ON THE TEMPERED SPECTRUM OF QUASI-SPLIT CLASSICAL GROUPS

David Goldberg*and Freydoon Shahidi**

§1 Introduction.

One of the most striking aspects of the Langlands Program is the conjectural relation between harmonic analysis and number theory. The proof of a conjecture of Langlands given in [**19**] shows that determining the poles of certain (conjectural) Langlands L-functions is equivalent to determining the non-discrete tempered spectrum of reductive p-adic groups. The theory of endoscopy [**15**] and twisted endoscopy [**12**,**13**] has proved particularly useful in giving a context within which to explore these problems, at least for classical groups (see the introduction and Section 3 of [**20**]).

Previously the authors separately determined the non-discrete tempered spectrum of classical groups supported in their Siegel parabolic subgroups ([**7**] and [**20**]), or equivalently, computed the symmetric square and the exterior square L-functions for $GL_n(F)$, [**20**], as well as the Asai L-functions for $GL_n(E)$, where E is a quadratic extension of F [**7**]. In fact, these L-functions were determined for an arbitrary irreducible admissible representation of $GL_n(F)$ or $GL_n(E)$, accordingly. Finally in [**21**], the second author addressed the problem for arbitrary maximal parabolic subgroups of split even special orthogonal groups. In terms of L–functions, the work in [**21**] determines the Rankin-Selberg product L–functions attached to irreducible admissible representations of $GL_n(F)\times SO_{2m}(F)$.

The purpose of the present paper is twofold. First is to generalize the work in [**21**] to symplectic and quasi-split special orthogonal groups and thus using [**8**],

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*Partially supported by National Science Foundation Postdoctoral Fellowship DMS9206246 and National Science Foundation Career Grant DMS9501868

**Partially supported by National Science Foundation Grant DMS9301040

eventually determine the non-discrete tempered spectrum of these groups, and the second is to remove a gap which existed in the proof of Theorems 7.8 and 8.1 of [**21**]. The final results of the present paper, Theorem 4.8 and Corollary 4.9, while having similar main (regular) term $R_{\mathbf{G}}$, have a different, and more complicated singular contribution than the singular terms given in Theorems 7.8 and 8.1 of [**21**]. However, in Proposition 5.2 of the present paper, we manage to relate the singular terms from the two different versions to each other (see Remark 4.11). The reader of [**21**] must therefore consider Theorem 4.8 and Corollary 4.9 of the present paper as correct versions of Theorems 7.8 and 8.1 of [**21**] and it is most efficient if Sections 4 and 5 of the present paper are substituted for Sections 6-9 of [**21**]. Sections 1-5 of [**21**], which were the main motivation for this whole project, are precise and correct. We are indebted to Robert Kottwitz for noticing this gap in an earlier version of the present paper and consequently in [**21**].

To explain our results, let $\mathbf{G}$ be a symplectic group or a quasi-split special orthogonal group of rank m defined over a p-adic field F of characteristic zero. Let τ and τ' be discrete series representations of $\mathbf{G}(F)$ and $GL_n(F)$, respectively. We denote by $\widetilde{\mathbf{G}}$ either the symplectic or the quasi-split orthogonal group of rank $m+n$ (with the same Witt rank and discriminant as $\mathbf{G}$ in the orthogonal case). Let $I(\tau'\otimes\tau)$ be the representation of $\widetilde{\mathbf{G}}(F)$ unitarily induced from the F-points of the parabolic subgroup whose Levi component is isomorphic to $GL_n\times\mathbf{G}$. The reducibility of $I(\tau'\otimes\tau)$ is governed by the residue of the standard intertwining operator $A(s,\tau'\otimes\tau,w_0)$ (cf. Section 2), at $s=0$.

Suppose n is even and W_n is the standard anti-diagonal matrix with respect to which one defines the symplectic or split even orthogonal group of rank $n/2$. We let $\varepsilon : GL_n \longrightarrow GL_n$ be the automorphism given by $\varepsilon(g) = W_n\,{}^tg^{-1}W_n^{-1}$. In the symplectic case, $\varepsilon=\theta^*$, and in the orthogonal case $\varepsilon=\theta$, where these automorphisms are defined in [**21**]. When one attempts to compute the residue of $A(s,\tau'\otimes\tau,w_0)$, one is led to consider a correspondence between certain ε-conjugacy classes of $GL_n(F)$ and conjugacy classes of $\mathbf{G}(F)$ (see equation (2.5) and Section 3). We call this the ε-norm correspondence. By Lemma 3.11 and Corollary 3.13, the norm correspondence is surjective whenever $n\geq 2m$. If $n<2m$, then the image of the correspondence contains no regular elliptic conjugacy classes. We show that the norm correspondence that we define agrees, up to a sign, with the norm map defined by Kottwitz and Shelstad on the set of strongly ε-regular conjugacy classes (cf. Lemma 3.18). That is, if $n = 2m$, and Y is a strongly ε-regular element of $GL_n(F)$, then, at least for almost all Y, the norm exhausts all those $\mathbf{G}(F)$ conjugacy classes which are $GL_n(F)$ conjugate to $-Y\varepsilon(Y)$. Since the norm correspondence has finite fibers (cf. Lemma 3.11, Corollary 3.13, and Lemma 4.7) we let $\mathcal{A}$ denote its one to finite inverse, extended by the non-square central elements (cf. Section 4). It agrees with the image map $\mathcal{A}_{\mathbf{G}/GL_n}$ of [**12**] from semisimple conjugacy classes in $\mathbf{G}(F)$ to ε-semisimple ε-conjugacy classes in $GL_n(F)$.

We restrict ourselves to the case where both τ and τ' are supercuspidal. There are two terms that appear in the residue of $A(s, \tau' \otimes \tau, w_0)$. The first we call the regular or main term, and is that associated to the regular elliptic conjugacy classes in $\mathbf{G}(F)$. If $n < 2m$ then this term does not occur. Suppose that f_τ is a matrix coefficient of τ and $f' \in C_c^\infty(GL_n(F))$ defines a matrix coefficient of τ' by descent. Then the regular term is given by

$$R_{\mathbf{G}}(f_\tau, f') = \sum_{\{\mathbf{T}_i\}} \mu(T_i)|W(\mathbf{T}_i)|^{-1} \int_{T_i} \Phi_\varepsilon(\mathcal{A}(\{\gamma\}), f')\Phi(\{\gamma\}, f_\tau)|D(\gamma)|\, d\gamma,$$

where the sum is over the conjugacy classes of elliptic tori in $\mathbf{G}(\ell)$, with $2\ell = \min(n, 2m)$, Φ_ε and Φ denote ε-twisted and ordinary orbital integrals, respectively, $\mu(T_i)$ is the measure of $T_i = \mathbf{T}_i(F)$, and $D(\gamma)$ is the usual discriminant of Harish-Chandra [**9**]. In fact, we should be more precise and say that

$$\Phi_\varepsilon(\mathcal{A}(\{\gamma\}), f') = \sum_{\{\gamma'\} \in \mathcal{A}(\{\gamma\})} \Phi_\varepsilon(\{\gamma'\}, f')\Delta(\{\gamma\}, \{\gamma'\}),$$

where $\Delta(\{\gamma\}, \{\gamma'\})$, is a transfer factor given in Section 4.

The singular term is more complicated and given as a limit

$$R_{\mathrm{sing}}(f_\tau, f') = \sum_i |W(\mathbf{T}_i)|^{-1} \lim_{\omega_i \to T_i} \operatorname*{Res}_{s=0} \int_{T_i' \setminus \omega_i} \psi_{\mathcal{A}}(s, \gamma)|D(\gamma)|\, d\gamma,$$

in which the sum is over conjugacy classes of *all* Cartan subgroups of $\mathbf{G}(\ell)$ and for each i, ω_i is a compact subset of the regular elements T_i' of $T_i = \mathbf{T}_i(F)$. The function $\psi_{\mathcal{A}}$, in which $\mathcal{A}$ is the image correspondence, is defined in Section 4. The fact that all Cartan subgroups must be included is critically important when $n < 2m$. The main result of this paper (Theorem 4.8 and Corollary 4.9) can be formulated as:

Theorem. *The intertwining operator* $A(s, \tau' \otimes \tau, w_0)$ *has a pole at* $s = 0$ *if and only if* $cR_{\mathbf{G}} + R_{\text{sing}} \neq 0$ *for some data, where* $c = (2n \log q)^{-1}$. *Consequently, suppose that* $\tau' \simeq \widetilde{\tau}'$. *Then*

(a) *The induced representation* $I(\tau' \otimes \tau)$ *is irreducible if and only if* $cR_{\mathbf{G}} + R_{\text{sing}} \neq 0$.

(b) *Assume* τ *is generic, i.e., has a Whittaker model. If* $I(\tau' \otimes \tau)$ *is irreducible then* $I(s, \tau' \otimes \tau)$, $s \in \mathbb{R}$ *is reducible exactly at* $s = \pm 1/2$ *or* $s = \pm 1$, *and at only one of these pairs.*

The term $R_{\mathbf{G}}$ can be easily expressed as a pairing between the character of τ and the ε–twisted character of τ' (Section 5). Its non-vanishing will then be the reason to call τ' the "ε–twisted endoscopic transfer" of τ (Definition 5.1). When $n \geq 2m$, the non-vanishing of $R_{\mathbf{G}}$ must be equivalent to the existence of a pole at $s = 0$ for the Rankin-Selberg product L–function $L(s, \tau' \times \tau)$ defined in [**19**].

On the other hand, if τ' comes from $SO_{n+1}(F)$ by twisted endoscopy as defined in [**20**] and $R_{\mathbf{G}} \equiv 0$ which we expect to be the case for any τ' as such, then $R_{\text{sing}} \not\equiv 0$ (Proposition 5.2). Thus, $R_{\text{sing}} \equiv 0$ must imply that τ' comes from $SO_n^*(F)$ (cf [**20**]). Consequently, the two terms $R_{\mathbf{G}}$ and R_{sing} pretty much separate the poles of the two L-functions $L(s, \tau' \times \tau)$ and $L(s, \tau', \wedge^2 \rho_n)$, respectively.

When $n < 2m$, $R_{\mathbf{G}} \equiv 0$, and therefore R_{sing} controls the poles of both L–functions and distinguishing them requires further analysis. But in all cases we have the following weaker result (Proposition 5.3) in which we may and will assume $\tau' \simeq \widetilde{\tau}'$:

Proposition.

(a) *Suppose* τ' *comes from* $SO_{n+1}(F)$ *(non-vanishing condition (5.2) of [20]) or equivalently* $L(s,\tau',\wedge^2\rho_n)$ *has a pole at* $s=0$. *Then* $L(s,\tau'\times\tau)$ *is holomorphic at* $s=0$.

(b) *Otherwise, i.e., if* τ' *does not come from* $SO_{n+1}(F)$, *then* $L(s,\tau'\times\tau)$ *has a pole at* $s=0$ *if and only if* $cR_{\mathbf{G}}+R_{\mathrm{sing}}\neq 0$ *for some data.*

Finally, we would like to thank Laurent Clozel and Robert Kottwitz for their suggestions and comments.

§**2 Preliminaries.** Let F be a p-adic field of characteristic zero. For an integer $l>0$, let

$$w_l=\begin{pmatrix} & & & & \cdot \\ & & & \cdot & \\ & & \cdot & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix} \in M_l(F),$$

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and

$$u_l=\begin{pmatrix} & & & & \cdot \\ & & & \cdot & \\ & & \cdot & & \\ & & -1 & & \\ & 1 & & & \\ -1 & & & & \end{pmatrix} \in M_l(F).$$

Let J_{2r} be a quasi-split symmetric or symplectic form of dimension $2r$, defined over F. If J_{2r} is symmetric and $V\simeq F^k$ is the maximal anisotropic subspace of F^{2r} under J_{2r}, then we assume that $J_{2r}=\begin{pmatrix} & & w_l \\ & \Lambda_k & \\ w_l & & \end{pmatrix}$, with $2r=2l+k$, and Λ_k an anisotropic form of dimension k. In the symplectic case we take $J_{2r}=u_{2r}$. For $X\in M_n(F)$, we let $\widetilde{\theta}(X)=w_n{}^tXw_n^{-1}$. If $Y\in GL_n(F)$, then we set $\theta(Y)=\widetilde{\theta}(Y^{-1})$. Similarly, we let $\widetilde{\theta}^*(X)=u_n{}^tXu_n^{-1}$, and $\theta^*=\widetilde{\theta}^*(Y^{-1})$. We set

$$\widetilde{\varepsilon}(X)=\begin{cases}\widetilde{\theta}^*(X), & \text{if } J_{2r} \text{ is symplectic;}\\ \widetilde{\theta}(X), & \text{otherwise,}\end{cases}$$

and write $\varepsilon(Y)$ for $\widetilde{\varepsilon}(Y^{-1})$.

Let $\widetilde{\mathbf{G}}$ be the special orthogonal, or symplectic group defined with respect to J_{2r}. Thus, $\widetilde{\mathbf{G}} = \{g \in GL_{2r} | \ {}^t g J_{2r} g = J_{2r}\}^\circ$, with the superscript indicating the connected component. In the symmetric case, we assume that $k > 0$, since $k = 0$ is dealt with in [**21**]. (Note that this implies that $k = 2$, and $\widetilde{\mathbf{G}}$ is then determined by the discriminant of the associated symmetric form [**17**].) If $\mathbf{G}$ is orthogonal, then we let E be the quadratic extension of F over which $\mathbf{G}$ splits.

If J_{2r} is symplectic, let $\mathbf{T}$ be the maximal split torus of diagonal elements in $\widetilde{\mathbf{G}}$. If J_{2r} is symmetric, then we take

$$\mathbf{T} = \left\{ \begin{pmatrix} x_1 & & & & & & & & \\ & x_2 & & & & & & & \\ & & \ddots & & & & & & \\ & & & x_l & & & & & \\ & & & & X & & & & \\ & & & & & x_l^{-1} & & & \\ & & & & & & \ddots & & \\ & & & & & & & x_2^{-1} & \\ & & & & & & & & x_1^{-1} \end{pmatrix} \mid x_i \in \mathbf{G}_m, X \in SO(\Lambda_2) \right\}.$$

Let $\mathbf{T}_d$ be the maximal split subtorus. Thus, in the symmetric case, $\mathbf{T}_d$ is the collection of elements of $\mathbf{T}$ with $X = I_2$. Let $\mathbf{B} = \mathbf{TU}$ be the Borel subgroup, with $\mathbf{U}$ the collection of upper triangular unipotent matrices in $\widetilde{\mathbf{G}}$. Let Δ be the set of simple roots of $\mathbf{T}$ in $\mathbf{U}$, and suppose that $\Theta = \Delta \setminus \{e_n - e_{n+1}\}$. Let

$$\mathbf{A} = \mathbf{A}_\Theta = (\bigcap_{\alpha \in \Theta} \ker \chi_\alpha)^\circ,$$

where χ_α is the root character attached to α. Then

$$\mathbf{A} = \left\{ \begin{pmatrix} xI_n & & \\ & I_{2m} & \\ & & x^{-1} I_n \end{pmatrix} \middle| x \in \mathbf{G}_m \right\}.$$

We will assume that n is even. If $\mathbf{M} = Z_{\mathbf{G}}(\mathbf{A})$, then $\mathbf{M} \simeq GL_n \times \mathbf{G}(m)$, where

$$\mathbf{G}(m) = \begin{cases} Sp_{2m}, & \text{if } \widetilde{\mathbf{G}} = Sp_{2r}; \\ SO_{J_{2m}}, & \text{if } \widetilde{\mathbf{G}} = SO_{J_{2r}}. \end{cases}$$

In fact,

$$\mathbf{M}=\left\{\begin{pmatrix} g & & \\ & h & \\ & & \varepsilon(g)\end{pmatrix} \mid g\in GL_n, h\in \mathbf{G}(m)\right\}.$$

Let $\mathbf{N}=\prod\limits_{\alpha\in\Phi^+\setminus\Sigma^+(\Theta)}\mathbf{N}_\alpha,$ where

$$\Sigma^+(\Theta)=\{\beta\in\Phi^+|\beta=\sum_{\Delta}c_\gamma\gamma,\ c_\gamma\geq 0\}.$$

We set $\mathbf{P}=\mathbf{MN}$. Let $P=\mathbf{P}(F)$, $M=\mathbf{M}(F)$, and $N=\mathbf{N}(F)$. We are interested in determining when $\mathrm{Ind}_P^{\widetilde{G}}(\pi)$ is reducible, if π is a discrete series representation of M. In order to do this, we will study the poles of the standard intertwining operators which arise, using the theory developed by Harish-Chandra [**23**]. We begin by looking at the structure of N to determine the types of integrals that need to be evaluated. The proof of the following lemma follows from a straightforward matrix computation.

