1.2. Note that, as we said above, given a finite number of ideals $I_1,\dots,I_r$ of $R = K[X_1,\dots,X_n]$ and a given term order τ it is possible to find a weight function $w = (w_1,\dots,w_n)$ from $\mathbb{Z}^n$ to $\mathbb{Z}$ such that $\operatorname{in}_\tau(I_i) = \operatorname{in}_w(I_i)$ for all i. Let $A = R[T]$ be the polynomial ring in one variable over R. In Section 1.2 for any $f \in R$ we defined $\tilde{f}$ as $T^a f(T^{-w_1}X_1,\dots,T^{-w_n}X_n)$, where a is the maximum weight of a monomial in the support of f. We defined $\tilde{I}$ to be the ideal of A generated by the elements $\tilde{f}$ for all $f \in I$.

Remark 7.1.1. From the definition it follows that $A/((T)+\tilde{I}) \cong R/\mathrm{in}_\tau(I)$ and $A/((T-1)+\tilde{I}) \cong S/I$. Moreover, the above isomorphisms are obtained just by specializing T at 0 and at 1 respectively. In general, for any $c \in K$ with $c \neq 0$ the ideal $\tilde{I}$ restricted at $T-c=0$ is just $D(I) \subset R$, where D is the change of coordinates on R induced by the diagonal matrix with diagonal $(c^{-w_1},\dots,c^{-w_n})$.

Proposition 7.1.2. *Let I be a homogenous ideal of $R=K[X_1,\dots,X_n]$. Assume that $|K|=\infty$, and let $\mathbf{I}$ be a set of ideals of R such that for any $H \in \mathbf{I}$ and any diagonal matrix $D \in Gl_n(K)$ we have $D(H) \in \mathbf{I}$. Given two integers i and j, let $J \in \mathbf{I}$ be an ideal such that*

$$\dim_K \mathrm{Tor}_i(R/I,R/J)_j = \min_{H \in \mathbf{I}} \dim_K \mathrm{Tor}_i(R/I,R/H)_j.$$

Then for any term order τ we have

$$\dim_K \mathrm{Tor}_i(R/I,R/J)_j \leq \dim_K \mathrm{Tor}_i(R/\mathrm{in}_\tau(I),R/J)_j.$$

Proof. Consider a weight function $w=(w_1,\dots,w_n)$ such that $\mathrm{in}_w(I)=\mathrm{in}_\tau(I)$. Let $I_{T=a}$ with $a \in K$ be the ideal of R obtained from $\tilde{I}$ by setting $T=a$. As we mentioned above, if $a=0$ then $I_{T=0}=\mathrm{in}_\tau(I)$ otherwise for $a \neq 0$ we have $I_{T=a}=D_a(I)$ where D_a is the diagonal matrix with diagonal $(a^{-w_1},\dots,a^{-w_n})$. We can calculate $\dim_K \mathrm{Tor}_i(R/I_{T=a},R/J)_j$ by taking a resolution of R/J and then tensoring with $R/I_{T=a}$. It is easy to see that we can find two matrices, say A_a and B_a, whose coefficients are rational functions of a such that $\dim_K \mathrm{Tor}_i(R/I_{T=a},R/J)_j = \dim_K(\ker A_a)-\dim_K(\mathrm{im}\, B_a)$. Therefore, there exists a non-empty Zariski open set $U \subset \mathbb{A}^1_K$ such that for any $a \in U$ the value $\dim_K(\ker A_a)-\dim_K(\mathrm{im}\, B_a)$ is constant and minimum. We have, for any $a \in U$ and $a \neq 0$, that:

$$\dim_K \mathrm{Tor}_i(R/\mathrm{in}_\tau(I),R/J)_j = \dim_K \mathrm{Tor}_i(R/I_{T=0},R/J)_j \geq \dim_K \mathrm{Tor}_i(R/I_{T=a},R/J)_j.$$

Since $I_{T=a} = D_a(I)$, we can use the change of coordinates induced by D_a^{-1} and obtain:

$$\dim_K \operatorname{Tor}_i(R/I_{T=a}, R/J)_j = \dim_K \operatorname{Tor}_i(R/I, R/D_a^{-1}(J))_j \geq \dim_K \operatorname{Tor}_i(R/I, R/J)_j,$$

where the last inequality depends on the fact that $D_a^{-1}(J) \in \mathbf{I}$ and on the choice of the ideal J. □

An immediate consequence of Proposition 7.1.2 is the next result, which provides an extension of formula 7.0.1.

Corollary 7.1.3. *Let I be a homogeneous ideal of $R = K[X_1, \dots, X_n]$. Assume that $|K| = \infty$ and let J be an ideal generated by r generic forms of degree $d_1, \dots, d_r$. Then for any i, j and any term order τ we have:*

$$\dim_K \operatorname{Tor}_i(R/I, R/J)_j \leq \dim_K \operatorname{Tor}_i(R/\operatorname{in}_\tau(I), R/J)_j.$$

Proof. We apply Proposition 7.1.2 by using as $\mathbf{I}$ the set of all the homogeneous ideals of R generated by r forms of degree $d_1, \dots, d_r$.We just have to note that since the forms generating J are generic we have:

$$\dim_K \operatorname{Tor}_i(R/I, R/J)_j = \min_{H \in \mathbf{I}} \dim_K \operatorname{Tor}_i(R/I, R/H)_j$$

for all i and j. □

An important remark to mention is that Corollary 7.1.3 is just a possible way to apply Proposition 7.1.2. For example, a similar result to Corollary 7.1.3 can be obtained by considering any ideal constructed, in a given way, by using sums and products of generic forms of fixed degrees. For instance let $I_1 = (f_1, \dots, f_a)$ and $I_2 = (g_1, \dots, g_b)$ be two ideals generated by generic forms of degrees $d_1, \dots, d_a$

and $h_1, \ldots, h_b$ respectively. Then, setting $J = I_1 I_2$, the formula of Corollary 7.1.3 still holds. In fact, we can employ Proposition 7.1.2 by using as **I** the set of all the homogeneous ideals of R obtained as product of two ideals: one generated by a forms of degrees $d_1, \ldots, d_a$ and one generated by b forms of degree $g_1, \ldots, g_b$.

7.2 Pardue's method and Macaulay estimate on the Hilbert function of standard graded algebras

The aim of this section is to show how to use Pardue's idea [Pa1] to derive at once his result and the well-known Macaulay estimate on the Hilbert function of standard graded algebra. Before going into the details of this proof, we give a brief introduction to polarizations (which sometimes are called distractions) and generic specializations. See [Pa1], [BH] Lemma 4.2.16, and [BCR] for further details.