Lemma 2.1. *Suppose that* $n\in\mathbf{N}$, *and* $n=\begin{pmatrix} I_n & X & Y\\ 0 & I_{2m} & X'\\ 0 & 0 & I_n\end{pmatrix}$. *Then we have*

$$X'=\begin{cases} u_{2m}{}^tXu_n, & \text{if } \widetilde{\mathbf{G}}=Sp_{2r};\\ -J_{2m}{}^tXw_n, & \text{if } \widetilde{\mathbf{G}}=SO_{J_{2r}},\end{cases}$$

and

$$Y+\widetilde{\varepsilon}(Y)=XX'. \qquad \square \tag{2.1}$$

Remark. If $n\in\mathbf{N}$ is as above, then we may denote n by $n(X,Y)$.

Let $\mathbf{G}'=GL_n$. For $\delta\in G'$, the set

$$\{g^{-1}\delta\varepsilon(g)\mid g\in G')\}$$

is called the ε-conjugacy class of δ in $GL_n(F)$. Furthermore, the group

$$\mathbf{G}'_{\varepsilon,\delta}=\{g\in\mathbf{G}'\mid g^{-1}\delta\varepsilon(g)=\delta\}$$

is called the ε-twisted centralizer of δ in $\mathbf{G}'$. We let $\mathcal{C}'$ be the set of ε-conjugacy classes in G', and denote the conjugacy classes in $G(m)$ by $\mathcal{C}$.

Let

$$g_\theta = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & -1 \end{pmatrix} \in GL_n(F).$$

Note that $u_n = g_\theta w_n$. If $n = 2m$, we rewrite (2.1) as

$$-Yg_\theta + \widetilde{\theta}(Yg_\theta) = Xg_\theta\widetilde{\theta}(X) \tag{2.2}$$

if $\widetilde{\mathbf{G}} = Sp_{2r}$, and

$$Y + \widetilde{\theta}(Y) = X(-J_n w_n)\widetilde{\theta}(X) \tag{2.3}$$

if $\widetilde{\mathbf{G}} = SO_{J_{2r}}$.

Let $\mathcal{N}$ be the collection of ε-conjugacy classes in $GL_n(F)$ for which (2.1) has F-rational solutions. Then (2.2) shows that, if $\widetilde{\mathbf{G}} = Sp_{2r}$, then

$$\mathcal{N} = \{\{Y\} \mid -Yg_\theta + \widetilde{\theta}(Yg_\theta) \sim g_\theta\},$$

where the equivalence is that of $\widetilde{\theta}$-skew symmetric forms. Since $-Yg_\theta + \widetilde{\theta}(Yg_\theta)$ is singular only on a set of measure zero in GL_n, and there is a unique equivalence class of non-degenerate $\widetilde{\theta}$-skew symmetric forms, we see that $\mathcal{N}$ is almost all of $\mathcal{C}'$. Similarly, (2.3) shows that if $\widetilde{\mathbf{G}} = SO_{J_{2r}}$ then, up to a set of measure zero, $\mathcal{N}$ is the set of θ-conjugacy classes for which $Y + \widetilde{\theta}(Y)$ is equivalent to $-J_n w_n$ as $\widetilde{\theta}$-symmetric forms.

Note that if $\{Y\} \in \mathcal{N}$, and (X, Y) is a solution to (2.1), then

$$Y^{-1} + \widetilde{\varepsilon}(Y^{-1}) = Y^{-1}XX'\widetilde{\varepsilon}(Y)^{-1} = (Y^{-1}X)(Y^{-1}X)'.$$

Therefore, $\mathcal{N}$ is closed under $\{Y\} \mapsto \{Y^{-1}\}$.

Let $\widetilde{w}_0$ be the non-trivial class in the Weyl group $W(\widetilde{\mathbf{G}}, \mathbf{A})$. Then $\widetilde{w}_0$ is represented by

$$w_0 = \begin{pmatrix} & & I_n \\ & I_{2m} & \\ I_n & & \end{pmatrix}.$$

The actions of w_0 on $\mathbf{M}$ and $\mathbf{A}$ are $(g,h)\mapsto(\varepsilon(g),h)$, and $x\mapsto x^{-1}$, respectively.

As in the work of Harish-Chandra, we have the following notation. For a connected reductive p-adic group G, we let $\mathcal{E}_c(G)$ be the collection of equivalence classes of irreducible admissible representations of G. We denote by $\mathcal{E}(G)$ those classes in $\mathcal{E}_c(G)$ which are unitarizable. We let $\mathcal{E}_2(G)$ be the discrete series classes. We use ${}^\circ\mathcal{E}_c(G)$ to denote the supercuspidal classes, and set ${}^\circ\mathcal{E}(G) = \mathcal{E}(G)\cap{}^\circ\mathcal{E}_c(G)$.

Let $(\tau',V')\in {}^\circ\mathcal{E}(G')$ and $(\tau,V)\in{}^\circ\mathcal{E}(G(m))$. Then $\tau'\otimes\tau$ is a unitary supercuspidal representation of M. Let

$$I(s,\tau'\otimes\tau)=\mathrm{Ind}_{MN}^{\widetilde{G}}((\tau'\otimes|\det()|^s)\otimes\tau\otimes\mathbf{1}_N).$$

We denote the space of $I(s,\tau'\otimes\tau)$ by $V(s,\tau'\otimes\tau)$. To understand the reducibility of $I(\tau'\otimes\tau)=I(0,\tau'\otimes\tau)$, one must determine the poles of the standard intertwining operator

$$A(s,\tau'\otimes\tau,w_0)f(g)=\int\limits_N f(w_0^{-1}ng)\,dn,$$

associated to $\tau'\otimes\tau$ (cf. [**23**]). Here $f\in V(s,\tau'\otimes\tau)$. By Bruhat's theorem (see [**10**]) we may assume that $w_0(\tau'\otimes\tau)\simeq\tau'\otimes\tau$. This is equivalent to assuming $\tau'\simeq\widetilde{\tau'}$ [**3**, §7].

Suppose that ${}^LM$ is the L-group of $\mathbf{M}$ and ${}^L\mathfrak{n}$ is the Lie algebra of ${}^LN$ (see [**4**]). If r is the adjoint representation of ${}^LM$ on ${}^L\mathfrak{n}$, then $r=r_1\oplus r_2$, where $r_2=\wedge^2\rho_n$, with ρ_n the standard representation of $GL_n(\mathbb{C})$, and r_1 is the tensor product of ρ_n with the standard representation of the L-group of $\mathbf{G}(m)$. Moreover, the labeling of these representations as r_1 and r_2 is consistent with the ordering given in [**18**]. Consequently, the results of [**19**] imply that if τ is generic, then the poles of $A(s,\tau'\otimes\tau,w_0)$ are the same as the poles of

$$L(s,\tau'\times\tau)L(2s,\tau',\wedge^2\rho_n),$$

where $L(s,\tau'\times\tau)=L(s,\tau'\otimes\tau,r_1)$. Furthermore, at most one of the two L-functions in the product can have a pole at $s=0$ (cf. [**19**, Theorem 7.6]). The poles of $L(s,\tau',\wedge^2\rho_n)$ were determined by Shahidi in [**20**]. Thus, our work here determines which representations τ' give rise to a pole of $L(s,\tau'\times\tau)$ at $s=0$ for some τ. As explained in [**19**] this completely determines $L(s,\tau'\times\tau)$ if τ and τ' are supercuspidal. The method of [**19**], as used in [**20**] and [**7**], can then be used to determine $L(s,\tau'\times\tau)$ for any pair of irreducible admissible representations τ and τ'.

Denote by $\bar{\mathbf{N}}$ the unipotent radical opposed to $\mathbf{N}$. We let

$$V(s,\tau'\otimes\tau)_0=\{h\in V(s,\tau'\otimes\tau)|\,\mathrm{supp}\,h\subset\bar{N}\text{ modulo }P\}.$$

By a lemma of Rallis, [**20**], it is enough to compute the poles that arise when $A(s,\tau'\otimes\tau,w_0)$ is applied to functions in $V(s,\tau'\otimes\tau)_0$ and evaluated at the identity. Thus, we note the following result.

Lemma 2.2. *Let $\bar{\mathbf{N}}$ be the unipotent radical opposed to $\mathbf{N}$. If $n=n(X,Y)\in N$, then $w_0^{-1}n\in P\bar{N}$ if and only if $Y\in GL_n(F)$, in which case*

$$w_0^{-1}n=\begin{pmatrix}\varepsilon(Y)&-Y^{-1}X&I_n\\0&I_{2m}-X'Y^{-1}X&X'\\0&0&Y\end{pmatrix}\begin{pmatrix}I_n&0&0\\(Y^{-1}X)'&I_{2m}&0\\Y^{-1}&Y^{-1}X&I_n\end{pmatrix}.\quad\square$$

We note, for future reference, that $(Y^{-1}X)'=X'\varepsilon(Y)$.

The proof of the following result is computational, and is based on equation (2.1).

Lemma 2.3. *Suppose that (X,Y) is an F-rational solution to (2.1). Then $I_{2m}-X'Y^{-1}X\in G(m)$.* $\square$

Let $h\in V(s,\tau'\otimes\tau)_0$. Fix open compact subsets $L\subset M_n(F)$ and $L'\subset M_{n\times 2m}(F)$. We assume that, for some $v'\in V'$, $v\in V$, we have

$$h\left(\begin{pmatrix}I&0&0\\X'\varepsilon(Y)&I&0\\Y^{-1}&Y^{-1}X&I\end{pmatrix}\right)=\xi_L(Y^{-1})\xi_{L'}(Y^{-1}X)(v'\otimes v),$$

where ξ_L and $\xi_{L'}$ are the characteristic functions of L and L', respectively. Choose $\widetilde{v}'\in\widetilde{V}'$ and $\widetilde{v}\in\widetilde{V}$. Let $\psi_{\tau'}$ and f_τ be the matrix coefficients of τ'

and τ given by the pairs $(v',\widetilde{v}')$ and $(v,\widetilde{v})$, respectively. Then, from Lemma 2.2, $<\widetilde{v}'\otimes\widetilde{v},A(s,\tau'\otimes\tau,w_0)h(e)>$ is equal to

$$\int\limits_{(Y,X)} \psi_{\tau'}(\varepsilon(Y))f_\tau(I-X'Y^{-1}X)|\det Y|^{-s-<\rho_{\mathbf{P}},\tilde{\alpha}>}\xi_L(Y^{-1})\xi_{L'}(Y^{-1}X)\ d(X,Y),$$

where the integral in over the collection of F-rational solutions (Y,X) of (2.1). Here, $\rho_{\mathbf{P}}=\frac{1}{2}\sum\limits_{\alpha\in\Phi^+\setminus\Sigma^+(\Theta)}\alpha$, $\widetilde{\alpha}$ is as in [**19**], and $d(X,Y)$ is a choice of Haar measure on N. Straightforward computations show that

$$<\rho_{\mathbf{P}},\widetilde{\alpha}>=\begin{cases}2m+(n+1) & \text{if } \mathbf{G}=Sp_{2r},\\ 2m+(n-1) & \text{otherwise.}\end{cases}$$

Consider the adjoint action of M on N. Note that $d^*(X,Y)=|\det Y|^{<\rho,\tilde{\alpha}>}d(X,Y)$ is an M–invariant measure on N. If (X,Y) represents a point in N, then its stabilizer under the adjoint action is

$$\Delta^\vee=\{(g,h)|g\in\mathbf{G}'_{\varepsilon,Y}(F),h\in\mathbf{G}(F),\ \text{with } gX=Xh\}.$$

For almost all (X,Y) the stabilizer $\Delta^\vee$ is a torus, whose conjugacy class is uniquely determined. Fix a measure $d\delta$ on each $\Delta^\vee$ compatible with this conjugation. Note that the orbit of (X,Y) and $(Y^{-1}X,\varepsilon(Y))$ are the same.

$\Delta^\vee$ compatible with this conjugation. Fix invariant measures d^*g and d^*h on $GL_n(F)$ and $\mathbf{G}(F)$ so that the measure $d^*(X,Y)$ on each orbit is given by the quotient of $d^*h\,d^*g$ by $d\delta$. Fix (X,Y) and consider the map $(g,h)\mapsto(gXh^{-1},gY\varepsilon(g)^{-1})$ from $GL_n(F)\times\mathbf{G}(F)$ to the orbit of (X,Y). Changing our representative for the orbit from (X,Y) to $(Y^{-1}X,\varepsilon(Y))$ changes (g,h) to (gY^{-1},h). Thus, the measures dg^* and dh^* are unchanged by this change of representative. Also, changing (X,Y) to $(Y^{-1}X,\varepsilon(Y))$ changes $\Delta^\vee$ to $(Y,I)\Delta^\vee(Y,I)^{-1}$, and therefore leaves $d\delta$ invariant. Thus, $d^*(X,Y)=d^*(Y^{-1}X,\varepsilon(Y))$.

Making this change of variables, the above integral can be expressed as

$$\text{(2.4)}\qquad \int\limits_{(Y,X)} \psi_{\tau'}(Y)f_\tau(I-X'Y^{-1}X)\xi_L(\varepsilon(Y)^{-1})\xi_{L'}(X)|\det Y|^s\ d^*(X,Y).$$

Let ω' be the central character of τ'. Since we are assuming that τ' is self-dual, ${\omega'}^2$ is trivial. We choose $f' \in C_c^\infty(G')$ so that

$$\psi_{\tau'}(g') = \int\limits_{Z(G')} f'(zg')\omega'(z^{-1})\ d^\times z.$$

Making this substitution, we rewrite (2.4) as

(2.5)

$$\int\limits_{F^\times}\int\limits_{(Y,X)} f'(zY)f_\tau(I - X'Y^{-1}X)\omega'(z)|\det Y|^s \xi_L(\varepsilon(Y)^{-1})\xi_{L'}(X)\ d^*(X,Y)\ d^\times z.$$

§3 **The norm correspondence and its consequences.** In this section, we study the properties of the map $(X,Y) \mapsto I_{2m} - X'Y^{-1}X$. In particular, we discuss the relation with the norm map defined by Kottwitz and Shelstad in [**12**]. The results here have the same flavor as Sections 4 and 5 of [**21**]. We let

$$\mathbf{G}^\vee(m) = \begin{cases} G(m), & \text{if } \widetilde{\mathbf{G}} = Sp_{2r}; \\ O_{J_{2m}}(F), & \text{if } \widetilde{\mathbf{G}} = SO_{J_{2r}}. \end{cases}$$

Lemma 3.1. *Suppose (X,Y) is a rational solution to (2.1). Further suppose that $Xg(Xg)' = XX'$, for some $g \in GL_{2m}(F)$. Then $Xg = Xh$ for some $h \in G^\vee(m)$.*

Proof. Consider F^n and F^{2m} as row vectors, and let $U = F^nX \subset F^{2m}$. Let $<,>$ be the symmetric (or symplectic) form given on F^{2m} by J_{2m}. Set

$$J'_{2m} = \begin{cases} J_{2m} = u_{2m}, & \text{if } \widetilde{\mathbf{G}} = Sp_{2r}; \\ -J_{2m}, & \text{if } \widetilde{\mathbf{G}} = SO_{J_{2r}}, \end{cases}$$

and

$$W_n = \begin{cases} u_n & \text{if } \widetilde{\mathbf{G}} = Sp_{2r}; \\ w_n & \text{if } \widetilde{\mathbf{G}} = SO_{J_{2r}}. \end{cases}$$

Note that, in each case, $X' = J'_{2m}\,{}^tXW_n$. Since $(Xg)(Xg)' = XX'$, we have $XgJ'_{2m}\,{}^tg\,{}^tXW_n = XJ'_{2m}{}^tXW_n$, which we rewrite as $XgJ'_{2m}{}^tgX = XJ'_{2m}{}^tX$. Let

$g^* : U \hookrightarrow F^{2m}$ be given by $g^*(v) = vg$. Now, suppose that $v_1, v_2 \in U$, and $v_i = y_i X$, with $y_i \in F^n$. Then

$$\begin{aligned} &< g^*(v_1), g^*(v_2) > = < v_1 g, v_2 g > \\ &= v_1 g J'_{2m} {}^t g {}^t v_2 = y_1 X g J'_{2m} {}^t g {}^t X y_2 \\ &= y_1 X J'_{2m} {}^t X y_2 = < v_1, v_2 > . \end{aligned}$$

Therefore, g^* is an isometry onto a subspace of F^{2m}. Consequently, by Witt's theorem for symmetric and symplectic forms, [**16**], we can choose $h \in G^\vee(m)$, with $vg = vh$ for all $v \in U$. Now, for each $y \in F^n$, we have $yXg = yXh$, and so $Xg = Xh$. $\square$

Let $\{Y^{-1}\}$ be the ε-conjugacy class of Y^{-1} in $GL_n(F)$. Suppose that (X, Y) satisfies (2.1), and $g \in GL_n(F)$. Let W_n and J'_{2m} be as in the proof of Lemma 3.1. Then $(gX)(gX)' = gXJ'_{2m} {}^t X {}^t g W_n = gXX'\varepsilon(g^{-1})$. Note that

$$\begin{aligned} gY\varepsilon(g^{-1}) + \widetilde{\varepsilon}(gY\varepsilon(g^{-1})) &= gY\varepsilon(g)^{-1} + W_n^{-1} {}^t(gYW_n^{-1} {}^t g W_n) W_n \\ &= gY\varepsilon(g)^{-1} + g\widetilde{\varepsilon}(Y)\varepsilon(g)^{-1} = (gX)(gX)'. \end{aligned}$$

Thus, the orbits $\{X\} \in GL_n(F)\backslash M_{n\times 2m}(F)$ parameterize the ε-conjugacy classes for which (2.1) has an F-rational solution.

Now suppose that $\{X\} \in GL_n(F)\backslash M_{n\times 2m}(F)$ parameterizes $\{Y^{-1}\} \in \mathcal{N}$. If we replace X with gX, then $I - X'Y^{-1}X$ is unchanged if we replace Y^{-1} with $\varepsilon(g)^{-1}Y^{-1}g$, which is also in $\{Y^{-1}\}$.

If X is changed to $X_1 = gX$, with $g \in GL_n(F)$, we then say that $\{X_1\}$ parameterizes $\{Y_1^{-1}\}$, rather than $\{X_1\}$ parameterizes $\{Y^{-1}\}$, to point out that (X_1, Y_1) satisfies (2.1), although the classes are the same.

Now suppose $\{X_1\}$, $X_1 = Xh$, $h \in GL_{2m}(F)$, also parameterizes $\{Y^{-1}\}$. Then by Lemma 3.1, $h \in G^\vee(m)$ and

$$\begin{aligned} I - X_1'Y^{-1}X_1 &= I - (Xh)'Y^{-1}Xh = I - (J'_{2m} {}^t h {}^t X W_n)Y^{-1}Xh \\ &= I - h^{-1}J'_{2m} {}^t X W_n Y^{-1} X h = h^{-1}(I - X'Y^{-1}X)h. \end{aligned}$$

Thus, the conjugacy class of $I - X'Y^{-1}X$ in $G^\vee(m)$ is unchanged.

Lemma 3.2. *Suppose* $\{X_1\}$, $X_1 = gXh$, $g \in GL_n(F)$, $h \in GL_{2m}(F)$, *parameterizes* $\{Y_1^{-1}\}$, $Y_1 = gY\varepsilon(g)^{-1}$. *Then* $X_1 \in GL_n(F)XG^\vee(m)$. $\square$

We denote the set of all conjugacy classes $\{I - X'Y^{-1}X\}$ for all possible X by $N_\varepsilon(\{Y^{-1}\})$. By Lemma 3.2 it defines a one to finite correspondence.

Lemma 3.3. *Suppose that* $n = n(X,Y) \in N$, *with* Y *invertible. Then*

(a) $(I - X'Y^{-1}X)X' = -X'Y^{-1}\varepsilon(Y^{-1})$, *and*

(b) $X(I - X'Y^{-1}X) = -\varepsilon(Y^{-1})Y^{-1}X$.

Proof. By Lemma 2.2,

$$\begin{pmatrix} 0 & 0 & I \\ 0 & I & X' \\ I & X & Y \end{pmatrix} = \begin{pmatrix} \varepsilon(Y) & -Y^{-1}X & I \\ 0 & I - X'Y^{-1}X & X' \\ 0 & 0 & Y \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ X'\varepsilon(Y) & I & 0 \\ Y^{-1} & Y^{-1}X & I \end{pmatrix}.$$

Comparing the (2,1)-entries, we see that

$$0 = (I - X'Y^{-1}X)X'\varepsilon(Y) + X'Y^{-1},$$

or

$$(I - X'Y^{-1}X)X'\varepsilon(Y) = -X'Y^{-1},$$

which gives us (a). We can rewrite this as

$$(I - X'Y^{-1}X)J'_{2m}\,{}^tXW_n = -J'_{2m}\,{}^tXW_nY^{-1}W_n\,{}^tYW_n^{-1},$$

and thus,

$$J_{2m}^{-1}(I - X'Y^{-1}X)J_{2m}\,{}^tX = -{}^tX\varepsilon({}^tY){}^tY.$$

Since $I - X'Y^{-1}X \in G(m)$, we have

$${}^t(I - X'Y^{-1}X)^{-1}\,{}^tX = -{}^tX\varepsilon({}^tY){}^tY,$$

or

$$X(I - X'Y^{-1}X)^{-1} = -Y\varepsilon(Y)X,$$

which proves (b). $\square$

Lemma 3.4. *Suppose that* $X \in M_{n\times 2m}(F)$ *and* $U = F^n X$. *Let*

$$H_X = \{h \in G(m) \,|\, Xh = g_h X \text{ for some } g_h \in GL_n(F)\}.$$

If $\{0\} \subsetneq U \subsetneq F^{2m}$, *then* $H_X \subsetneq G(m)$.

Proof. Suppose that $h \in H_X$. Let $u \in U$, and choose $v \in F^n$ with $u = vX$. Then $uh = vXh = vg_h X \in U$. Thus, U is an H_X invariant subspace of F^{2m}. If $H_X = G(m)$, then we know that $\{0\}$ and F^{2m} are the only invariant subspaces. □

Lemma 3.5. *Suppose that* n *is even, and* $\{X\} \in GL_n(F)\backslash M_{n\times 2m}(F)/G^\vee(m)$. *If* (X,Y) *is a rational solution to (2.1), with* Y *invertible, then*

$$X(I - X'Y^{-1}X) = -\varepsilon(Y^{-1})Y^{-1}X,$$

and $I - X'Y^{-1}X \in H_X$.

Proof. That $X(I - X'Y^{-1}X) = -\varepsilon(Y^{-1})Y^{-1}X$ follows from Lemma 3.3. Since $X(I - X'Y^{-1}X) = gX$, for some $g \in GL_n(F)$, we see that $I - X'Y^{-1}X \in H_X$. □

Lemma 3.6. *Fix* X *in* $M_{n\times 2m}(F)$, *and let* $U = F^n X$. *We consider* F^{2m} *as a symplectic or symmetric space with respect to the form* J_{2m}. *If* U *is non-degenerate, then* H'_X, *the right stabilizer of* U *in* $G(m)$, *is the stabilizer of an involution of* $G^\vee(m)$. *If* U *is degenerate, then* H'_X *is contained in a proper parabolic subgroup of* $G(m)$.

Proof. If U is non-degenerate, then Lemma 42.4 of [**16**] implies that $F^{2m} = U \perp W$. Of course, $W = U^\perp$. Since $Uh = U$, and $h \in G(m)$, we know h stabilizes W as well. Therefore, h centralizes $-1_U \perp 1_W$. Conversely, suppose $h \in \mathrm{Stab}_{-1_U \perp 1_W} \cap G(m)$. For each $u \in U$, we let $uh = u_1 + w_1$, be the decomposition of u with respect to U and W. Then $u(-1_U \perp 1_W)h = -uh = -u_1 - w_1$, while $uh(-1_U \perp 1_W) = -u_1 + w_1$. Thus, $w_1 = 0$, and so $Uh = U$, i.e. $h \in H'_X$.

If U is degenerate, then we let $U = W \perp \mathrm{Rad}\, U$. If $u \in \mathrm{Rad}\, U$, then, for all $u_1 \in U$, we have $< uh, u_1 >=< u, u_1 h >= 0$, so h stabilizes $\mathrm{Rad}\, U$. Since $\mathrm{Rad}\, U$ is totally isotropic, $H'_X \subset P_X = M_X N_X$, where P_X is the parabolic subgroup stabilizing precisely $\mathrm{Rad}\, U$. □

Proposition 3.7. *Suppose that $n < 2m$, and take $X \in M_{n\times 2m}(F)$. Fix an invertible Y with (X,Y) an F-rational solution to (2.1). Then $I - X'Y^{-1}X$ belongs to a proper parabolic subgroup, or a proper centralizer of a singular elliptic element of $G^\vee(m)$. Moreover, $\{N_\varepsilon(Y)\}$ is never regular elliptic.*

Proof. Since $n < 2m$, we have $U = F^nX \subsetneq F^{2m}$, and the first result follows from Lemmas 3.3(b) and 3.6. The last assertion is a consequence of $I - X'Y^{-1}X$ having at least $2m-n$ eigenvalues equal to 1. $\square$

Lemma 3.8. *Suppose that $I + S \in G(n/2)$. Then there is some $Y \in GL_n(F)$, and a projection $X \in M_n(F)$ with*

$$S = -X'Y^{-1}X = -X'Y^{-1} = \begin{cases} Y^{-1}X, & \text{if } \widetilde{\mathbf{G}} = Sp_{2r}; \\ J_n w_n Y^{-1}X, & \text{if } \widetilde{\mathbf{G}} = SO_{J_{2r}}. \end{cases}$$

Proof. First suppose that J_{2r} is symmetric. For $S \in M_n(F)$, we let $S^\vee = J_n{}^tSJ_n^{-1}$. Then $S \mapsto S^\vee$ is an anti-involution. Suppose $I + S \in SO_{J_n}$. Then

$$(I+S)J_n{}^t(I+S) = J_n,$$

so

$$SJ_n + J_n{}^tS + SJ_n{}^tS = 0.$$

Therefore,

$$S + S^\vee + SS^\vee = 0,$$

or

$$S^\vee = -(I+S)^{-1}S.$$

Now, applying Lemma 5.6 of [**21**], with $A = -(I+S)^{-1}$, we can find $Y_1 \in GL_n(F)$, and a projection $X \in M_n(F)$, with $S = X^\vee Y_1^{-1}X = X^\vee Y_1^{-1} = Y_1^{-1}X$. Let $Y = Y_1J_nw_n^{-1}$. Then $S = -X'Y^{-1}X = -X'Y^{-1}$, and $S = Y_1^{-1}X = J_nw_nY^{-1}X$.

Now suppose that J_{2r} is symplectic. Then we repeat the above argument with $S^\vee = u_n{}^tSu_n = -\widetilde{\theta}^*(S)$, to get the result of the lemma. $\square$

Definition 3.9. *Suppose the pair* (Y, X) *satisfies the hypotheses of Lemma 3.8. Then* $\{Y^{-1}\}$ *is called the* **canonical section** *of the norm correspondence over* $\{Z\}$, *where* $Z = I - X'Y^{-1}X$ (cf. [**21**]).

Remark. We have called this the canonical section since it is the natural extension of the unique section defined at every *generic* point Z, i.e., those Z for which $X = I$, through $I + Y^{-1} = Z$. Observe that this applies to almost all Z.

Lemma 3.10. *Suppose that* $\widetilde{\mathbf{G}} = SO_{J_{2r}}$ *and* $g \in G(n)$. *Then the dimension of the* g*-fixed vectors in* F^{2n} *is even.*

Proof. We know that $\mathbf{G} = \mathbf{G}(n)$ splits over a quadratic extension E/F. Thus, $\mathbf{G}(E) \simeq SO_{2n}(E)$. By Lemma 5.8 of [**21**], we know that the dimension of the fixed point set of g in E^{2n} is even. Suppose that V is the fixed point set of g in F^{2n}. Let $v \in E^{2n}$, with $gv = v$. We write $v = \sum c_i v_i$, with $v_i \in F^{2n}$, and $c_i \in E$. We may assume that $\{v_i\}$ and $\{c_i\}$ are linearly independent over F. Extending $\{v_i\}$ to an F-basis of F^{2n}, we write $gv_i = \sum \alpha_{ij} v_j$, with $\alpha_{ij} \in F$. Since $gv = \sum c_i g v_i = v$, we see that $\sum c_i v_i = \sum \sum c_j \alpha_{ji} v_j$. Comparing coefficients, and using the linear independence of $\{c_i\}$ over F, we see that (α_{ij}) is the identity matrix, i.e., $v_i \in V$, for each i. Thus, $\dim_F(V) = \dim_E(V \otimes_F E)$, and this last is the dimension of the g fixed points in E^{2n}, and hence is even. $\square$

Lemma 3.11. *Suppose that* $n = 2m$. *Then* $N_\varepsilon : \mathcal{N} \longrightarrow \mathcal{C}$ *is surjective with finite fibers.*

Proof. First suppose that J_n is symmetric. Suppose $I - S \in G(m)$, and choose a projection $X \in M_n(F)$ with $S = X'Y^{-1}X = X'Y^{-1} = -J_n w_n Y^{-1} X$, as in Lemma 3.8.

Now, we have

$$(I + J_n w_n Y^{-1} X) J_n (I - {}^tX {}^tY {}^tX') = J_n,$$

which implies

$$J_n w_n Y^{-1} X J_n - J_n {}^tX {}^tY^{-1} {}^tX' = J_n w_n Y^{-1} X J_n {}^tX {}^tY^{-1} {}^tX'.$$

Thus

$$Y^{-1}X + \widetilde{\theta}(Y^{-1})X = -Y^{-1} X J_n {}^tX {}^tY^{-1} w_n X = Y^{-1} X X' \widetilde{\theta}(Y^{-1}) X.$$

Note that if v is in the image of X, then $v = Xv$, so $Y^{-1}v + \widetilde{\theta}(Y^{-1})v = Y^{-1}XX'\widetilde{\theta}(Y^{-1})v$, or $(Y + \widetilde{\theta}(Y))v = XX'v$.

So we need to show that we can choose (X, Y) satisfying (2.1) on $\ker X$. Since $\ker X \subset \ker(Y^{-1}XX'\widetilde{\theta}(Y^{-1})X) = \ker(Y^{-1}XX'\widetilde{\theta}(Y^{-1}))$, we need to show that we can choose (Y, X) so that $Y + \widetilde{\theta}(Y) = 0$ on $\ker X$. We know that $J_n w_n Y^{-1}X = -X'Y^{-1}$, and $J_n w_n \widetilde{\theta}(Y^{-1})X = -X'\widetilde{\theta}(Y^{-1})X$. Note that $\widetilde{\theta}(X)$ is also a projection, and $\widetilde{\theta}(X) = -w_n J_n^{-1} X'$. Thus, both $Y^{-1}|_{\ker X}$ and $\widetilde{\theta}(Y^{-1})|_{\ker X}$ are isomorphisms onto $\ker \widetilde{\theta}(X)$. (Note that $-\widetilde{\theta}(X)$ is the matrix denoted by X' in [**21**].) For $v \in F^n$, we let $\widetilde{\theta}(v) = {}^t(w_n v)$. Note that $\widetilde{\theta}(v)\widetilde{\theta}(Y^{-1}) = {}^tv\,{}^tY^{-1}w_n = \widetilde{\theta}(Y^{-1}v)$. Choose bases $\mathcal{B}$ and $\mathcal{B}'$ for $\ker X$ and $\ker \widetilde{\theta}(X)$, respectively. Then we see that the matrix of $Y^{-1}|_{\ker X}$ with respect to $\mathcal{B}, \mathcal{B}'$, is equal to the matrix of $\widetilde{\theta}(Y^{-1})|_{\ker X}$, with respect to the bases $\widetilde{\theta}(\mathcal{B})$, and $\widetilde{\theta}(\mathcal{B}')$. Note that $\ker X$ is precisely the fixed point space of $I - X'Y^{-1}X$ in F^n, and thus, by Lemma 3.10, we can choose a unique (up to θ-conjugacy) Y with $Y^{-1}|_{\ker X}$ a $\widetilde{\theta}$-skew symmetric matrix. For such a choice of (Y, X), we now have $Y^{-1} + \widetilde{\theta}(Y^{-1})X = Y^{-1}XX'\widetilde{\theta}(Y^{-1})$, and thus (2.1) is satisfied.