7.2.1 Polarizations and specializations of monomial ideals

In the literature, polarizations are often used to obtain from a monomial ideal, a new one which is square-free. For example, given $I = (X_1^3, X_1^2 X_2^2) \subset R = K[X_1, X_2]$ by polarizing we get the ideal $J = (T_1 T_2 T_3, T_1 T_2 T_4 T_5) \subseteq S = K[X_1, X_2, T_1, \ldots, T_5]$. This ideal is square-free and encodes all the information of I, in the sense that there exists a regular sequence for S/J , precisely: $T_1 - X_1, T_1 - X_2, T_3 - X_3, T_4 - X_2, T_5 - X_2$, such that $I = (J + (T_1 - X_1, T_1 - X_2, T_3 - X_3, T_4 - X_2, T_5 - X_2)) \cap R$. In other words, by substituting back X_1 for T_1 and so on, we obtain the ideal I.

This is the kind of polarization considered by Pardue. For our purposes we prefer to use certain *partial polarizations* that do not modify the original ideal too much.

For instance, let $I \subseteq R = K[X_1,\dots,X_n]$. Choose a distinguished variable, say X_a, and fix an exponent, say b. Let S be the polynomial ring obtained from R by adjoining the new variable $T_{(a,b)}$. Let $m_1,\dots,m_s$ be the minimal system of monomial generators of J. If m_i is divisible by X_a^b define $n_i = (m_i/X_a)T_{(a,b)}$, otherwise set $n_i = m_i$. The polarization of I obtained by the choice of (a,b) is the ideal $J = (n_1,\dots,n_s) \subset S$. From J it is possible to re-obtain I as $I = (J + (T_{(a,b)} - X_a)) \cap R$. Moreover $T_{(a,b)} - X_a$ is a regular element for S/J.

For example, the polarization of $I = (X_1^3, X_1^2X_2^2) \subset R = K[X_1,X_2]$, with respect to the variable X_1 and the exponent 3 (i.e. the pair $(1,3)$), is

$$(X_1^2T_{(1,3)}, X_1^2X_2^2) \subset S = K[X_1,X_2,T_{(1,3)}].$$

We want to describe these polarizations in a slightly greater generality.

Definition 7.2.1. Let $I \subset R = K[X_1,\dots,X_n]$ be a monomial ideal and let $G(I)$ be the set of its minimal monomial generators. Define Q to be the following set of integers pairs: $\{(i,j)|0 \le i \le n, 0 < j\}$ and let $P \subset Q$ be a finite subset. Let $R[P]$ be the polynomial ring over R defined as $R[T_{(i,j)}|(i,j) \in P]$. Given a monomial $m \in R$ we set the monomial $P(m) \in R[P]$ to be:

$$m \cdot \left(\prod_{\substack{(i,j)\in P \text{ and} \\ X_i^j | m}} \frac{T_{(i,j)}}{X_i} \right).$$

The *polarization of I with respect to P* is defined as

$$P(I) = (P(m)|m \in I).$$

It is easy to verify that $P(I)$ is also equal to $(P(m)|m \in G(I))$.

We recall some facts about polarizations that are quite easy to prove.

Remark 7.2.2. Let $P \subset \{(i,j)|0 \leq i \leq n, 0 < j\}$ be a polarization and let I and J be monomial ideals.

(1) Note that, by definition, if $I \subset J$ then $P(I) \subset P(J)$.

(2) Each polarization can be factored as a composition of simpler ones. In fact it is easy to find a sequence of polarizations $P_1, \dots, P_s$ each of them consisting of only one pair, such that $P(I) = P_s \circ P_{s-1} \circ \cdots \circ P_1(I)$.

(3) In general, $P(I+J) = P(I) + P(J)$ and $P(I \cap J) = P(I) \cap P(J)$.

(4) The elements $T_{(i,j)} - X_i$, for all $(i,j) \in P$, form a regular sequence for the ring $R[P]/P(I)$. For a proof of this fact one can read, for example, the proof of Lemma 4.2.16. in [BH]. Another possible way to see this fact is the following: write $I = \cap I_i$ where the I_i are irreducible monomial ideals, i.e. generated by pure powers of the variables. By part (2) we can assume that the polarization consists of only one pair, say (a,b). By part (3) we know that $P(I) = \cap P(I_i)$. Each one the $P(I_i)$ has at most two associated primes, and clearly $T_{(a,b)} - X_a$ does not belongs to any of these. Therefore, $T_{(a,b)} - X_a$ is not contained in any prime associated to $P(I)$.

Remark 7.2.3. For any of the elements $T_{(i,j)} - X_i$ choose a variable, say $X_{l_{(i,j)}}$. Since K is an infinite field, the elements $T_{(i,j)} - X_i - \lambda_{(i,j)} X_{l_{(i,j)}}$, where the $\lambda_{(i,j)}$ are generic elements in K, are also a regular sequence for $R[P]/P(I)$. Here generic means that there exists a non-empty Zariski open set of $U \subset \mathbb{A}^r_K$ such that the $\lambda_{(i,j)}$ can be taken to be the entries of any point in U. This open set is, for example, the one at which the Hilbert function of $P(I) + (T_{(i,j)} - X_i - \lambda_{(i,j)} X_{l_{(i,j)}})$ is maximal and constant. We will denote the ideal $(T_{(i,j)} - X_i - \lambda_{(i,j)}X_{l_{(i,j)}})$ by W, and by $P(I)_W$ the ideal $(P(I)+(W))\cap R$, obtained by polarizing and then by specializing using W.

Remark 7.2.4. The idea of Pardue is partially based on the isomorphisms below. Let P be a polarization and let $S = R[P]$. Set $\mathbf{m}$ and $\mathbf{n}$ to be the homogeneous maximal ideals of R and S respectively. Let V be the ideal of S generated by the $T_{(i,j)} - X_i$'s , and let W be the ideal of S generated by the $T_{(i,j)} - X_i - \lambda_{(i,j)}X_{l_{(i,j)}}$'s . We have:

$$\mathrm{Tor}_i^{R/I}(R/J, R/\mathbf{m}) \cong \mathrm{Tor}_i^{S/(P(I)+(V))}(S/(P(J)+(V)), S/(\mathbf{m}+(V)).$$

Since $(\mathbf{m}+(V)) = \mathbf{n}$ and V is generated by a regular sequence for both $S/P(I)$ and $S/P(J)$, we get:

$$\mathrm{Tor}_i^{S/(P(I)+(V))}(S/(P(J)+(V)), S/(\mathbf{m}+(V)) \cong \mathrm{Tor}_i^{S/P(I)}(S/P(J), S/(\mathbf{n})).$$