Suppose now that J_n is symplectic. If $I - S \in Sp_{2n}(F)$, then we can choose a projection $X \in M_n(F)$ and $Y \in GL_n(F)$ so that $S = X'Y^{-1}X = X'Y^{-1} = -Y^{-1}X$, as in Lemma 3.8. Using an argument similar to the one above, we see that

$$Y^{-1}X + \widetilde{\theta}^*(Y^{-1})X = Y^{-1}XX'\widetilde{\theta}^*(Y^{-1})X.$$

Then we again have (2.1) on the image of X. Note that, in this case, $X' = u_n{}^tXu_n = -\widetilde{\theta}^*(X)$, and $-X' = \widetilde{\theta}^*(X)$ is a projection. Moreover, $-X'Y^{-1} = Y^{-1}X$, and $-X'\widetilde{\theta}^*(Y^{-1}) = \widetilde{\theta}^*(Y^{-1})X$. Thus, both Y^{-1} and $\widetilde{\theta}^*(Y^{-1})$ carry $\ker X$ isomorphically onto $\ker(-X')$. Thus, the argument reduces to showing that we can choose Y so that

$$Y|_{\ker X} + \widetilde{\theta}^*(Y|_{\ker X}) = 0.$$

Note that $\widetilde{\theta}^*(Y) = -Y$ if and only if $Yg_\theta w_n = {}^t(Yg_\theta w_n)$. However, on any vector space, one can always find a symmetric invertible transformation. For example, taking $Y|_{\ker X} = (w_n g_\theta)|_{\ker X} = u_n^{-1}|_{\ker X}$, we get the desired element of GL_n.

To prove that N_ε has finite fibers we use the following result.

Lemma 3.12. *If (Y,X) is an F-rational solution to (2.1), then $N_\varepsilon(\{Y^{-1}\})$ determines the semisimple part of $\{\varepsilon(Y^{-1})Y^{-1}\}$ in $GL_n(F)$ uniquely.*

Proof. Changing Y to an ε-conjugate, we can assume that X is a projection. Suppose that v is in the left image $\operatorname{Im} X$ of X. Then $vX = v$, so by Lemma 3.3, $v\varepsilon(Y^{-1})Y^{-1}X = -v(I - X'Y^{-1}X)$. Suppose v is in the left kernel $\ker X$ of X. Then $v(Y + \widetilde{\varepsilon}(Y)) = vXX' = 0$, so $v\varepsilon(Y^{-1})Y^{-1} = -v$. Therefore, the matrix of $\varepsilon(Y^{-1})Y^{-1}$ with respect to a basis of F^n which respects $F^n = \operatorname{Im} X \oplus \ker X$ is (acting on the right)

$$\begin{pmatrix} -(I - X'Y^{-1}X)|_{\operatorname{Im} X} & * \\ 0 & -I \end{pmatrix}.$$

Thus, the semisimple part of $\{\varepsilon(Y^{-1})Y^{-1}\}$ is uniquely determined by $N_\varepsilon(\{Y^{-1}\})$, completing the lemma. $\square$

Now, the finiteness of the fibers is as in [**21**]. $\square$

Corollary 3.13. *If $n > 2m$, then the statement of Lemma 3.11 is true.*

Proof. Let $j = (n/2) - m$, and consider the injection

$$h \mapsto \begin{pmatrix} I_j & & \\ & h & \\ & & I_j \end{pmatrix}$$

of $\mathbf{G}(m)$ into $\mathbf{G}(n/2)$. By Lemma 3.11, there are $X \in M_n(F)$ and $Y \in GL_n(F)$ with

$$\begin{pmatrix} I_j & & \\ & h & \\ & & I_j \end{pmatrix} = I_n - X'Y^{-1}X.$$

Let $X = \begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix}$, with X_1, $X_3 \in M_{n\times j}(F)$, and $X_2 \in M_{n\times 2m}(F)$. Then

$$X' = \begin{pmatrix} X_3'' \\ X_2' \\ X_1'' \end{pmatrix},$$

where $X_i'' = -W_j\,{}^tX_iW_n^{-1}$, and W_ℓ is as in the proof of Lemma 3.1. We now have

$$I - X'Y^{-1}X = I - \begin{pmatrix} X_3''Y^{-1} \\ X_2'Y^{-1} \\ X_1''Y^{-1} \end{pmatrix} \begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix}.$$

Thus, by inspection, $I_{2m} - X_2'Y^{-1}X_2 = h$. The finiteness of the fiber follows from that of the case $n = 2m$. $\square$

Finally, let N_ε^{-1} denote the canonical section of the norm correspondence defined before. Then, by Lemma 3.11, $N_\varepsilon^{-1} : \mathcal{C} \longrightarrow \mathcal{C}'$.

Lemma 3.14. *The map* $N_\varepsilon^{-1} : \mathcal{C} \longrightarrow \mathcal{C}'$ *is continuous.*

Proof. The proof is similar to the proof of Lemma 5.16 of [**21**].

Note that θ^* fixes the standard splitting of GL_n, while θ does not. However, $\theta^* = \mathrm{Int}(g_\theta)\theta$, where $g_\theta = \mathrm{diag}\{1, -1, 1, -1 \ldots, -1\}$. Suppose that $F = \bar{F}$. Let $\mathbf{T}_0'$ be the torus of diagonal matrices in GL_n. We define $\overline{N}_{\theta^*}$ by $\overline{N}_{\theta^*}(Y) = Y\theta^*(Y)$, for $Y \in \mathbf{T}_0'$.

If $Y = \mathrm{diag}\{a_1, \ldots, a_n\}$, then $\theta^*(Y) = \mathrm{diag}\{a_n^{-1} \ldots, a_2^{-1}, a_1^{-1}\}$. Therefore, $Y\theta^*(Y) = \mathrm{diag}\{a_1a_n^{-1}, a_2a_{n-1}^{-1}, \ldots, a_na_1^{-1}\}$. Consequently,

$$\ker \overline{N}_{\theta^*} = \{\mathrm{diag}\{a_1, a_2, \ldots, a_{n/2}, a_{n/2}, \ldots, a_2, a_1\}\}.$$

Let $Y_0 = \mathrm{diag}\{a_1, \ldots, a_{n/2}, 1, 1, \ldots, 1\}$. Then

$$(I - \theta^*)(Y_0) = Y_0\theta^*(Y_0)^{-1} = \mathrm{diag}\{a_1, \ldots, a_{n/2}, a_{n/2}, \ldots, a_2, a_1\} \in \ker \overline{N}_{\theta^*}.$$

Therefore, $\ker \overline{N}_{\theta^*} = (I - \theta^*)\mathbf{T}_0'$. (Of course, this is the same as in [**21**], since $SO_{J_{2n}}$ is split over $\bar{F}$. For Sp_{2n}, it also follows from [**21**], since SO_{2n} and Sp_{2n} share the diagonal Cartan.)

Now suppose that F is not necessarily algebraically closed. Let $\mathbf{T_H}$ be a Cartan subgroup of $\mathbf{G}(n/2)$, defined over F. Choose a θ^*-stable pair $(\mathbf{B}', \mathbf{T}')$ of GL_n, with $\mathbf{T}'$ defined over F, such that there is an isomorphism $\mathbf{T_H} \xrightarrow{\sim} \mathbf{T}'_{\theta^*}$ is defined over F [**12**].

Lemma 3.15. *The map* $Y \mapsto Y\theta^*(Y)$ *from* $\mathbf{T}'$ *to* $\mathbf{T}'$ *has*

$$(\mathbf{T}')^{\theta^*} = \{t | \theta^*(t) = t\}$$

as its image, can be identified with the projection of $\mathbf{T}'$ *onto* $\mathbf{T}'_{\theta^*}$, *and is defined over* F.

Proof. The proof follows from the above observations, and the argument of Lemma 5.17 of [**21**]

Lemma 3.16. *Assume* $n = 2m$.

(a) *Suppose that* F *is algebraically closed. Let* $\{Y\} \in \mathcal{N} = \mathcal{C}'$ *be* ε*-semisimple, with* Y *in an* ε*-stable Cartan subgroup of* GL_n. *Then, there is an* $X \in M_n(F)$ *which satisfies (2.1) with* Y *so that* $I - X'Y^{-1}X$ *is semisimple in* $G(n/2)$. *Moreover,* $\varepsilon(Y^{-1})Y^{-1}$ *belongs to* $G(n/2)$, *and* $I - X'Y^{-1}X$ *is* $GL_n(F)$*-conjugate to* $-\varepsilon(Y^{-1})Y^{-1}$. *Furthermore, every* $GL_n(F)$*-conjugate of* $-\varepsilon(Y^{-1})Y^{-1}$ *belongs to the image of* $\{Y^{-1}\}$ *under* N_ε.

(b) *Suppose* F *is not necessarily algebraically closed. If there is an* X *in* $M_n(F)$ *such that* $I - X'Y^{-1}X$ *is semisimple in* $G(n/2)$, *then all the remaining statements of (a) are valid as well. If* $\{Y^{-1}\} \in \mathcal{N}$ *corresponds to* $\{X\} = \{I\}$, *i.e., for almost all* Y, *then the* $GL_n(F)$*-conjugates of* $-\varepsilon(Y^{-1})Y^{-1}$ *exhaust the image of* $\{Y^{-1}\}$ *under the norm map* N_ε.

(c) *In either case, the semisimple part of every conjugacy class in* $N_\varepsilon(\{Y\})$ *is* $GL_n(F)$ *conjugate to* $\{-\varepsilon(Y^{-1})Y^{-1}\}$.

Proof. Assume that F is algebraically closed. Suppose that J_n is symmetric, and that $A \in GL_n(F)$ satisfies $AJ_n\,{}^tA = w_n$. Since Y is θ-semisimple, there is an $X_1 \in M_n(F)$ with $I - X_1''Y^{-1}X_1$ semisimple in $SO_n(F)$ (the split form). Here $X'' = -w_n\,{}^tX_1 w_n$. Let $X = X_1A$. Then

$$\begin{aligned} I - X'Y^{-1}X &= I + J_n\,{}^t(X_1A)w_nY^{-1}X_1A = I + J_n\,{}^tA\,{}^tX_1w_nY^{-1}X_1A \\ &= I + A^{-1}w_n\,{}^tX_1w_nY^{-1}X_1A = A^{-1}(I - X_1''Y^{-1}X_1)A. \end{aligned}$$

Since $I - X_1''Y^{-1}X_1$ is semisimple in $SO_n(F)$, we see that $I - X'Y^{-1}X$ is semisimple in $SO_{J_n}(F)$.

On the other hand, if Y is θ^*-semisimple, then there is an $h \in Sp_{2n}(F)$ with $Y_1 = hYh^{-1} = \mathrm{diag}\{a_1, \ldots, a_n\}$. Then

$$Y_1 + \widetilde{\theta}^*(Y_1) = \mathrm{diag}\{a_1 + a_n, a_2 + a_{n-1}, \ldots, a_1 + a_n\}$$

Let $i = \sqrt{-1}$, and

$$X_1 = i\,\mathrm{diag}\{a_1 + a_n, a_2 + a_{n-1} \ldots, a_{n/2} + a_{n/2+1}, 1, \ldots, 1\}.$$

Then $X_1' = -i\,\mathrm{diag}\{1, \ldots, 1, a_{n/2} + a_{n/2+1}, \ldots, a_1 + a_n\}$. Therefore, $X_1X_1' = Y_1 + \widetilde{\theta}^*(Y_1)$. Note that

$$I - X_1'Y^{-1}X_1 = -\,\mathrm{diag}\{a_1^{-1}a_n, a_2^{-1}a_{n-1}, \ldots, a_n^{-1}a_1\},$$

which is semisimple. Letting $X = h^{-1}X_1$, we have

$$I - X'Y^{-1}X = I - X_1'\theta^*(h)h^{-1}Y_1h(h^{-1}X_1) = I - X_1'Y_1^{-1}X_1$$

is semisimple.

So, in either case, we can choose $X \in M_n(F)$ with the desired property. Now, since Y belongs to a ε-stable Cartan, we see that $\varepsilon(Y^{-1})Y^{-1}$ is semisimple in $G(n/2)$. Choose Y_2 which is ε-conjugate to Y and a projection X_2 satisfying (2.1) with Y, such that $I - X_2'Y_2^{-1}X_2 = I - X'Y^{-1}X$. Then $\varepsilon(Y_2^{-1})Y_2^{-1}$ is conjugate to $\varepsilon(Y^{-1})Y^{-1}$. By the proof of Lemma 3.12, we see that $-\varepsilon(Y^{-1})Y^{-1}$ has matrix $\begin{pmatrix} (I - X_2'Y_2^{-1}X_2)|_{\mathrm{Im}\ X} & * \\ 0 & I \end{pmatrix}$ with respect to a basis which respects $F^n = \mathrm{Im}\ X_2 \oplus \ker X_2$. Thus, the eigenvalues of $-\varepsilon(Y^{-1})Y^{-1}$ different from 1 are among those of $I - X_2'Y_2^{-1}X_2$. Since Y belongs to an ε-stable Cartan subgroup of $GL_n(F)$, we see that $Y^{-1}\varepsilon(Y^{-1}) = \varepsilon(Y^{-1})Y^{-1}$, and thus $Y_2^{-1}\varepsilon(Y_2^{-1})$ and $\varepsilon(Y_2^{-1})Y_2^{-1}$ have the same eigenvalues. Consequently, the eigenvalues of $I - X'Y^{-1}X = I - X_2Y_2^{-1}X_2$ which are not equal to one, are among those of $-\varepsilon(Y_2^{-1})Y_2^{-1}$, and therefore, $I - X'Y^{-1}X$ and $-\varepsilon(Y^{-1})Y^{-1}$ are $GL_n(F)$-conjugate. This part(a). Parts (b) and (c) follow from part (a). $\square$

Lemma 3.17. *Assume* $n = 2m$. *Suppose that* $Y + \widetilde{\varepsilon}(Y) = XX'$, *and* $Z = I - X'Y^{-1}X$. *Let* $g \in G'_{\varepsilon,Y}(F)$, *and suppose that there is an* $h \in G(n/2)$ *with* $gX = Xh$. *Then* h, *whose class modulo the right stabilizer of* X *is uniquely determined, belongs to* $G_Z(F)$. *Conversely, suppose* $h \in G_Z(F)$, *and* (X,Y) *gives the canonical section over* Z. *If there is some* $g \in G'(F)$ *with* $gX = Xh$, *then we can choose such a* g *in* $G'_{\varepsilon,Y}(F)$.

Proof. We only need to prove the converse. Recall that

$$W_n = \begin{cases} u_n, & \text{if } \widetilde{\mathbf{G}} = Sp_{2r}; \\ w_n, & \text{if } \widetilde{\mathbf{G}} = SO_{J_{2r}}. \end{cases}$$

Then in each case, $X' = -J_n\, {}^tXW_n^{-1}$, and from Lemma 3.8 we have

$$X'Y^{-1}X = X'Y^{-1} = -J_nW_n^{-1}Y^{-1}X.$$

Set $X'' = -W_n\, {}^tXW_n^{-1} = -\widetilde{\varepsilon}(X)$. Then $X'' = W_n^{-1}J_n^{-1}X'$, and thus,

$$X''Y^{-1}X = X''Y^{-1} = -Y^{-1}X.$$

Further note that $-X''$ is a projection, since X is.