Now, because we also have that $\mathbf{n} = (\mathbf{m}+(W))$, in the same way, we deduce that $\mathrm{Tor}_i^{S/P(I)}(S/P(J), S/(\mathbf{n})) \cong \mathrm{Tor}_i^{S/(P(I)+(W))}(S/(P(J)+(W)), S/(\mathbf{m}+(W))$. By denoting $P(I)_W$ and $P(J)_W$ the ideals $(P(I)+(W))\cap R$ and $(P(J)+(W))\cap R$ we obtain:

$$\mathrm{Tor}_i^{R/I}(R/J, K) \cong \mathrm{Tor}_i^{R/P(I)_W}(R/P(J)_W, K).$$

Taking the initial ideal with respect to the lexicographic term order: lex, and applying Lemma 7.0.13, we deduce:

$$\dim_K \mathrm{Tor}_i^{R/I}(R/J, K)_j \leq \dim_K \mathrm{Tor}_i^{R/\,\mathrm{in}_{\mathrm{lex}}(P(I)_W)}(R/\,\mathrm{in}_{\mathrm{lex}}(P(J)_W), K)_j. \quad (7.2.1)$$

Remark 7.2.5. An important point to make is that a polarization followed by a specialization is, in a certain way, similar to performing a change of coordinates (actually any change of coordinate can be obtained in this way), with the advantages that the characteristic of the base field does not create too many problems.

For example, let $I = (X_1^2, X_2^2) \subset K[X_1, X_2]$ with $\mathrm{char}(K) = 2$. This ideal is fixed under any change of coordinates. On the other hand, the polarization $P = \{(2,1)\}$ followed by the specialization $W = (T_{(2,1)} - X_2 - \lambda_{(2,1)} X_1)$ gives the ideal $P(I)_W = (X_1^2, X_2(X_2 + \lambda_{(2,1)} X_1))$ and in particular $\mathrm{in}_{\mathrm{lex}}(P(I)_W) = (X_1^2, X_1X_2, X_2^3)$.

7.2.2 *A total order on the monomial ideals*

We now want to consider a total order on the monomial ideals of R induced by a given term order τ. It will be useful to compare, using τ, monomials of different degree. For this purpose set $m > n$ whenever $\deg(m) < \deg(n)$.

Definition 7.2.6. Let I and J be two monomial ideals. Let $m_1 > m_2 > \cdots > m_r$ and $n_1 > n_2 > \cdots > n_s$ be the minimal monomial generators, ordered with respect to τ, of I and J respectively. We say that $I >_\tau J$ if $I \subsetneq J$ or $m_1 = n_1, \ldots, m_i = n_i$ and $m_{i+1} > n_{i+1}$.

Lemma 7.2.7. *An increasing sequence of monomial ideals, with respect to the total order above, eventually stabilizes.*

Proof. Given a monomial ideal I we denote by $I_{|i}$ the ideal generated by the i greatest minimal monomial generators of I. When the number of minimal generators of I is less than i we define $I_{|i} = I$.

Let $I_1 \leq I_2 \leq \ldots$ be a sequence of monomial ideals. For a given monomial m there are only a finite number of monomials greater than m and therefore $(I_1)_{|1} \leq (I_2)_{|1} \leq (I_3)_{|1} \leq \ldots$ eventually stabilizes. By induction we can assume that $(I_1)_{|a} \leq (I_2)_{|a} \leq (I_3)_{|a} \leq \ldots$ stabilizes, say at $(I_b)_{|a}$. As above, we observe that there are only a finite number of monomials greater than the $(a+1)^{\mathrm{th}}$ generator of I_b and, therefore $(I_1)_{|a+1} \leq (I_2)_{|a+1} \leq (I_3)_{|a+1} \leq \ldots$ also becomes stable.

We have constructed the chain $J_1 \subseteq J_2 \subseteq J_3 \subseteq \dots$ of the ideals J_i at which the restriction to the first i generators becomes stable. Since R is Noetherian the chain of the J_i's also becomes stable and so does the one of the I_i's . □

Before proving the next lemma it is useful to discuss a different way to compare two monomial ideals by using $>_\tau$.

Remark 7.2.8. Let I and J be two monomial ideals such that none of them contains another. Order decreasingly all the monomials in I, and respectively in J, by τ. In this way we obtain two sequences of monomial $\{m\}_{i\in\mathbb{N}}$ and $\{n\}_{i\in\mathbb{N}}$. Then $I >_\tau J$ if and only if $m_1 = n_1, \dots, m_j = n_j$ and $m_{j+1} > n_{j+1}$ for some j.

Consider now the task of comparing the initial ideals of two given homogeneous ideals I and J of $R = K[X_1, \dots, X_n]$. First of all, note that a possible way to describe all the monomials of $\text{in}(I)$ of degree d is the following. Let $f_1, \dots, f_r$ be a system of generators for the polynomials in I of degree d. Let $t_1 > t_2 > \dots > t_s$ be all the monomials of degree d of R. Write any f_i as $\sum_i^s a_{i,j} t_j$, and let A be the matrix given by the $a_{i,j}$. Define $r_{(d,j)}$ to be the rank of the submatrix consisting of the first j columns of A. Then $t_j \in (\text{in}_\tau I)_d$ if and only if $r_{(d,j-1)} < r_{(d,j)}$. Given a homogeneous ideal I we can construct the sequence $\{p_i\}_{i\in\mathbb{N}}$ of non-negative integers consisting of all the ranks:

$$r_{(1,1)}, \dots, r_{(1,dimR_1)}, r_{(2,1)}, \dots, r_{(2,dimR_1)} \cdots, r_{(d,1)}, \dots, r_{(d,dimR_d)} \cdots$$

Similarly, given another homogenous ideal J we can construct its sequence of the ranks: $\{q_i\}_{i\in\mathbb{N}}$. Assuming there is no containment between $\text{in}_\tau(I)$ and $\text{in}_\tau(J)$ we have $\text{in}_\tau(I) >_\tau \text{in}_\tau(J)$ if and only if $p_1 = q_1, \dots, p_j = q_j$ and $p_{j+1} > q_{j+1}$ for some index j.