Take $U_1 = XF^n$, $U_1' = (-X'')F^n$, $U_2 = \{v|Xv = 0\}$, and $U_2' = \{v'|X''v = 0\}$. Then $Y^{-1} : U_1 \longrightarrow U_1'$, and $Y^{-1} : U_2 \longrightarrow U_2'$. If we set $\widetilde{\varepsilon}(v) = {}^t(W_nv) = {}^tvW_n^{-1}$, then this is an isomorphism of F^n which takes U_1 to U' and U_1' to U, where $U = F^nX$ and $U' = F^n(-X'')$. Now $\varepsilon(Y)$ takes U' to U.

Suppose that $h^{-1}X'Y^{-1}Xh = X'Y^{-1}X$. Then

$$J_n\ {}^t(h^{-1}X'Y^{-1}Xh)W_n = J_n\ {}^t(X'Y^{-1}X)W_n,$$

which gives

$$h^{-1}X'\varepsilon(Y)Xh = X'\varepsilon(Y)X.$$

A straightforward computation shows that

$$G'_{\varepsilon,\varepsilon(Y)} = G'_{\varepsilon,Y^{-1}} = \varepsilon\left(G'_{\varepsilon,Y}\right).$$

Note that $I + X''$ and $I - X$ are projections, and

$$(I + X'')Y^{-1}(I - X) = (I + X'')Y^{-1} = Y^{-1}(I - X).$$

We now take $g \in G'_{\varepsilon,Y}$ with $gX = Xg$. Then $\varepsilon(g)^{-1}Y^{-1}g = Y^{-1}$, and so

$$(I + X'')\varepsilon(g)^{-1}Y^{-1}g(I - X) = (I + X'')Y^{-1}(I - X), \tag{3.1}$$

which implies that

$$\varepsilon(g)^{-1}(I + X'')Y^{-1}(I - X)g = (I + X'')Y^{-1}(I - X). \tag{3.2}$$

Let $g^\vee$ be defined by $g^\vee|_{U_1} = Xh|_{U_1}$ and $g^\vee|_{U_2} = (I-X)g|_{U_2}$. If $v \in U_1$, then $g^\vee Xv = g^\vee v = Xhv$. If $v \in U_2$, then $J_nW_nY^{-1}Xhv = hJ_nW_nY^{-1}Xv = 0$, and so, $Xhv = 0 = g^\vee Xv$. Thus, $g^\vee X = Xh$.

If $v \in U_1$, then we know that $Y^{-1}v \in U_1'$. Note that

$$\begin{aligned}
-X''\varepsilon(g^\vee)^{-1}Y^{-1}g^\vee v &= \varepsilon(h)^{-1}(-X'')Y^{-1}Xhv \\
= -\varepsilon(h)^{-1}W_n^{-1}J_n^{-1}X'Y^{-1}Xhv &= -\varepsilon(h)^{-1}W_n^{-1}J_n^{-1}hX'Y^{-1}v \\
= -\varepsilon(h)^{-1}W_n^{-1}\,{}^th^{-1}J_n^{-1}X'Y^{-1}v &= \varepsilon(h)^{-1}\varepsilon(h)(-X'')Y^{-1}v = -X''Y^{-1}v.
\end{aligned}$$

Furthermore, we see that

$$\begin{aligned}
(I + X'')\varepsilon(g^\vee)^{-1} &= W_n(I - X)W_n^{-1}W_n{}^tg^\vee W_n^{-1} \\
= W_n{}^t(g^\vee(I - X))W_n^{-1} &= W_n{}^t((I - X)g)W_n^{-1} = \varepsilon(g)^{-1}(I + X'').
\end{aligned}$$

Thus, we have

$$(I + X'')\varepsilon(g^\vee)^{-1}Y^{-1}g^\vee v = \varepsilon(g)^{-1}(I + X'')Y^{-1}Xhv = 0 = (I + X'')Y^{-1}v,$$

since $Y^{-1}Xhv \in U_1'$.

Now suppose that $v \in U_2$. Then

$$\begin{aligned}
-X''\varepsilon(g^\vee)^{-1}Y^{-1}g^\vee v &= \varepsilon(h)(-X'')Y^{-1}(I - X)gv \\
&= \varepsilon(h)(-X'')(I + X'')Y^{-1}gv = 0 = -X''Y^{-1}v.
\end{aligned}$$

Finally, we see that

$$\begin{aligned}
(I + X'')\varepsilon(g^\vee)^{-1}Y^{-1}g^\vee v &= \varepsilon(g)^{-1}(I + X'')Y^{-1}(I - X)gv \\
&= (I + X'')Y^{-1}(I - X)v = (I + X'')Y^{-1}v,
\end{aligned}$$

the next to last equality coming from (3.2), and the final one from the definition of U_2. So, $g^\vee$ is the desired element. $\square$

We can now state the final result of this section. This will allow us to determine the residues of the intertwining operator given in (2.5) via the theory of twisted endoscopy.

Lemma 3.18. *Suppose that* $n = 2m$.

(a) *For* $\widetilde{\mathbf{G}} = SO_{J_{2r}}$, *the norm correspondence* N_θ *agrees with the norm map of Kottwitz and Shelstad on the intersection of* $\mathcal{N}$ *with strongly* θ*-regular* θ*-semisimple conjugacy classes in* GL_n.

(b) *For* $\widetilde{\mathbf{G}} = Sp_{2r}$, *the norm correspondence* N_{θ^*} *agrees with the negative of the norm map of Kottwitz and Shelstad on the intersection of* $\mathcal{N}$ *with strongly* θ^**-regular* θ^**-semisimple conjugacy classes in* GL_n.

Proof. (a) follows from the argument of Lemma 5.21 of [**21**], since $SO_{J_{2n}}$ splits over L. For (b) we simply note that if $\mathbf{T_H}$ is a Cartan subgroup of Sp_{2n} defined over F, and $\mathbf{T}'$ is a θ^*-stable Cartan of GL_n, with $\mathbf{T_H} \longrightarrow \mathbf{T}'_{\theta^*}$ defined over F, then $N_{\theta^*}(\{Y\}) = \{-Y^{-1}\theta^*(Y^{-1})\}$, which is the negative of the norm map defined in [**12**]. $\square$

Remark. The appearance of the minus sign in (b) appears somewhat incongruous with the results of [**21**]. However, this will not affect the location of the poles of the intertwining operator, because it will only have the affect of multiplying the residue by a scalar, namely $\omega(-1)$, where ω is the central character of τ. However, the sign may be of some arithmetic significance, and this question should be addressed in the future.

§4 **The poles of Intertwining Operators.** We return now to the computation of the poles of the intertwining operator $A(s, \tau' \otimes \tau, w_0)$, discussed in Section 2. We will use the results of Section 3 to determine the residues of these integrals. We begin with the following proposition.

Proposition 4.1. *Assume* $n = 2m$. *Suppose the* ε *–conjugacy class* $\{Y^{-1}\}$ *is* ε *–regular. Then* $N_\varepsilon(\{Y^{-1}\})$ *consists of a single conjugacy class in* $G^\vee$ *of a* *regular semisimple element in G. Assuming that Y and $\varepsilon(Y)$ commute, i.e., that $Y^{-1}\varepsilon(Y^{-1})$ is in G, then the converse is true, i.e., if $N_\varepsilon(\{Y^{-1}\})$ is regular, then $\{Y^{-1}\}$ is ε –regular (and hence ε –semisimple).*

Proof. Suppose the ε-conjugacy class $\{Y^{-1}\}$ is ε –regular. Then up to $GL_n(\bar{F})$ –conjugation, $\varepsilon(Y^{-1})Y^{-1}$ is a regular semisimple element of $\mathbf{G}$. Choose Y_2^{-1}, ε –conjugate to Y^{-1}, and a projection X_2 satisfying (2.1) with Y_2, such that $I - X_2'Y_2^{-1}X_2 = I - X'Y^{-1}Y$. By Lemma 5.10 of [**21**], the eigenvalues of $\varepsilon(Y^{-1})Y^{-1}$ different from 1 are among those of the semisimple part of $I - X_2Y_2^{-1}X_2$. Since $Y^{-1}\varepsilon(Y^{-1})$ is $GL_n(F)$ –conjugate to $\varepsilon(Y^{-1})Y^{-1}$, one sees that the eigenvalues of $-\varepsilon(Y_2)^{-1}Y_2^{-1}$ and $-Y_2^{-1}\varepsilon(Y_2)^{-1}$ are the same. Therefore, one can apply the argument of Lemma 5.10 of [**21**] to equation in Lemma 3.3(a) to show that the eigenvalues of the semisimple part of $I - X_2'Y_2^{-1}X_2$ which are not 1 are also among those of $-\varepsilon(Y_2)^{-1}Y_2^{-1}$. Then the semisimple parts of $I - X'Y^{-1}X$ and $-\varepsilon(Y)^{-1}Y^{-1}$ are $GL_n(\bar{F})$ –conjugate. But $-\varepsilon(Y)^{-1}Y^{-1}$ is $GL_n(\bar{F})$ –conjugate to a regular element in $\mathbf{G}$, and therefore, $I - X'Y^{-1}X$ must be semisimple and regular.

Suppose now that $Y + \tilde{\varepsilon}(Y) = XX'$, with $Y^{-1}\varepsilon(Y)^{-1} \in \mathbf{G}(F)$, and assume $N_\varepsilon(\{Y^{-1}\})$ contains a regular semisimple element $\{I - X'Y^{-1}X\}$. Again by Lemma 3.3(a), and the argument of Lemma 3.12, the conjugacy class of $I - X'Y^{-1}X$ is completely determined by the semisimple part of $\{-Y^{-1}\varepsilon(Y^{-1})\}$, a conjugacy class in $\mathbf{G}(F)$. That is, the eigenvalues of the first are among those of the second. Moreover, by Lemma 3.12, the semisimple part of the conjugacy class of $-\varepsilon(Y^{-1})Y^{-1}$ is completely determined by $I - X'Y^{-1}X$. Since $\{I - X'Y^{-1}X\}$ is regular semisimple in $\mathbf{G}(F)$, and $Y^{-1}\varepsilon(Y^{-1}) \in \mathbf{G}(F)$, we conclude that $Y^{-1}\varepsilon(Y^{-1})$ is regular and semisimple. Therefore, $\{Y^{-1}\}$ is ε –regular. In fact, let $\widetilde{Y} = (Y, \varepsilon)$ represent an element in the non-identity component of $GL_n \rtimes \{1, \varepsilon\}$. Write $\widetilde{Y} = su$, with s semisimple and u unipotent. Then $\widetilde{Y}^2 = s^2u^2 = Y^{-1}\varepsilon(Y^{-1})$. If $Y^{-1}\varepsilon(Y^{-1})$ is semisimple, then $u^2 = u = 1$, and thus Y is ε –semisimple and ε –regular. $\square$

Corollary 4.2. *Suppose $n > 2m$. For almost all regular elliptic conjugacy classes*

$\{h\} \in G(m)$, the collection of ε –conjugacy classes, $N_\varepsilon^{-1}(\{h\})$ is parameterized by a unique ε –regular ε –conjugacy class in $GL_{2m}(\bar{F})$.

Proof. For almost all regular semisimple classes in $G(m)$, there is a choice of $Y_2 \in GL_{2m}(F)$ which satisfies (2.1) with $X_2 = I_{2m}$, so that $I - X_2'Y_2^{-1}X_2 \in \{h\}$. In particular $Y_2 + \tilde{\varepsilon}(Y_2) = I'_{2m}$. By Proposition 4.1, the ε –conjugacy class of Y_2 is ε –regular and uniquely determined by h. Let $j = (n-2m)/2$, as in Corollary 3.13. Let $X = \begin{pmatrix} 0_{j\times 2m} \\ I_{2m} \\ 0_{j\times 2m} \end{pmatrix} \in M_{n\times 2m}(F)$. Then $X' = \begin{pmatrix} 0 & I' & 0 \end{pmatrix}$, and

$$XX' = \begin{pmatrix} 0_j & 0 & 0 \\ 0 & I_{2m} & 0 \\ 0 & 0 & 0_j \end{pmatrix}.$$

Let

$$Y = \begin{pmatrix} I_j & & \\ & Y_2 & \\ & & -I_j \end{pmatrix}.$$

If $\mathbf{G}$ is orthogonal, then

$$\begin{aligned}\tilde{\varepsilon}(Y) &= \begin{pmatrix} & & w_j \\ & w_{2m} & \\ w_j & & \end{pmatrix} \begin{pmatrix} I_j & & \\ & {}^tY_2 & \\ & & -I_j \end{pmatrix} \begin{pmatrix} & & w_j \\ & w_{2m} & \\ w_j & & \end{pmatrix} \\ &= \begin{pmatrix} -I_j & & \\ & \tilde{\varepsilon}(Y_2) & \\ & & I_j \end{pmatrix}.\end{aligned}$$

Similarly, if $\mathbf{G}$ is symplectic, then we have

$$\tilde{\varepsilon}(Y) =$$

$$\begin{aligned}&\begin{pmatrix} & & (-1)^j u_j \\ & (-1)^j u_{2m} & \\ u_j & & \end{pmatrix} \begin{pmatrix} I_j & & \\ & {}^tY_2 & \\ & & -I_j \end{pmatrix} \begin{pmatrix} & & u_j^{-1} \\ & (-1)^j u_{2m} & \\ (-1)^j u_j & & \end{pmatrix} \\ &= \begin{pmatrix} -I_j & & \\ & \tilde{\varepsilon}(Y_2) & \\ & & I_j \end{pmatrix}.\end{aligned}$$

Therefore,

$$Y+\widetilde{\varepsilon}(Y)=\begin{pmatrix}0_j & 0 & 0\\ 0 & I'_{2m} & 0\\ 0 & 0 & 0_j\end{pmatrix}=XX'.$$

Moreover we see that $I_{2m}-X'Y^{-1}X=I_{2m}-I'_{2m}Y_2^{-1}=h$.

It remains to check that for almost all Y_2, the class of Y satisfying (2.1) is, up to $GL_n(\bar{F})-\varepsilon$-conjugacy, of the form given in the previous paragraph. First observe that for almost all Y satisfying (2.1) with $X=\begin{pmatrix}0\\ I\\ 0\end{pmatrix}$, Y acts semisimply on the direct sum of the image and the kernel of XX', both of which are invariant under ε, since Y_2 is ε-regular. Moreover, Y must be ε-skew symmetric on $\ker(XX')$ and can therefore be given in the form $\mathrm{diag}(J_1,Y_2,J_2)$ with $\mathrm{diag}(J_1,J_2)$ ε-symmetric ($\mathbf{G}$ symplectic) or ε-skew symmetric ($\mathbf{G}$ orthogonal), proving our assertion. □

Lemma 4.3. *Suppose that* $\alpha\in F^{\times^2}\backslash F^{\times}$. *Let* $\alpha_0=\mathrm{diag}\{\alpha,1,\alpha,1,\dots,1\}\in GSp_{2m}$. *Then*

$$N_\varepsilon(\{\alpha\gamma'\})=\alpha_0^{-1}N_\varepsilon(\{\gamma'\})\alpha_0,$$

for any $\gamma'\in\mathcal{N}$. *Similarly, let* $\alpha\in NE^{\times}$. *Take* $g_0\in GL_k(F)$ *with* $g_0\Lambda_k{}^tg_0=\alpha\Lambda_k$. *Let* $\alpha_0=\mathrm{diag}\{\alpha I_{m-(k/2)},g_0,I_{m-(k/2)}\}\in GO_{J_{2m}}$. *Then*

$$N_\varepsilon(\{\alpha\gamma'\})=\alpha_0^{-1}N_\varepsilon(\{\gamma'\})\alpha_0,$$

for all $\{\gamma'\}\in\mathcal{N}$.