Lemma 7.2.9. *Let I be a monomial ideal and τ be a term order. For any polarization P and any generic specialization $W = ((T_{(i,j)} - X_i - \lambda_{(i,j)} X_{l_{(i,j)}}) | (i,j) \in P)$ we have:*

$$I \leq_\tau \mathrm{in}_\tau(P(I)_W).$$

Proof. First of all note that $\mathrm{in}_\tau(P(I)_W)$ is constant for a generic choice of the coefficients $\lambda_{(i,j)}$. To simplify the notation let $J = \mathrm{in}_\tau(P(I)_W)$. Since I and J have the same Hilbert function it is impossible that one ideal is strictly contained in another. Using the notation of the above Remark 7.2.8, let $\{p_i\}_{i\in\mathbb{N}}$ and $\{q_i\}_{i\in\mathbb{N}}$ be the sequences of ranks of I and J respectively. If, by contradiction, $I >_\tau J$ then $p_1 = q_1, \ldots, p_j = q_j$ and $p_{j+1} > q_{j+1}$ for some index j. If all the $\lambda_{(i,j)}$ were zero we would have $p_{j+1} = q_{j+1}$. On the other hand, choosing generic $\lambda_{(i,j)}$, the ranks q_{j+1} can only increase and therefore $p_{j+1} \leq q_{j+1}$. This contradicts our assumption. □

7.2.3 Results of Macaulay and Pardue

We are finally ready to prove the Theorems of Macaulay and Pardue.

Definition 7.2.10. We say that a monomial ideal I is a *lex-segment ideal* if, in any degree d, the vector space I_d is spanned by a lex-segment, i.e. the greatest $\dim_K(I_d)$ monomials of degree d with respect to the lexicographic term order.

The next lemma is the keystone of the whole chapter.

Lemma 7.2.11. *Let $I \subseteq R = K[X_1, \ldots, X_n]$ be a monomial ideal, which is not a lex-segment ideal. Then there exists a polarization P and a generic specialization W such that* $\mathrm{in}(P(I)_W) >_{\mathrm{lex}} I$.

Proof. By Lemma 7.2.9 it is enough to show that there exists a polarization P and a generic specialization W such that $\mathrm{in}(P(I)_W) \neq I$. We consider two cases.

First, assume that I is not strongly stable. Then there exists a monomial m and variables X_i and X_j with $j < i$ such that $X_i | m$ but $(m/X_i)X_j \notin I$. Set $P = \{(i,1)\}$ and let the generic specialization be $W = (T_{(i,1)} - X_i - \lambda_{(i,1)}X_j)$. The ideal $P(I)_W$ contains $(m/X_i)(X_i + \lambda_{(i,1)}X_j)$ and, therefore, $(m/X_i)X_j \in \mathrm{in}_{\mathrm{lex}}(P(I)_W)$.

Second, assume that I is strongly stable but not a lex-segment ideal. Let $m_1 > m_2 > \cdots > m_r$ be the minimal monomial generators of I ordered reverse lexicographically. Since I is not a lex-segment ideal there exists a monomial m_s such that $(m_1, m_2, \ldots, m_{s-1})$ is a lex-segment ideal while $J = (m_1, m_2, \ldots, m_s)$ is not. By Remark 7.2.2 part (1), without loss of generality, we can assume that $I = J$. Let X_i be the lowest variable dividing m_s and write $m_s = mX_{i-1}^a X_i^b$, with m belonging to $K[X_1, \ldots, X_{i-2}]$. The lowest monomial greater than m_s is $mX_{i-1}^{a+1}X_n^{b-1}$, which is not in I since I is not a lex-segment ideal. Let V be the vector space spanned by all the monomials u of $S = K[X_{i-1}, X_i, X_n]$ of degree $a+b$ such that $um \in I$. Note that (V) is strongly stable but it is not a lex-segment ideal of S. It is enough to show that there exists a polarization P and a generic specialization W for the ring S such that $\mathrm{in}_{\mathrm{lex}}(P((V))_W) \neq (V)$. Similarly, without loss of generality, we can assume that $a = 0$. We know that $X_i^b \in V$ and that $X_{i-1}X_n^{b-1} \notin V$. Let c be the greatest integer less than b such that $X_{i-1}X_i^c X_n^{b-1-c} \notin V$, and let U be the vector space spanned by all the monomials u of $S = K[X_{i-1}, X_i, X_n]$ of degree $b-c$ such that $X_i^c u \in V$. By substituting V for U, without loss of generality, we can assume that $c = 0$. After these reductions, we know that $X_i^b \in V$ while $X_{i-1}X_n^{b-1} \notin V$ and all the monomials of degree b which are greater than $X_{i-1}X_n^{b-1}$ belong to V. Set $P = \{(i,j) | 1 \leq j \leq b\}$ and let the generic specialization to be $W = (T_{(i,1)} - X_i - \lambda_{(i,1)}X_{i-1}) + (T_{(i,j)} - X_i - \lambda_{(i,j)}X_n | 1 < j \leq b)$. We see that $X_{i-1}X_n^{b-1} \in \mathrm{in}_{\mathrm{lex}}(P((V))_W)$. Thus $\mathrm{in}_{\mathrm{lex}}(P((V))_W) \neq (V)$. □

Theorem 7.2.12. *(Macaulay) Let $I \subseteq R = K[X_1, \dots, X_n]$ be a homogeneous ideal. Then there exists a lex-segment ideal J with the same Hilbert function of I.*

Proof. By taking the initial ideal with respect to the lexicographic term order, without loss of generality, we can assume that I is monomial. Using Lemma 7.2.7, we know that there exists a monomial ideal, say J, maximal with respect to $<_{\mathrm{lex}}$ among the monomial ideals with the same Hilbert function of I. By Lemma 7.2.11 J has to be a lex-segment ideal. □

We can now give the definition below.

Definition 7.2.13. Let $I \subseteq R = K[X_1, \dots, X_n]$ be a homogeneous ideal. We denote by I^{lex} the only lex-segment ideal with the same Hilbert function of I.

Theorem 7.2.14. *(Pardue-Iyengar) Let $R = K[X_1, \dots, X_n]$ and let I, J be homogeneous ideals such that $I \subseteq J$. Then*

$$\dim_K \mathrm{Tor}_i^{R/I}(R/J, K)_j \leq \dim_K \mathrm{Tor}_i^{R/I^{\mathrm{lex}}}(R/J^{\mathrm{lex}}, K)_j.$$

Proof. By Lemma 7.0.13 we can assume that both I and J are monomial ideals. Let $\mathbf{X}$ be the set of all the pairs of ideals (H, L) such that $H \subseteq L$, H and L have the same Hilbert function of I and J respectively, and they satisfy:

$$\dim_K \mathrm{Tor}_i^{R/I}(R/J, K)_j \leq \dim_K \mathrm{Tor}_i^{R/H}(R/L, K)_j.$$

By Lemma 7.2.7, let (H, L) be a maximal pair in $\mathbf{X}$ with respect to the partial order induced by $<_{\mathrm{lex}}$. If H or L is not a lex-segment ideal, we can use Lemma 7.2.11 and formula (7.2.1) to contradict the maximality of (H, L). □

Chapter 8

Variations on a Theorem of Eakin and Sathaye and on Green's Hyperplane Restriction Theorem

This chapter has been developed around our observation, presented in [Ca3], that the following theorem according to Eakin and Sathaye, can be viewed, after some standard reductions, as a corollary of Green's Hyperplane Restriction Theorem, which gives an estimate of the Hilbert function of a generic hyperplane restriction of a standard graded algebra.

Theorem (Eakin-Sathaye)**.** *Let (R,m) be a quasi-local ring with infinite residue field. Let I be an ideal of R, and let i and p be positive integers. If the number of minimal generators of I^i, denoted by $v(I^i)$, satisfies*

$$v(I^i) < \binom{i+p}{p},$$

then there are elements $h_1,\ldots,h_p$ in I such that $I^i = (h_1,\ldots,h_p)I^{i-1}$.

A recent generalization of the above result according to L.O'Carroll [O] gave us the motivation for the following study (see also [HT] for an interesting proof of the Eakin-Sathaye Theorem). The idea is that, since it is possible to extend the Theorem of Eakin and Sathaye, it should be also possible to modify Green's Hyperplane Restriction Theorem to recover, as corollaries, the result of O'Carroll. In fact, this is the case.

One interesting part of this approach is that we can actually obtain some further generalizations of the work of O'Carroll and at the same time give a general method for deriving variations of the Theorem of Eakin and Sathaye.

The modification of the Hyperplane Restriction Theorem that we present follows very closely Green's original proof [Gr2], with the only difference that we underline the key-properties he needs to build up the inductive step of his proof.

The chapter is divided as follows. First, we give a short introduction to Macaulay representation of integer numbers. This is needed for the understanding of the Hyperplane Restriction Theorem.

Second, we prove a more general version of the Hyperplane Restriction Theorem, putting some emphasis on certain special cases where it can be applied. In fact, our main goal is to obtain a restriction theorem that does not have to deal necessarily with a *generic* hyperplane, as it is in Green's original statement. Roughly speaking, if we know some extra informations about the algebra, it is possible to obtain the same estimate on the Hilbert function as Green did by restricting to some *partially generic* hyperplane.

Finally, we apply the previous work to obtain variations on the Eakin-Sathaye Theorem.

8.1 Macaulay representation of integer numbers

The following are, nowadays, quite standard facts. An interested reader may look at [Sta],[Gr1] or [Gr2] for more details on the subject.

Let d be a positive integer. Any positive integer c can then be uniquely expressed as

$$c = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_1}{1},$$

where the k_i's are non-negative and strictly increasing, i.e. $k_d > k_{d-1} > \cdots > k_1 \geq 0$. This way of writing c is called the d'th *Macaulay representation* of c, and the k_i's are called the d'th *Macaulay coefficients* of c. For instance, setting $c = 13$ and $d = 3$ we get $13 = \binom{5}{3} + \binom{3}{2} + \binom{0}{1}$.

Remark 8.1.1. An important property of Macaulay representation is that the usual order on the integers corresponds to the lexicographical order on the arrays of Macaulay coefficients. In other words, given two positive integers $c_1 = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_1}{1}$ and $c_2 = \binom{h_d}{d} + \binom{h_{d-1}}{d-1} + \cdots + \binom{h_1}{1}$ we have $c_1 < c_2$ if and only if $(k_d, k_{d-1}, \ldots, k_1)$ is smaller lexicographically than $(h_d, h_{d-1}, \ldots, h_1)$.

Definition 8.1.2. Let c and d be positive integers. We define $c_{<d>}$ to be

$$c_{<d>} = \binom{k_d - 1}{d} + \binom{k_{d-1} - 1}{d-1} + \cdots + \binom{k_1 - 1}{1}$$

where $k_d, \ldots, k_1$ are d'th Macaulay coefficients of c. We use the convention that $\binom{a}{b} = 0$ whenever $a < b$.

Remark 8.1.3. It is easy to verify that if $c_1 \leq c_2$ then $c_{1<d>} \leq c_{2<d>}$. This property, as we see further, allows us to iteratively apply the Restriction Theorem to derive Corollary 8.2.7.

Defining $\delta = \min\{m | k_m \geq m\}$ we have the alternate representations $c = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_\delta}{\delta}$ and $c_{<d>} = \binom{k_d - 1}{d} + \binom{k_{d-1}-1}{d-1} + \cdots + \binom{k_\delta - 1}{\delta}$. Note also that if $k_\delta \neq \delta$ then $(c-1)_{<d>} < c_{<d>}$.

8.2 Green's Hyperplane Restriction Theorem

Let R be a standard graded algebra over an infinite field K. We can write R as $K[X_1, \ldots, X_n]/I$ where I is a homogeneous ideal. The result of Mark Green we discuss below gives an upper bound for the dimension of the d^{th} graded component of $R/(l)$, where l is a linear form, in terms of dimension of the d^{th} graded component of R. This bound is satisfied generically, in the sense that it holds for any linear form in a certain non-empty Zariski open set $U \subseteq \mathbb{P}(R_1)$. In other words we say that Green's estimate is satisfied by a generic linear form. In general when we say that a property (P) is satisfied by r generic linear forms we mean that there exists a non-empty Zariski open set of $U \subseteq \mathbb{P}(R_1)^r$ such that any r-tuple in U consists of r linear forms satisfying (P).

Green's result is the following:

Theorem 8.2.1 (Green's Hyperplane Restriction Theorem)**.** *Let R be a standard graded algebra over an infinite field K, and let l be a generic linear form of R. Setting S to be $R/(l)$, we have*

$$\dim_k S_d \leq (\dim_K R_d)_{<d>}.$$

The Hyperplane Restriction Theorem first appeared in [Gr2] where it was proved with no assumption on the characteristic of the base field K.

A more combinatorial proof can be found in [Gr1] where the characteristic zero assumption is a working hypothesis. A person interested in reading this last proof can observe that, with a few minor changes, the arguments in [Gr1] also work in positive characteristic.

It is important to recall that the numerical bound of Theorem 8.2.1 can be also interpreted in the following way: let $A = K[X_1, \ldots, X_n]$ and let $I \subset A$ be a homogeneous ideal. Define $J = I^{\text{lex}} \subset A$ to be the lex-segment ideal with the same Hilbert function as I. Let c be the dimension, as a K-vector space, of $(A/I)_d$. By definition we also have that $\dim_K(A/J)_d = c$. It is possible to show that $\dim_K(A/(J+$

$(X_n))_d = c_{<d>}$. Then, for any degree d, Theorem 8.2.1 is equivalent to

$$\dim_K(A/I+(l))_d \leq \dim_K(A/I^{\text{lex}}+(X_n))_d.$$

Recently, in this respect, Aldo Conca has proved the following stronger result that we have discussed in the previous chapter, see formula (7.0.2): assume the characteristic of K to be zero and let $l_n,\dots,l_r$ be generic linear forms of A, then $\dim_K \mathrm{Tor}_i(A/I, A/(l_n,\dots,l_r))_j \leq \dim_K \mathrm{Tor}_i(A/I^{\text{lex}}, A/(X_n,\dots,X_r))_j$. Conca's result, when the index of the Tor is equal to zero, gives Theorem 8.2.1 in characteristic zero.