Proof. Let $\alpha^\vee=\alpha I_{2m}$. In the symplectic case, $\alpha^\vee=\alpha_0\widetilde{\varepsilon}(\alpha_0)$, while in the orthogonal case, $\alpha^\vee=J_n^{-1}\alpha_0J_n{}^t\alpha_0$. Let $Y^{-1}\in\{\gamma'\}$. Then $X\alpha^\vee X'=\alpha Y+\widetilde{\varepsilon}(\alpha Y)$, while $X\alpha^\vee X'=(X\alpha_0)(X\alpha_0)'$. Since $N_\varepsilon(\{\alpha\gamma'\})$ is the conjugacy class given by

$$I-(X\alpha_0)'(\alpha Y)^{-1}(X\alpha_0).$$

we have

$$I-(X\alpha_0)'(\alpha Y)^{-1}(X\alpha_0)=\alpha_0^{-1}(I-X'Y^{-1}X)\alpha_0.\quad\square$$

Let $\mathcal{C}$ and $\mathcal{C}'$ be the sets of conjugacy classes in $G(m)$ and ε-conjugacy classes in $GL_n(F)$, respectively. Suppose that $n = 2m$. Let $\mathbf{T}$ be a Cartan subgroup of $\mathbf{G}(n/2)$, defined over F. By Lemma 3.3.B of [**12**], there is a θ^*-stable Cartan subgroup $\mathbf{T}'$ of GL_n with an isomorphism $\mathbf{T} \xrightarrow{\sim} \mathbf{T}'_{\theta^*}$, which is defined over F. Furthermore, this isomorphism induces the *image map* $\mathcal{A}_{\mathbf{G}(n/2)/GL_n}$ between semisimple classes in $\mathcal{C}$, and θ^*-semisimple θ^*-conjugacy classes in GL_n. We set

$$g_\varepsilon = \begin{cases} I, & \text{if } \mathbf{G}(n/2) \text{ is symplectic,} \\ g_\theta, & \text{if } \mathbf{G}(n/2) \text{ is orthogonal.} \end{cases}$$

(See [**21**, §7].) Let $h : \mathbf{T}' \longrightarrow \mathbf{T}' g_\varepsilon$ be given by right multiplication by g_ε. Then, up to a sign, we see that the diagram

$$\begin{tikzcd} \mathbf{T}' \arrow[d, "N_{\theta^*}"'] \arrow[dr, "h"] & \\ \mathbf{T} \cong \mathbf{T}'_{\theta^*} & \mathbf{T}' g_\varepsilon \arrow[l, "N_\varepsilon"] \end{tikzcd}$$

commutes on the set of strongly θ^*-regular elements of $\mathbf{T}'$. (Recall that an element t of $\mathbf{T}'$ is called strongly θ^*-regular if $\mathbf{G}'_{\theta^*,t}$ is abelian.) Since $\mathbf{T}'$ is θ^*-stable, we can, in the language of §3.1 of [**12**], take $\psi = 1$, to see that h^{-1} is the map m of [**12**].

Lemma 4.4. *Let* $\mathbf{T}$ *be a Cartan subgroup of* $\mathbf{G}(n/2)$ *defined over* F. *Then there is a* θ^**-stable Cartan* $\mathbf{T}'$ *of* $\mathbf{G}'(n)$, *such that the diagram*

$$\begin{tikzcd} \mathbf{T}' \arrow[d, "N_{\theta^*}"'] \arrow[dr, "m^{-1}"] & \\ \mathbf{T} \cong \mathbf{T}'_{\theta^*} & \mathbf{T}' g_\varepsilon \arrow[l, "N_\varepsilon"] \end{tikzcd}$$

commutes up to a sign on all strongly θ^**-strongly regular* θ^**-semisimple elements of* $\mathbf{T}'(F)$. *Furthermore,* $\mathbf{T} \simeq \mathbf{T}'_{\theta^*}$ *induces the image map* $\mathcal{A}_{\mathbf{G}/\mathbf{G}'}$. *If* $\delta^* \in \mathbf{T}'$ *is*

θ^*-strongly regular, then $\mathrm{Cent}_{\theta^*}(\delta^*, \mathbf{G}') = (\mathbf{T}')^{\theta^*}$.

Proof. We only need to prove the last statement. If δ^* is strongly θ^*-regular, then $\mathrm{Cent}_{\theta^*}(\delta^*, \mathbf{G}')$, is a maximal torus in $\mathbf{G}'$, stable under $\mathrm{Int}(\delta^*) \circ \theta^*$. Since $\mathbf{T}'_{\delta^*} = \mathbf{T}'$, is a maximal torus, and $\mathbf{T}'$ is θ^*-stable, we have $\mathrm{Cent}_{\theta^*}(\delta^*, \mathbf{G}') = (\mathbf{T}')^{\theta^*}$. □

Suppose that $d\gamma'$ is a measure on $(\mathbf{T}')^{\theta^*}(F)$. By Lemma 3.15, $Y \mapsto Y\theta^*(Y)$ has $(\mathbf{T}')^{\theta^*}$ as its image, and since $\gamma' \in (\mathbf{T}')^{\theta^*}(F)$ is semisimple, Lemma 3.16 implies that, for almost all Y, the $GL_n(F)$-conjugates (up to a sign) of $\theta^*(Y^{-1})Y^{-1}$ exhaust $N_\varepsilon(\{Y^{-1}\})$. Observe that $|D_{\theta^*}(\gamma')|\, d\gamma'$ is a measure on the θ^*-conjugacy classes of $GL_n(F)$ which meet $\mathbf{T}'(F)$ (cf. [**22**]). Here

$$D_{\theta^*}(\gamma') = \det(Ad(\gamma') \circ \theta^* - 1)|_{\mathfrak{g}/\mathfrak{g}_{\theta^*,\gamma'}},$$

as described in [**12**]. Using the map m, we see that $|D_{\theta^*}(\gamma')|\, d\gamma'$ transfers to a measure on the ε-conjugacy classes of $GL_n(F)$ intersecting $\mathbf{T}'(F)g_\varepsilon$. We also know that $|D(\gamma)|\, d\gamma$ is a measure on the semisimple classes in $\mathcal{C}$ which meet $\mathbf{T}(F)$. By Lemma 4.5.A of [**12**], $\kappa_1(\gamma, \gamma') = |D_{\theta^*}(\gamma')|/|D(\gamma)|$ is continuous on $\{(\{N_{\theta^*}(\gamma')\}, \{\gamma'\})\}$. Suppose $\{\gamma\}$ is regular and semisimple. Define

$$\kappa(\{\gamma\}, \{\gamma'\}) = \begin{cases} \kappa_1(\gamma, \gamma'), & \text{if } \{\gamma\} = N_\varepsilon(\{\gamma'\}) \text{ and } \gamma' \text{ is regular;} \\ 0, & \text{otherwise.} \end{cases}$$

For each regular semisimple conjugacy class $\{\gamma\} \in \mathcal{C}$, define

$$\mathcal{A}(\{\gamma\}) = \left\{ \{\alpha\gamma'\} \mid \alpha \in F^{\times^2} \backslash F^\times, \{\gamma\} \in N_\varepsilon\left(\{(\gamma')^{-1}\}\right) \right\}.$$

We set

$$\Delta(\{\gamma\}, \{\alpha\gamma'\}) = \begin{cases} \omega'(\alpha)\kappa(\{\gamma\}, \{\gamma'\}), & \text{if } \{\gamma\} \in N_\varepsilon(\{(\gamma')^{-1}\}); \\ 0, & \text{otherwise.} \end{cases}$$

Finally let

$$\Phi_\varepsilon(\mathcal{A}(\{\gamma\}), f') = \sum_{\{\gamma'\} \in \mathcal{A}(\{\gamma\})} \Delta(\gamma, \gamma')\Phi_\varepsilon(\gamma', f').$$

The main part of the residue of the intertwining operator at $s = 0$ comes from the regular elliptic elements in the form of the Weyl integration formula applied to the class function $\Phi_\varepsilon(\mathcal{A}(\{\gamma\}), f')$, i.e. as a pairing between characters and twisted characters (cf. Section 5). More precisely, the contribution from these regular elliptic classes has the form

$$R_{\mathbf{G}}(f_\tau, f') = \sum_{\{\mathbf{T}_i\}} \mu(T_i)\, |W(T_i)|^{-1} \int_{T_i} \Phi(\{\gamma\}, f_\tau) \Phi_\varepsilon(\mathcal{A}(\{\gamma\}), f')\, |D(\gamma)|\, d\gamma,$$

where $\{\mathbf{T}_i\}$ runs over the conjugacy classes of elliptic Cartan subgroups of $\mathbf{G} = \mathbf{G}(m)$ and $T_i = \mathbf{T}_i(F)$. For each i, $\mu(T_i)$ is the measure of T_i. Observe that by the way that our transfer factor Δ is defined, this is in fact an integration over ε –conjugacy classes in $\mathcal{N}$ (since the norm correspondence is surjective).

Let us first check the convergence of $R_{\mathbf{G}}$. It is enough to show that

$$\int_{\mathbf{T}(F)} \Phi(\gamma, f_\tau) \Phi_\varepsilon(\mathcal{A}(\{\gamma\}), f') |D(\gamma)|\, d\gamma,$$

converges for any elliptic torus $\mathbf{T}$ of $\mathbf{G}(n/2)$. By Theorem 14 of [**9**],the function $|D(\gamma)|^{1/2}\Phi(\gamma, f_\tau)$ is bounded on the intersection of $\mathbf{T}(F)$, with the regular set of G. Therefore, we are reduced to proving the convergence of

$$\int_{\mathbf{T}(F)} \Phi_\varepsilon(\mathcal{A}(\{\gamma\}), f') |D(\gamma)|^{1/2}\, d\gamma.$$

It is enough to show that each integral in the sum converges absolutely, i.e., that

$$\int_{\substack{\mathbf{T}(F) \\ \mathbf{N}_{\theta *}(\{\gamma'\}) = \{\gamma\}}} |\Phi_\varepsilon(\alpha\gamma' g_\varepsilon, f')|\, \kappa(\{\gamma\}, \{\gamma'\})\, |D(\gamma)|^{1/2}\, d\gamma$$

converges for every $\alpha \in F^{\times^2} \backslash F^\times$. We are therefore reduced to showing the convergence of

$$\int_{N_{\theta *}(\{\gamma'\}) = \{\gamma\}} |\Phi_{\theta *}(\gamma', R_{g_\varepsilon \alpha} f')|\, |D_{\theta *}(\gamma')|\, |D(\gamma)|^{-1/2}\, d\gamma'. \tag{4.1}$$

Since $\Phi_{\theta^*}(-,-)$ is a tempered distribution, the function $\Phi_{\theta^*}(\gamma', f')|D_{\theta^*}(\gamma')|^{1/2}$ is bounded on the intersection of $\mathbf{T}_{\theta^*}(F)$ with the θ^*-regular set [**5**,**9**]. Now the convergence of (4.1) follows from the boundedness of $\kappa(\{\gamma\},\{\gamma'\})^{1/2}$ (Lemma 4.5.A of [**12**]).

Let $M^\vee = GL_n(F) \times G^\vee$ act on N by the adjoint action, where $\mathbf{G}^\vee = \mathbf{G}^\vee(m) = O_{J_{2m}}$ if $\mathbf{G}(m) = SO_{J_{2m}}$ and $\mathbf{G}^\vee(m) = Sp_{2m}$, otherwise. Then the orbit of $n(X,Y) \in N$ consists of pairs $(gXh^{-1}, gY\varepsilon(g)^{-1}) \in N$, with $g \in G'$ and $h \in G^\vee$. The stabilizer $\Delta^\vee$ of the action at $n(X,Y)$ consists of all those pairs (g,h) with $g \in \mathbf{G}'_{\varepsilon,Y}(F)$ for which $gX = Xh$. Then $h \in \mathbf{G}^\vee_Z(F)$, where $Z = I - X'Y^{-1}X$ and $\mathbf{G}^\vee_Z$ is the centralizer of Z in $\mathbf{G}^\vee(m)$. By abuse of both notation and terminology, identify $\Delta^\vee$ as a subgroup of both $\mathbf{G}^\vee_Z(F)$ and $\mathbf{G}'_{\varepsilon,Y}(F)$ through projection onto its components.

There is a map from the orbit of $n(X,Y)$ under $M^\vee$ onto $G'/\Delta^\vee \times \Delta^\vee\backslash G^\vee$ whose fiber is homeomorphic to to $X\Delta^\vee$. The integration over the orbit is then equal to the integration over the product of $G'/\Delta^\vee \times \Delta^\vee\backslash G^\vee$ with $X\Delta^\vee$. The contribution to (2.5) from the orbit of $n(X,Y)$ under the action of $M^\vee$ is then

$$\widetilde{\psi}(s,Z) = \sum_{\alpha \in (F^{\times 2})\backslash F^\times} \omega'(\alpha) \cdot$$

$$\int_{g \in G'/\Delta^\vee} \int_{h \in \Delta^\vee\backslash G^\vee} \int_{X\Delta^\vee} f'(\alpha g Y \varepsilon(g)^{-1}) f_\tau(h^{-1}Zh)\,|\det(gY\varepsilon(g)^{-1})|^s \, dg\, dh\, d(Xh_0) \cdot$$

$$\int_{Z(G')} \xi_{\tilde{L}}(z^{-2}gY\varepsilon(g)^{-1})\xi_{L'}(z^{-1}gXh_0h)\,|\det z|^{-2s}\, d^\times z.$$

Observe that we have suppressed the dependence of $\widetilde{\psi}$ on other parameters X,

Y, f, f_τ, L, and L'. Breaking the variables further:

$$\widetilde{\psi}(s,Z)=\sum_{\alpha}\omega'(\alpha)\cdot$$

$$\int\limits_{g\in G'/G'_{\varepsilon,Y}}\int\limits_{h\in G^\vee_Z\backslash G^\vee}\int\limits_{g_0\in G'_{\varepsilon,Y}/\Delta^\vee}\int\limits_{XG^\vee_Z} f'(\alpha gY\varepsilon(g)^{-1})f_\tau(h^{-1}Zh)|\det(gY\varepsilon(g)^{-1})|^s\,dg\,dh\,dg_0\,dX\,h_0$$

$$\int\limits_{Z(G')}\xi_{\tilde{L}}(z^{-2}gY\varepsilon(g)^{-1})\xi_{L'}(z^{-1}gg_0Xh_0h)\,|\det z|^{-2s}\,d^\times z, \tag{4.2}$$

where $G'_{\varepsilon,Y}=\mathbf{G}'_{\varepsilon,Y}(F)$ and $G^\vee_Z=\mathbf{G}^\vee_Z(F)$.

Let Z_j's be representatives for the G–orbits inside the $G^\vee$–orbit of Z. Note that there are either one or two G–orbits in each $G^\vee$–orbit. Denote by

$$\Phi(Z_j,f_\tau)=\int\limits_{\mathbf{G}_{Z_j}(F)\backslash G} f_\tau(h^{-1}Z_jh)\,dh,$$

and

$$\Phi(Z,f_\tau)=\int\limits_{\mathbf{G}^\vee(F)\backslash G^\vee} f_\tau(h^{-1}Zh)\,dh$$

the corresponding orbital integrals. Finally, let

$$\Phi_{\varepsilon,s}(\alpha Y,f')=\int\limits_{G'/\mathbf{G}'_{\varepsilon,Y}(F)} f'(\alpha gY\varepsilon(g)^{-1})\,|\det(gY\varepsilon(g)^{-1})|^s\,dg.$$

Clearly,

$$\begin{aligned}\lim_{s\to 0}\Phi_{\varepsilon,s}(\alpha Y,f')&=\Phi_\varepsilon(\alpha Y,f')\\&=\int\limits_{G'/\mathbf{G}'_{\varepsilon,Y}(F)} f'(\alpha gY\varepsilon(g)^{-1})\,dg\end{aligned}$$

is the corresponding twisted orbital integral.

Suppose first that $n=2m$. Assume Y is so that (Y,I) satisfies (2.1) and $Z=I-I'Y^{-1}$. Note that $I'=-I$ if $\mathbf{G}$ is symplectic, and is $-J_nw_n$, otherwise.

(If $\mathbf{G}$ is a split orthogonal group then, again, $I' = -I$, and $Z = I + Y^{-1}$.) Then $\mathbf{G}_Z(F) \simeq \mathbf{G}'_{\varepsilon,Y}(F) \simeq \Delta^\vee$. Let

$$\psi(s,Z) = \sum_{\alpha} \omega'(\alpha) \cdot$$

$$\int\limits_{g \in G'/\mathbf{G}'_{\varepsilon,Y}(F)} \int\limits_{h \in G_Z^\vee \backslash G^\vee} \int\limits_{h_0 \in G_Z^\vee} f'(\alpha g Y \varepsilon(g)^{-1}) f_\tau(h^{-1} Z h) \,|\det(gY\varepsilon(g)^{-1})|^s \, dh dg \, dh_0 \cdot$$

$$\int\limits_{Z(G')} \xi_{\tilde{L}}(z^{-2} g Y \varepsilon(g)^{-1}) \xi_{L'}(z^{-1} g h_0 h) \,|\det z|^{-2s} \, d^\times z. \tag{4.3}$$

Then $\psi(s,Z) = \widetilde{\psi}(s,Z)$.