We want to modify Green's result in a rather different direction. Our main focus is to substitute the genericity condition for the linear form with some weaker assumption. Assume, for example, that the standard graded algebra in Green's result is a quotient of the following toric algebra:

$$S = K[X_iY_j | 0 \leq i \leq n_1, 0 \leq j \leq n_2] \cong K[T_1,\dots,T_{n_1n_2}]/I.$$

It is reasonable to think that, for such an algebra, the product of a generic linear forms in the X_i's and a generic linear forms in the Y_j's may satisfy the bound of Theorem 8.2.1. This is in fact true, as we will see further, even though such an element is not generic. The forms of this type belong in fact to a non-trivial Zariski closed set of the projective space of linear forms.

Before proving our version of the Hyperplane Restriction Theorem we have to introduce some notation.

Definition 8.2.2. Let R be a standard graded algebra and let $\{l_1,\dots,l_r\}$ be a set of linear forms. Let $\mathbf{I_{0,0}} = \{(0)\}$ and $\mathbf{I_{1,0}} = \{(l_1)\}$. For $0 < i \leq r$ and $0 \leq j < d$ inductively define the set $\mathbf{I_{i,j}}$ to be

$$\mathbf{I_{i,j}} = \{(I : l_i) | I \in \mathbf{I_{i-1,j-1}}\} \cup \{I + (l_i) | I \in \mathbf{I_{i-1,j}}\}.$$

Note that the first index keeps track of how many linear forms have been used and the second one indicates the number of colons that have been performed.

We say that $l_1, \dots, l_r$ are *suitable for the Hyperplane Restriction Theorem in degree d*, property *(Gr,d)*, if they satisfy the next three conditions:

(1) For any $I \in \mathbf{I_{i-1,j}}$, with $0 < i \le r$ and $\mathbf{m} \not\subseteq I$, we have $l_i \notin I$.

(2) For any $I \in \mathbf{I_{r,j}}$, we have $\mathbf{m} \subseteq I$ where $\mathbf{m}$ is the homogeneous maximal ideal.

(3) For any $I \in \mathbf{I_{i,j}}$, with $i \le r-2$ we have:

$$\dim_K((I : l_{i+1}) + (l_{i+2}))_{d-j-1} \le \dim_K((I + l_{i+1}) : l_{i+2})_{d-j-1}.$$

Remark 8.2.3. Let $n = \dim_K R_1$. If property (1) holds, then property (2) is automatically satisfied if $r \ge n+d-1$. Property (3) is implied by the next stronger condition.

(3') For any $I \in \mathbf{I_{i,j}}$ with $0 \le i \le r-2$, the Hilbert function of $((I : l_{i+1}) + l_{i+2})$ agrees with the Hilbert function of $((I : l_{i+2}) + l_{i+1})$.

In fact if (3') is satisfied, for any degree a we have

$$\dim_K((I : l_{i+1}) + (l_{i+2}))_a = \dim_K((I : l_{i+2}) + (l_{i+1}))_a \le \dim_K((I + (l_{i+1})) : l_{i+2})_a$$

where the last inequality comes from $((I : l_{i+2}) + (l_{i+1})) \subseteq ((I + (l_{i+1})) : l_{i+2})$.

Example 8.2.4. At a first sight the properties (Gr,d) may not seem too easy to verify. However there are several examples for which it is not hard to find linear forms, not generic, satisfying our condition. We give a list of the most significant ones for the variations of the Theorem of Eakin-Sathaye we want to prove.

Let R be a standard graded algebra, $\dim_K R_1 = n$, $|K| = \infty$. The following are examples of r linear forms with $r \ge d+n-1$, satisfying (Gr,d).

(A) With no further assumptions on R the r linear forms can be taken to be generic.

(B) Assume R to be the homomorphic image of the Segre ring:

$$S = K[X_{1,i_1} \cdot X_{2.i_2} \cdots X_{s,i_s} | 1 \le i_1 \le n_1, \ldots, 1 \le i_s \le n_s],$$

under a map sending the monomials $X_{1,i_1} \cdot X_{2.i_2} \cdots X_{s,i_s}$ to linear forms. Then the r linear forms can be taken to be the images of $l_1 \cdots l_s$, where l_i is a generic linear form of $K[X_{i,1}, \ldots, X_{i,n_i}]$.

(C) Assume that $\mathrm{char}(K) = 0$ and that R is the homomorphic image of the Veronese ring: $S = K[X_1^{a_1} \cdots X_s^{a_s} | \sum_{i=1}^{s} a_i = b$ and $a_i \ge 0]$ under a map sending the monomials $X_1^{a_1} \cdots X_s^{a_s}$ to linear forms. Then the r linear forms can be taken to be the images of l^b, where l is a generic linear form of $K[X_1, \ldots, X_s]$.

(D) Assume that $\mathrm{char}(K) = 0$ and that R is the homomorphic image of Segre products of Veronese rings:

$$S = K\left[\prod_{\substack{1 \le i \le s \\ 1 \le j \le n_i}} X_{i,j}^{a_{i,j}} \text{ such that } \sum_j a_{i,j} = b_i \text{ and } a_{i,j} \ge 0\right]$$

under a map sending the monomials $\prod X_{i,j}^{a_{i,j}}$ to linear forms. Then the r linear forms can be taken to be the images of $l_1^{b_1} \cdots l_s^{b_s}$, where l_i is a generic linear form of $K[X_{i,1}, \ldots, X_{i,n_i}]$.

(E) Assume that $\mathrm{char}(K) = 0$ and that R is the homomorphic image of the following toric ring:

$$S = K[X_{i_1} \cdot X_{i_2} \cdots X_{i_s} | 1 \le i_1 \le n_1, \ldots, 1 \le i_s \le n_s \text{ and } n_1 \le n_2 \le \cdots \le n_s]$$

under a map sending the monomials $X_{i_1} \cdot X_{i_2} \cdots X_{i_s}$ to linear forms. Then the r linear forms can be taken to be the images of $l_1(l_1+l_2)\cdots(l_1+l_2+\cdots+l_s)$, where l_i is a generic linear form of $K[X_{n_{i-1}},\dots,X_{n_i}]$.