Let $L(\mathbf{1},s) = (1-q^{-s})^{-1}$ denote the local Hecke L–function attached to the trivial character $\mathbf{1}$. We have

Lemma 4.5. *Let $n = 2m$. Assume that Y is ε–regular. Then $\widetilde{\psi}(s,Z)$ converges absolutely for $Re\ s > 0$. Suppose further that (Y,I) satisfies (2.1), i.e., for almost all twisted conjugacy classes in $\mathcal{N}$. Then $\psi(s,Z)$ also converges absolutely for $Re\ s > 0$ and there exists a function $E_Z(s) = E(s,Z,Y,f_\tau,f',L,L')$, which as a function of s is entire, such that $\psi(s,Z) = E_Z(s)$ if Z is regular and non-elliptic, and*

$$\psi(s,Z) = E_Z(s) +$$

$$\Big(\sum_{\alpha \in (F^\times)^2 \backslash F^\times} \sum_j \Phi_{\varepsilon,s}(\alpha Y, f') \Phi(Z_j, f_\tau) \mu(\mathbf{G}_{Z_j}(F))\Big) q^{b(Y,Z)s} L(\mathbf{1}, 2ns)$$

for $Re\ s > 0$, if Z is regular elliptic. Here $b(Y,Z)$ is an integer depending on the data f_τ, f', L, L', as well as Y and Z. In particular,

$$\operatorname*{Res}_{s=0} \psi(s,Z) = (2n \log q)^{-1} \sum_{\alpha \in (F^\times)^2 \backslash F^\times} \sum_j \Phi_\varepsilon(\alpha Y, f') \Phi(Z_j, f_\tau) \mu(\mathbf{G}_{Z_j}(F))$$

if Z is regular elliptic, and

$$\operatorname*{Res}_{s=0} \psi(s,Z) = 0$$

if Z *is regular but non-elliptic.*

Proof. We will prove the lemma when $X = I$. The convergence of $\widetilde{\psi}(s, Z)$ can be proved the same way. We may assume L (therefore $\widetilde{L}$) and L' are both basic neighborhoods of 0. Clearly $f'(\alpha g Y\varepsilon(g)^{-1})\xi_{\tilde{L}}(z^{-2}gY\varepsilon(g)^{-1}) = 0$ unless $gY\varepsilon(g)^{-1} \in z^2\widetilde{L} \cap \alpha^{-1}\operatorname{supp}(f')$ for some α. A standard argument using the support of f' then implies that $|\det z|$ must be bounded below with the bound depending only on L and f'. Since Y and Z are ε –semisimple and semisimple, respectively, we may assume g and h in (4.3) belong to compact sets $\operatorname{supp}(g)$ and $\operatorname{supp}(h)$ of $G'/\mathbf{G}'_{\varepsilon,Y}(F)$ and $\mathbf{G}^\vee_Z(F)\backslash G^\vee$, depending upon Y, Z, f_τ, and f', respectively.

To study the convergence of (4.3), we may assume h_0 is in the F –points of the split component of $\mathbf{G}^\vee_Z$ (i.e., the connected component of $\mathbf{G}_Z$), which we may further assume are given by diagonal matrices. Write $z = \operatorname{diag}(z, z, \ldots, z)$, and

$$h_0 = \operatorname{diag}(a_1, a_1 \ldots, a_1, a_2, \ldots, a_2, \ldots, a_2^{-1}, \ldots, a_2^{-1}, a_1^{-1}, \ldots, a_1^{-1}).$$

By the compactness of $\operatorname{supp}(g)$ and $\operatorname{supp}(h)$, $\xi_{L'}(z^{-1}gh_0h) = 0$, unless $z^{-1}h_0$ belongs to a compact subset of $M_n(F)$, i.e., $|za_i| \geq \kappa$ and $|za_i^{-1}| \geq \kappa$, for some $\kappa > 0$. Consequently, $|z| \geq \kappa$. Moreover, there exists a $\kappa_1 > 0$ such that if $|z^{-1}a_i^{-1}| \leq \kappa_1$, and $|z^{-1}a_i| \leq \kappa_1$, then $\operatorname{supp}(g)z^{-1}h_0T\operatorname{supp}(h) \subset L'$, since both $\operatorname{supp}(g)$ and $\operatorname{supp}(h)$ are compact. Here T is the compact part of $G^\vee_Z$. Observe that $\kappa_1 \geq \kappa$. Consequently, with such a z and h_0, h will be free to change over all its compact support leading to $\Phi(Z, f_\tau)$, which vanishes if the split component of $\mathbf{G}^\vee_Z$ is non-trivial (cf. [**21**]). Otherwise $\mathbf{G}^\vee_Z(F)$ is compact.

Let η be the lower bound given by f' and L on $|z|$. We will use κ instead of η if $\kappa \geq \eta$. We may assume κ_1/κ is a non-negative integral power of q. Choose a non-negative integer m so that $|\varpi^{-m}| = \kappa_1/\kappa$. Then $|\varpi^{-m}za_i| \geq \kappa_1$, $|\varpi^{-m}za_i^{-1}| \geq \kappa_1$, and $|\varpi^{-m}z| \geq \eta\kappa_1/\kappa \geq \eta$. Consequently, for all z with $|z| \geq \eta\kappa_1/\kappa$, we can drop both $\xi_{\tilde{L}}$ and $\xi_{L'}$ and integrating over all $z = \beta\varpi^{-m}$, with $\beta \in \mathfrak{o}^\times$, and all of $\mathbf{G}^\vee_Z(F)$ we get

$$\sum_{\alpha \in (F^\times)^2\backslash F^\times} \sum_j \Phi(Z_j, f_\tau)\Phi_{\varepsilon,s}(\alpha Y, f') \sum_{m \geq \ell} q^{-2mns}\mu(m). \tag{4.4}$$

Here $|\varpi^{-\ell}| = \eta\kappa_1/\kappa$ and $\mu(m)$ is equal to

$$\prod_i \int_{q^{-(m-d)} \leq |a_i| \leq q^{(m-d)}} d^\times a_i$$

times the measure of the compact part of $\mathbf{G}_Z^\vee(F)$. Moreover, d is given by $\kappa_1 = q^d$. The series clearly converges for $\operatorname{Re} s > 0$. If $\mathbf{G}_Z^\vee(F)$ is not compact, then (4.4) vanishes. The remaining values of z, i.e., $\eta\kappa_1/\kappa > |z| \geq \eta$, only lead to an entire function in s. The lemma is now complete. $\square$

Corollary 4.6. *Let $\mathbf{T}_i$ be a Cartan subgroup of $\mathbf{G}$. Denote by T_i' the subset of regular elements of of $T_i = \mathbf{T}_i(F)$. Let ω_i be a compact subset of T_i'. Then given f_τ, f', L and L', $b = b(Y, Z)$ and E_Z can be chosen independent of Y and Z for all $Z \in \omega_i$.*

Proof. This follows from the corollary to Lemma 19 of [**9**], which implies that the compact sets $\operatorname{supp}(g)$ and $\operatorname{supp}(h)$ of the proof of Lemma 4.5 can in fact be chosen be the same for all $Z \in \omega_i$. $\square$

To calculate the residue for the intertwining operator, we must now integrate over all the orbits of the action of $M^\vee$ on N. We do this by integrating over all the ε-regular ε–conjugacy classes in $\mathcal{N}$.

First assume $n = 2m$. We must integrate $\widetilde{\psi}(s, Z)$ over orbits of N under the action of $M^\vee$. Almost all of these orbits are parameterized by ε–regular conjugacy classes in $\mathcal{N}$. Therefore removing a set of measure zero from these orbits, we may integrate $\psi(s, Z)$ over ε–regular (and ε–semisimple) ε–conjugacy classes $\{Y\}$ in $\mathcal{N}$. Then, by Proposition 4.1, $\{Z\} = N_\varepsilon(\{Y^{-1}\})$ is regular and semisimple in $\mathbf{G}^\vee(F)$.

Let $\{\mathbf{T}_i\}$ denote a complete set of conjugacy classes of Cartan subgroups of $\mathbf{G}$ defined over F. For each $\mathbf{T}_i$ choose a θ^*–stable Cartan subgroup $\widetilde{\mathbf{T}}_i$ of GL_n with an isomorphism $\mathbf{T}_i \simeq (\widetilde{\mathbf{T}}_i)_{\theta^*}$ defined over F. Since $n = 2m$, every $\widetilde{T}_i g_\varepsilon$, with $\widetilde{T}_i = \widetilde{\mathbf{T}}_i(F)$, will intersect $\mathcal{N}$ and outside a set of measure zero, $\cup_i(\mathcal{N} \cap \widetilde{T}_i g_\varepsilon)$ is equal to the set of all the ε–regular conjugacy classes in $\mathcal{N}$. As we discussed before, given $\widetilde{\gamma}_i \in (\mathbf{T}_i)^{\theta^*}(F)$, the measure $|D_{\theta^*}(\widetilde{\gamma}_i)|\, d\gamma_i$ can be transformed to a measure on the ε-regular ε-conjugacy classes of $GL_n(F)$ which intersect $\widetilde{T}_i g_\varepsilon$. Using the surjectivity of the norm correspondence, we instead integrate over $\cup_i T_i$. Then

$$|D_{\theta^*}(\widetilde{\gamma}_i)|\, d\widetilde{\gamma}_i = \kappa_1(\gamma_i, \widetilde{\gamma}_i)\, |D(\gamma_i)|\, d\gamma_i$$

where $\widetilde{\gamma}_i \mapsto \gamma_i$ through the isomorphism $(\widetilde{\mathbf{T}}_i)_{\theta^*} \simeq \mathbf{T}_i$. The measures are transferred through $(\widetilde{\mathbf{T}}_i)^{\theta^*} \simeq \mathbf{T}_i$.

Now suppose $n > 2m$. Then for almost all $\{Y\} \in \mathcal{N}$ we can choose a representative $\mathrm{diag}(J_1, g, J_2)$, with $g \in GL_{2m}(F)$ ε-regular in $GL_{2m}(F)$ as in Corollary 4.2. More precisely, we may assume $X \in M_{n\times 2m}(F)$ has the form

$$X = \begin{pmatrix} 0_{j\times 2m} \\ I_{2m} \\ 0_{j\times 2m} \end{pmatrix}$$

with $j = (n/2) - m$. Moreover, as in Corollary 4.2, we may assume Y preserves the kernel and the image of XX' for almost all Y, and therefore Y must be ε-skew symmetric on $\ker(XX')$. Consequently, $\mathrm{diag}(J_1, J_2)$ must be either symmetric ($\mathbf{G}$ symplectic) or skew symmetric ($\mathbf{G}$ orthogonal) on this kernel. Outside of a set of measure zero, the twisted conjugacy classes in $\mathcal{N}$ now form a fiber bundle with finite fibers coming from the twisted conjugacy classes of possible $\mathrm{diag}(J_1, J_2)$. The base of the fiber bundle is parameterized by ε- conjugacy classes in $GL_{2m}(F)$ which are rational solutions of (2.1) with $G(m)$ and we may use θ^*-stable conjugacy Cartan subgroups of GL_{2m} and their F-isomorphisms with $\mathbf{T}_i$'s as in the case $n = 2m$. Here $\{\mathbf{T}_i\}_i$ is a complete set of representatives for the conjugacy classes of Cartan subgroups of $\mathbf{G}$. We then get measures $\kappa_1(\gamma_i, \widetilde{\gamma}_i)|D(\gamma_i)|\, d\gamma_i$ on the T_i's and by surjectivity we need to integrate over $\cup_i T_i$ for each fiber. The integral over the whole fiber bundle is then achieved by using the image correspondence $\mathcal{A}$ defined precisely as in the case $n = 2m$, but still integrating over $\cup_i T_i$. Still assuming $n \geq 2m$ and choosing $X = X_0 = \begin{pmatrix} 0_{j\times 2m} \\ I_{2m} \\ 0_{j\times 2m} \end{pmatrix}$, equation (4.2) can now be written as

$$\widetilde{\psi}(s,Z) = \sum_{\alpha} \omega'(\alpha)\cdot$$

$$\int_{g\in G'/\mathbf{G}'_{\varepsilon,Y}(F)} \int_{h\in \mathbf{G}_Z^\vee(F)\backslash G^\vee} \int_{h_0\in\mathbf{G}_Z^\vee(F)} f'(\alpha g Y\varepsilon(g)^{-1}) f_\tau(h^{-1}Zh)|\det(gY\varepsilon(g)^{-1})|^s\, dg\, dh\, dh_0\cdot$$

$$\int_{Z(G')} \xi_{\tilde{L}}(z^{-2}gY\varepsilon(g)^{-1})\xi_{L'}(z^{-1}gX_0h_0h)|\det z|^{-2s}\, d^\times z.$$

Lemma 4.5 is now valid for $\widetilde{\psi}(s,Z)=\psi(s,Z)$ even if $n>2m$, and in what follows we may assume $n\geq 2m$.

For each G –conjugacy class of Cartan subgroups of $\mathbf{G}$, choose a representative $\mathbf{T}_i$. Given i, let T_i' be the subset of regular elements in $T_i=\mathbf{T}_i(F)$. For $\gamma\in T_i'$, let

$$\psi_{\mathcal{A}}(s,\gamma)=\sum \psi(s,\gamma),$$

where the sum is over all $\{Y\}$ for which $N_\varepsilon(\{Y^{-1}\})$ contains $\{\gamma\}$. Let ω_i be a compact subset of T_i'. By the absolute convergence of intertwining operators for Re $s>0$ (cf. [**23**]) we see that (2.5) is equal to

$$\sum_i |W(\mathbf{T}_i)|^{-1}\int_{\omega_i} \psi_{\mathcal{A}}(s,\gamma)\,|D(\gamma)|\,d\gamma + \sum_i |W(\mathbf{T}_i)|^{-1}\int_{T_i'\backslash\omega_i} \psi_{\mathcal{A}}(s,\gamma)\,|D(\gamma)|\,d\gamma.$$

Let

$$R_{\omega_i}(s) = |W(\mathbf{T}_i)|^{-1}\int_{\omega_i} \psi_{\mathcal{A}}(s,\gamma)\,|D(\gamma)|\,d\gamma.$$

By Corollary 4.6,

$$R_{\omega_i}(s) = h_i(s) + |W(\mathbf{T}_i)|^{-1}\int_{\omega_i} \Phi(\gamma,f_\tau)\Phi_\varepsilon(\mathcal{A}(\{\gamma\}),f')\,|D(\gamma)|\,d\gamma\cdot\mu(T_i)q^{bs}L(\mathbf{1},2ns),$$

if $\mathbf{T}_i$ is anisotropic, where $h_i(s)$ is an entire function of s. Otherwise, $R_{\omega_i}(s)$ is entire. Clearly,

$$\operatorname*{Res}_{s=0}\sum_i R_{\omega_i}(s) = (2n\log q)^{-1}{\sum_i}' \mu(T_i)|W(\mathbf{T}_i)|^{-1}\int_{\omega_i}\Phi(\gamma,f_\tau)\Phi_\varepsilon(\mathcal{A}(\{\gamma\}),f')|D(\gamma)|\,d\gamma,$$

where the sum $\sum_i{}'$ is understood to be over conjugacy classes of anisotropic Cartan subgroups $\mathbf{T}_i$ of $\mathbf{G}$. Thus, the residue of the operator at $s=0$ equals:

$$(2n\log q)^{-1}{\sum_i}' \mu(T_i)|W(\mathbf{T}_i)|^{-1}\int\limits_{\omega_i}\Phi(\gamma,f_\tau)\Phi_\varepsilon(\mathcal{A}(\{\gamma\}),f')\,|D(\gamma)|\,d\gamma$$

$$+\operatorname*{Res}_{s=0}\sum_i|W(\mathbf{T}_i)|^{-1}\int\limits_{T_i'\backslash\omega_i}\psi_{\mathcal{A}}(s,\gamma)\,|D(\gamma)|\,d\gamma.$$

Letting $\omega_i\to T_i'$ for all i, the residue is then equal to

$$cR_{\mathbf{G}}(f_\tau,f')+\sum_i|W(\mathbf{T}_i)|^{-1}\lim_{\omega_i\to T_i}\operatorname*{Res}_{s=0}\int\limits_{T_i'\backslash\omega_i}\psi_{\mathcal{A}}(s,\gamma)\,|D(\gamma)|\,d\gamma,$$

where $c=(2n\log q)^{-1}$. Here $R_{\mathbf{G}}$ is defined the same way as in the case $n=2m$.