Proof. It is easy to see that in all the examples above the elements of S we consider belong to some products of projective spaces. For example, in (B) an element written as $l_1\cdots l_s$, where l_i is a linear form of $K[X_{i,1},\dots,X_{i,n_i}]$, corresponds, up to scalars, to a point in $V = \mathbb{P}^{n_1-1}\times\mathbb{P}^{n_2-1}\times\cdots\times\mathbb{P}^{n_s-1}$. We have to show that there exists a non-empty open set of $W = V^r$ whose r-tuple satisfy (GR,d). First of all, note that (2) is satisfied since there exists an open set of $U \subseteq W$ such that any n entries of the points of U are generators for the homogeneous maximal ideal. Here is where we need the characteristic zero assumption for some of the examples, otherwise our linear forms may not generate the homogeneous maximal ideal. Thus we can find a non-empty open set where (1) and (2) are satisfied.

Part (3) needs more explanation. Once we fixed an r-tuple, i.e. a point in W, any ideal in $\mathbf{I}_{\mathbf{i},\mathbf{j}}$ can be identified with a precise sequence of operations: sums and colons. For any sequence there exists a non-empty Zariski open set of W for which the Hilbert function of the ideal constructed using such a sequence is constant. Since r is fixed, the number of possible sequences of sums and colons is finite. Therefore, there exists a non-empty Zariski open set $U \subset W$ such that any ideal in any of the $\mathbf{I}_{\mathbf{i},\mathbf{j}}{}'$s has a Hilbert function constant on U, which depends only on the sequence of operations defining the ideal. Since $U \subset W = V^r$ is open and non-empty, we can find an open and non-empty subset of U closed under any permutation of the r entries of its elements. Thus, property (3') is satisfied by the r-tuples of such open set. In particular, by Remark 8.2.3, condition (3) holds. □

Remark 8.2.5. Note that the characteristic assumption in (C),(D) and (E) is essential. Let $R = K[X_1^2, X_1X_2, X_2^2]/(X_1^2, X_2^2) \cong K[Y_1, Y_2, Y_3]/(Y_1, Y_3, Y_2^2 - Y_1Y_3)$ and assume $\mathrm{char}(K) = 2$. This correspond to the case $s = 2$ and $b = 2$ of example (C). The square of a generic linear form of $K[X_1, X_2]$ can be written as $X_1^2 + \lambda X_2^2$ and it has a zero image in R. Property (2) of (Gr,d) is not satisfied. Moreover, a zero linear form clearly does not satisfy Green's estimate.

We can now prove the Hyperplane Restriction Theorem. The structure of the proof is exactly the same as in Green's paper [Gr2].

Theorem 8.2.6. *Let R be a standard graded algebra and let $l_1, \ldots, l_r$ be linear forms satisfying (Gr,d). Then*

$$\dim_K(R/(l_1))_d \leq (\dim_K(R_d))_{<d>}.$$

Proof. Since $(0) \in \mathbf{I_{0,0}}$, in order to prove the theorem it is enough to show:

Claim. *For any $I \in \mathbf{I_{i,j}}$ with $i < r$ we have:*

$$\dim_K(R/(I + (l_{i+1})))_{d-j} \leq (\dim_K(R/I)_{d-j})_{<d-j>}. \tag{8.2.1}$$

First of all, we show that the claim holds for all the ideals in $\mathbf{I_{r-1,j}}$ and in $\mathbf{I_{i,d-1}}$. By part (2) of (Gr,d), since $I \in \mathbf{I_{r-1,j}}$ then $\mathbf{m} \subseteq (I + (l_r)) \in \mathbf{I_{r,j}}$. Because $j < d$ we have $(R/(I + (l_r)))_{d-j} = 0$ and, therefore, the inequality (8.2.1) holds. If $I \in \mathbf{I_{i,d-1}}$ the inequality (8.2.1) becomes $\dim_K(R/(I + (l_{i+1})))_1 \leq \dim_K(R/I)_1 - 1$ and it follows from the part (1) of (Gr,d).

We do a decreasing induction on the double index of $\mathbf{I_{i,j}}$.

Let $I \in \mathbf{I_{a,d-b}}$ with $a < r - 1$ and $b > 1$. By induction we know that (8.2.1) holds for $(I + (l_{a+1})) \in \mathbf{I_{a+1,d-b}}$ and for $(I : l_{a+1}) \in \mathbf{I_{a+1,d-b+1}}$. Consider the sequence below:

$$0 \to \frac{R}{(I + (l_{a+1})) : l_{a+2}}(-1) \xrightarrow{\cdot l_{a+2}} \frac{R}{I + (l_{a+1})} \to \frac{R}{I + (l_{a+1}) + (l_{a+2})} \to 0.$$

By looking at the graded component of degree b we get:

$$\dim_K \left(\frac{R}{I+(l_{a+1})}\right)_b = \dim_K \left(\frac{R}{(I+(l_{a+1})):l_{a+2}}\right)_{b-1} + \\ + \dim_K \left(\frac{R}{I+(l_{a+1})+(l_{a+2})}\right)_b.$$

Property (3) of (Gr,d) implies

$$\dim_K \left(\frac{R}{(I+(l_{a+1})):l_{a+2}}\right)_{b-1} \leq \dim_K \left(\frac{R}{(I:l_{a+1})+(l_{a+2})}\right)_{b-1},$$

and by using the inductive assumption on $I:l_{a+1}$ and on $I+(l_{a+1})$ we know that $\dim_K \left(\frac{R}{I+(l_{a+1})}\right)_b \leq \left(\dim_K \left(\frac{R}{I:l_{a+1}}\right)_{b-1}\right)_{<b-1>} + \left(\dim_K \left(\frac{R}{I+(l_{a+1})}\right)_b\right)_{<b>}$.

To simplify the notation, set $c = \dim_K(R/I)_b$ and $c_H = \dim_K(R/(I+(l_1)))_b$. From the short exact sequence

$$0 \to \frac{R}{I:l_{a+1}}(-1) \xrightarrow{l_{a+1}} \frac{R}{I} \to \frac{R}{I+(l_{a+1})} \to 0$$

we know that $\dim_K \left(\frac{R}{I:l_{a+1}}\right)_{b-1} = c - c_H$, therefore the above upper bound for $\dim_K \left(\frac{R}{I+(l_{a+1})}\right)_b$ becomes:

$$c_H \leq (c_H)_{<b>} + (c-c_H)_{<b-1>}. \tag{8.2.2}$$

Write $c_H = \binom{k_b}{b} + \binom{k_{b-1}}{b-1} + \cdots + \binom{k_\delta}{\delta}$. The inequality of the claim, i.e. $c_H \leq c_{<b>}$, is equivalent to $c \geq \binom{k_b+1}{b} + \binom{k_{b-1}}{b-1} + \cdots + \binom{k_\delta+1}{\delta}$.

If the claim fails we have:

$$c - c_H < \binom{k_b}{b-1} + \binom{k_{b-1}}{b-2} + \cdots + \binom{k_\delta}{\delta-1}. \tag{8.2.3}$$

We use (8.2.2) to derive a contradiction. There are two cases to consider.

If $\delta = 1$ then (8.2.3) becomes $c - c_H \leq \binom{k_b}{b-1} + \binom{k_{b-1}}{b-2} + \cdots + \binom{k_2}{1}$.

Thus

$$(c-c_H)_{<b-1>} \leq \binom{k_b-1}{b-1} + \binom{k_{b-1}-1}{b-2} + \cdots + \binom{k_2-1}{1} \qquad \text{and}$$

$$(c_H)_{<b>} \leq \binom{k_b-1}{b} + \binom{k_{b-1}-1}{b-1} + \cdots + \binom{k_2-1}{2} + \binom{k_1-1}{1}.$$

By adding these two inequalities, (8.2.2) gives

$$c_H \leq \binom{k_b}{b} + \binom{k_{b-1}}{b-1} + \cdots + \binom{k_1-1}{1} < c_H,$$

which is a contradiction.

If $\delta > 1$ then the equation (8.2.3) is $c - c_H < \binom{k_b}{b-1} + \binom{k_{b-1}}{b-2} + \cdots + \binom{k_\delta}{\delta-1}$ and since $k_\delta - 1 > \delta - 1$ applying ${}_{b-1}$ the strict inequality is preserved and gives

$$(c-c_H)_{<b-1>} < \binom{k_b-1}{b-1} + \binom{k_{b-1}-1}{b-2} + \cdots + \binom{k_\delta-1}{\delta-1}.$$

Adding the last inequality with $(c_H)_{<b>} \leq \binom{k_b-1}{b} + \binom{k_{b-1}-1}{b-1} + \cdots + \binom{k_\delta-1}{\delta}$ we obtain the following contradiction

$$c_H < \binom{k_b}{b} + \binom{k_{b-1}}{b-1} + \cdots + \binom{k_1}{1} = c_H.$$

□

A direct consequence of Theorem 8.2.6 is the corollary below.

Corollary 8.2.7. *Let R be a standard graded algebra and let $l_1, \ldots, l_r$ be linear forms satisfying (Gr,d), and let the Macaulay representation of $\dim_K(R_d)$ be $\binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_1}{1}$. Then for any p such that $1 \leq p \leq r$ we have*

$$\dim_K(R/(l_1, \ldots, l_p))_d \leq \binom{k_d-p}{d} + \binom{k_{d-1}-p}{d-1} + \cdots + \binom{k_1-p}{1}.$$

Proof. By Theorem 8.2.6 we have $\dim_K(R/(l_1))_d \leq \binom{k_d-1}{d} + \binom{k_{d-1}-1}{d-1} + \cdots + \binom{k_1-1}{1}$. Note that the images of $l_2, \ldots, l_r$ satisfy (Gr,d) for $R/(l_1)$. We can, therefore, apply Theorem 8.2.6 and obtain the result by induction. □

8.3 Variations of the Theorem of Eakin and Sathaye

We now prove a general version of the Theorem of Eakin and Sathaye.

Theorem 8.3.1. *Let (A,m) be a quasi-local ring with infinite residue field K. Let I be an ideal of A. Let i and p be positive integers. If the number of minimal generators of I^i, denoted by $v(I^i)$, satisfies $v(I^i) < \binom{i+p}{i}$ then*

(a) (Eakin-Sathaye) There are $h_1,\dots,h_p$ in I such that $I^i = (h_1,\dots,h_p)I^{i-1}$.

Moreover:

(b) (O'Carroll) If $I = I_1 \cdots I_s$, where I_j's are ideals of R, then we can find the elements h_j's of the form $l_1 \cdots l_s$ with $l_i \in I_i$.

(c) Assume $\operatorname{char}(K) = 0$. *If $I = J^b$, where J is an ideal of A, then we can find the elements h_j's of the form l^b with $l \in I$.*

(d) Assume $\operatorname{char}(K) = 0$. *If $I = I_1^{b_1} \cdots I_s^{b_s}$, where I_j's are ideals of A, then we can find the elements h_j's of the form $l_1^{b_1} \cdots l_s^{b_s}$ with $l_i \in I_i$.*

(e) Assume $\operatorname{char}(K) = 0$. *If $I = I_1(I_1+I_2)\cdots(I_1+\cdots+I_s)$, where I_j's are ideals of A, then we can find the elements h_j's of the form $l_1(l_1+l_2)\cdots(l_1+\cdots+l_s)$ with $l_i \in I_i$.*

Proof. First of all, note that since $v(I^i)$ is finite, without loss of generality we can assume that I is also finitely generated: in fact, if $H \subseteq I$ is a finitely generated ideal such that $H^i = I^i$ the result for H implies the one for I. Similarly, we can also assume that the ideals I_j of (b),(d) and (e) and the ideal J of (c) are finitely generated. By the use of Nakayama's Lemma, we can replace I by the homogeneous maximal ideal of the fiber cone $R = \bigoplus_{i\geq 0} I^i/mI^i$. Note that R is a standard graded algebra finitely generated over the infinite field $R/m = K$. Moreover, the algebras R of (a),(b),(c),(d),(e) satisfy the properties of the Example 8.2.4 parts (A),(B),(C),(D), and (E) respectively. Let $l_1,\dots,l_r$ as in Example 8.2.4 and assume also that $p \leq r$. The theorem is proved if we can show that $(R/(l_1,\dots,l_p))_i = 0$. Note that $\dim_K R_i \leq \binom{i+p}{i} - 1 = \binom{i+p-1}{i} + \binom{i+p-2}{i-1} + \cdots + \binom{i+p-j}{i-j+1} + \cdots + \binom{r}{1}$. This can be proved directly or by using Remark 8.1.1. In fact, one can first order the array of Macaulay coefficients using the lexicographic order and then note that the previous array of $(i+p,0,\dots,0)$ is given by $(i+p-1,i+p-2,\dots,p)$. By Corollary 8.2.7 we deduce

$$\dim_K(R/(l_1,\dots,l_p))_i \leq \binom{i-1}{i} + \binom{i-2}{i-1} + \cdots + \binom{0}{1}.$$

The term on the right hand side is zero and therefore the theorem is proved. □

The general principle behind this proof is that in passing to the fiber cone of R the properties of the ideal I allow us to conclude that R is the quotient of some particularly nice toric ring. In such a ring it is quite easy to find interesting $l_1,\dots,l_r$ satisfying (Gr,d). Finally, a pre image of $l_1,\dots,l_r$ gives an interesting reduction for the ideal I.

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