Now suppose that $n<2m$, and consider the injection

$$h\mapsto\begin{pmatrix}I_{m-(n/2)}&&\\&h&\\&&I_{m-(n/2)}\end{pmatrix}$$

of $\mathbf{G}(n/2)$ into $\mathbf{G}(m)$. Let $\mathbf{N}_\varepsilon:\mathcal{N}\longrightarrow\mathcal{C}$ be the ε-norm correspondence from ε-conjugacy classes $\mathcal{N}$ in $GL_n(F)$ to conjugacy classes in $G(m)$. Suppose that $X\in M_{n\times 2m}(F)$, and $Y\in GL_n(F)$ satisfy $Y+\widetilde{\varepsilon}(Y)=XX'$. Note that $\operatorname{rank}X'Y^{-1}X\le n$, and so at most n eigenvalues of $I-X'Y^{-1}X$ are different from one. Thus, the semisimple part of the conjugacy class $\{I-X'Y^{-1}X'\}$ has a representative in $G(n/2)$. Now let $\mathcal{C}^\vee$ be the subset of $\mathcal{C}$ consisting of those conjugacy classes of $G(m)$ whose semisimple parts meet $G(n/2)$. Then we see that $N_\varepsilon:\mathcal{N}\longrightarrow\mathcal{C}^\vee$.

Lemma 4.7. *If $n<2m$, then the norm correspondence N_ε has finite fibers.*

Proof. We only need to show that if $Y\in GL_n(F)$, and $X\in M_{n\times 2m}(F)$ satisfy (2.1), then $I-X'Y^{-1}X$ determines the semisimple part of $\varepsilon(Y^{-1})Y^{-1}$. But, by Lemma 3.3, we have $\varepsilon(Y^{-1})Y^{-1}X=-X(I-X'Y^{-1}X)$. We may suppose that X is in row echelon form, with the last $n-r$ rows of X identically zero. We thus have the decomposition $F^n = F^r \oplus F^{n-r}$, with $X|_{F^r}$ an injection into F^{2m}, and $X|_{F^{n-r}} = 0$. Consequently, the matrix of $\varepsilon(Y^{-1})Y^{-1}$ with respect to a basis respecting the above decomposition of F^n is $\begin{pmatrix} A & * \\ 0 & -I \end{pmatrix}$, with A determined by $I - X'Y^{-1}X$. Thus, N_ε has finite fibers. $\square$

Continuing with our study of the case $n < 2m$, we notice that again almost all of the ε –conjugacy classes in $\mathcal{N}$ can be parameterized by regular semisimple conjugacy classes in $\mathcal{C}^\vee$ or by regular semisimple conjugacy classes in $G(n/2)$. More precisely, for $\mathbf{G} = SO_J$, we set $X_1 = (\, 0_{n\times j} \quad I_n \quad 0_{n\times j}\,)$, with $j = (2m - n)/2$. Then

$$X_1' = -\begin{pmatrix} & & w_j \\ & J_n & \\ w_j & & \end{pmatrix} \begin{pmatrix} 0_{j\times n} \\ I_n \\ 0_{j\times n} \end{pmatrix} w_n$$
$$= \begin{pmatrix} 0_{j\times n} \\ -J_n w_n \\ 0_{j\times n} \end{pmatrix}.$$

Thus, $X_1X_1' = -J_nw_n = Y + \tilde{\varepsilon}(Y)$. If $\mathbf{G} = Sp_{2r}$, and j is even, then we take $X_1 = (\, 0 \quad I_n \quad 0\,)$, as above. We have $X_1X_1' = -I_n = Y + \tilde{\varepsilon}(Y)$. If j is odd, then we take $X_1 = (\, 0 \quad u_n \quad 0\,)$. Then again we see that $X_1X_1' = -I_n = Y + \tilde{\varepsilon}(Y)$. In all but this last case, the identity

$$I_{2m} - X_1'Y^{-1}X_1 = Z = \begin{pmatrix} I_j & & \\ & Z_1 & \\ & & I_j \end{pmatrix},$$

with Z_1 in the image of the norm correspondence for Y^{-1} in the case $n = 2m$ is now obvious. Thus, the parameterization of almost all the ε –conjugacy classes in $\mathcal{N}$ is as claimed in these cases. In the case where $\mathbf{G} = Sp_{2r}$ and j is odd, then

$$I_{2m} - X_1'Y^{-1}X_1 = \begin{pmatrix} I_j & & \\ & I_n - u_nY^{-1}u_n & \\ & & I_j \end{pmatrix}$$
$$= \begin{pmatrix} I_j & & \\ & \tilde{\varepsilon}({}^t(I_n + Y^{-1})) & \\ & & I_j \end{pmatrix}.$$

Since $Y \in \mathcal{N}$ satisfies (2.1) with $X = I_n$, we know that $Z_1 = I_n + Y^{-1} \in Sp_{2n}$, and thus $\varepsilon(Z_1) = Z_1$, or $\widetilde{\varepsilon}(Z_1) = Z_1^{-1}$. Therefore, $\widetilde{\varepsilon}({}^t(I + Y^{-1})) = {}^tZ_1^{-1}$. Thus, $I - X_1' Y^{-1} X_1 = \begin{pmatrix} I_j & & \\ & {}^tZ_1^{-1} & \\ & & I_j \end{pmatrix}$, with Z_1 as before.

We again choose a complete set of representatives $\{\mathbf{T}_i\}_i$ for the conjugacy classes of Cartan subgroups in $\mathbf{G}(n/2)$ and fix $\{\widetilde{\mathbf{T}}_i\}_i$ and the corresponding F – isomorphisms with ε – stable Cartan subgroups $\widetilde{\mathbf{T}}_i$ of GL_n. None of the $\mathbf{T}_i$ will be elliptic in $\mathbf{G} = \mathbf{G}(m)$. By Lemma 3.2, the contribution from almost all the orbits is given by

$$\psi(s, Z) = \sum_\alpha \omega'(\alpha) \cdot$$

$$\int_{g \in G'/\mathbf{G}'_{\varepsilon,Y}(F)} \int_{h \in \mathbf{G}^\vee_Z(F)\backslash G^\vee} \int_{X_1 \mathbf{G}^\vee(F)} f'(\alpha g Y \varepsilon(g)^{-1}) f_\tau(h^{-1} Z h) |\det(g Y \varepsilon(g)^{-1})|^s \, dg\, dh\, d(X_1 h_0) \cdot$$

$$\int_{Z(G')} \xi_{\tilde{L}}(z^{-2} g Y \varepsilon(g^{-1})) \xi_{L'}(z^{-1} g X_1 h_0 h) |\det z|^{-2s} \, d^\times z,$$

where X_1 is as above and $Z = I - X_1' Y^{-1} X_1$. Moreover, $X_1 \mathbf{G}^\vee_Z(F) \simeq \mathbf{G}'_{\varepsilon,Y}(F)$ and therefore the formula for $\psi(s, Z)$ can be written as

$$\psi(s, Z) = \sum_\alpha \omega'(\alpha) \cdot$$

$$\int_{g \in G'/\mathbf{G}'_{\varepsilon,Y}(F)} \int_{h \in \mathbf{G}^\vee_Z(F)\backslash G^\vee} \int_{g_0 \in \mathbf{G}'_{\varepsilon,Y}(F)} f'(\alpha g Y \varepsilon(g^{-1})) f_\tau(h^{-1} Z h) |\det(g Y \varepsilon(g^{-1}))|^s \, dg\, dh\, dg_0 \cdot$$

$$\int_{Z(G')} \xi_{\tilde{L}}(z^{-2} g Y \varepsilon(g^{-1})) \xi_{L'}(z^{-1} g g_0 X_1 h) |\det z|^{-2s} \, d^\times z$$

which is exactly the same expression as the one given in the case $n > 2m$, with the roles of $\mathbf{G}^\vee(F)$ and $\mathbf{G}'_{\varepsilon,Y}(F)$, as well as h_0 and g_0, interchanged. Using the image correspondence, again we define $\psi_{\mathcal{A}}(s, Z)$ as in the other cases. The integration of $\psi_{\mathcal{A}}(s, \gamma)$ will now be over $\bigcup_i T_i$ and an argument similar to those used before applies. But this time $R_{\mathbf{G}}(f_\tau, f') = 0$, since $\Phi(\gamma, f_\tau) = 0$ for all elliptic γ. Observe that although the integration is only over regular semisimple conjugacy classes in $G(n/2)$, the orbital integrals $\Phi(-, f_\tau)$ are computed over all of $G = G(m)$.

We state our result as follows:

Theorem 4.8. *Let $2\ell = \min(n, 2m)$ and denote by $\{\mathbf{T}_i\}$ a complete set of representatives for the conjugacy classes of Cartan subgroups of $\mathbf{G}(\ell)$. For each i, let ω_i denote a compact subset of T_i', the set of regular elements of T_i. Then the intertwining operator $A(s, \tau' \otimes \tau, w_0)$ has a pole at $s = 0$ if and only if*

$$cR_{\mathbf{G}}(f_\tau, f') + \sum_i |W(\mathbf{T}_i)|^{-1} \lim_{\omega_i \to T_i} \operatorname*{Res}_{s=0} \int\limits_{T_i' \setminus \omega_i} \psi_{\mathcal{A}}(s, \gamma) |D(\gamma)| \, d\gamma \neq 0,$$

for some choice of the data $f_\tau, f', L,$ and L. Here $c = (2n \log q)^{-1}$. If $n < 2m$, then $R_{\mathbf{G}}(f_\tau, f') = 0$ for all f_τ and f' and therefore it is the non-vanishing of

$$\sum_i |W(\mathbf{T}_i)|^{-1} \lim_{\omega_i \to T_i} \operatorname*{Res}_{s=0} \int\limits_{T_i' \setminus \omega_i} \psi_{\mathcal{A}}(s, \gamma) |D(\gamma)| \, d\gamma$$

which determines the pole of the operator at $s = 0$.

Corollary 4.9. *Suppose that $\tau' \simeq \widetilde{\tau}'$.*

(a) *The induced representation $I(\tau' \otimes \tau)$ is irreducible if and only if*

$$cR_{\mathbf{G}}(f_\tau, f') + \sum_i |W(\mathbf{T}_i)|^{-1} \lim_{\omega_i \to T_i} \operatorname*{Res}_{s=0} \int\limits_{T_i' \setminus \omega_i} \psi_{\mathcal{A}}(s, \gamma) |D(\gamma)| \, d\gamma \neq 0,$$

for some choice of the data.

(b) *Assume τ is generic. If $I(\tau' \otimes \tau)$ is irreducible, then $I(s, \tau' \otimes \tau)$, $s \in \mathbb{R}$, is reducible exactly at $s = \pm 1/2$ or $s = \pm 1$, and at only one of these pairs.*

Remark 4.10. When $n \geq 2m$, the residue is the sum of $cR_{\mathbf{G}}$ and the more complicated term

$$\sum_i |W(\mathbf{T}_i)|^{-1} \lim_{\omega_i \to T_i} \operatorname*{Res}_{s=0} \int\limits_{T_i' \setminus \omega_i} \psi_{\mathcal{A}}(s, \gamma) \, |D(\gamma)| \, d\gamma.$$

In the next section, we will relate the non-vanishing of the second term to the simple non-vanishing condition of Proposition 5.1 of [**20**], i.e., in terms of the theory of twisted endoscopy which discusses the equivalence of this non-vanishing with τ' coming from $SO_{n+1}(F)$.

Remark 4.11. Clearly Theorem 4.8 and Corollary 4.9 are valid for the split even special orthogonal groups and are the correct versions of what was originally stated as Theorems 7.8 and 8.1 in [**21**]. In fact, although the main term $R_{\mathbf{G}}$ of Theorem 7.8 of [**21**] is correct, the singular terms given there are too optimistic and are consequences of a gap in the proof of Theorem 7.8 of [**21**]. However, in Proposition 5.2 of the next section, using L–functions, we relate the singular terms from the two different versions to each other. We refer the reader to the introduction for further comments.

§5. The connection with twisted endoscopy and L–functions. We now show how the results from Section 4 can be related to the theory of twisted endoscopy and L–functions.

Let χ_τ be the distribution character of τ. By the work of Harish-Chandra, [**9**, **10**], we know that χ_τ is given by a locally integrable function, which we also denote by χ_τ. From [**11**], and [**6**], we can choose a matrix coefficient f_τ for τ such that $\Phi(\gamma, f_\tau) = \chi_\tau(\gamma)$, for all regular semisimple $\gamma \in G$. The matrix coefficient f_τ is then called a *pseudo-coefficient.*

Since $\tau' \simeq \widetilde{\tau}' \simeq (\tau')^\varepsilon$, we see that τ' extends to a representation of the disconnected group $GL_n(F) \ltimes \{1, \varepsilon\}$. This comes from fixing an equivalence $\tau'(\varepsilon)$ between τ' and $(\tau')^\varepsilon$. The ε-twisted distribution character of τ' is defined by $\chi^\varepsilon_{\tau'}(f') = \mathrm{trace}(\tau'(f)\tau'(\varepsilon))$. In [**5**] Clozel showed that there is a locally integrable function, also denoted by $\chi^\varepsilon_{\tau'}$, on the ε-regular elements such that

$$\chi^\varepsilon_{\tau'}(f') = \int\limits_{Z'\backslash G'} \chi^\varepsilon_{\tau'}(g) f'(g)\, dg,$$

with Z' the center of G'. Kottwitz and Rogawski [**14**] proved the existence of ε-twisted pseudo coefficients. That is, there exists a matrix coefficient $\psi_{\tau'}$ of τ' such that $\Phi_\varepsilon(\gamma', \psi_{\tau'}) = \chi^\varepsilon_{\tau'}(\gamma')$ for every ε-regular element $\gamma' \in GL_n(F)$. Thus, by choosing $f' \in C_c^\infty(GL_n(F))$ which defines $\psi_{\tau'}$, we have

$$\Phi_\varepsilon(\mathcal{A}(\{\gamma\}), f') = \sum_{\gamma' \in \mathcal{A}(\{\gamma\})} \Delta(\gamma, \gamma')\chi^\varepsilon_{\tau'}(\gamma'),$$

and we denote this by $\chi^\varepsilon_{\tau'}(\mathcal{A}(\{\gamma\}))$.

Let $n = 2m$. Then, by the above observations, we can choose f_τ and f' so that the regular term becomes

$$R_{\mathbf{G}}(f_\tau, f') = \sum_{\{\mathbf{T}_i\}} \mu(\mathbf{T}_i)|W(\mathbf{T}_i)|^{-1} \int_{\mathbf{T}_i} \chi_\tau(\gamma)\chi^\varepsilon_{\tau'}(\mathcal{A}(\{\gamma\}))|D(\gamma)|\, d\gamma.$$

Thus, $R_{\mathbf{G}}$ defines a pairing between the character χ_τ of τ, and the ε-twisted character $\chi^\varepsilon_{\tau'}$ of τ'. Consequently, we expect that the non-vanishing of $R_G(f_\tau, f')$ must in part point towards τ' coming from τ via twisted endoscopy [**1**,**2**,**12**,**13**]. For our purposes, we define the notion of endoscopic transfer below. Observe that $R_{\mathbf{G}}$ makes sense and is convergent for any discrete series τ.

Definition 5.1. *A self-dual irreducible supercuspidal representation τ' of G' is said to be the ε-twisted endoscopic transfer of a discrete series representation τ of $G(n/2)$ if $R_{\mathbf{G}}(f_\tau, f') \neq 0$, for some matrix coefficient f_τ of τ, and some $f' \in C_c^\infty(GL_n(F))$ defining one for τ'.*

Assume $n \geq 2m$, and resume our assumption that τ and τ' are supercuspidal. We expect that the Rankin-Selberg product L–function $L(s, \tau' \times \tau)$, which was formally defined in [**19**] must satisfy the following (defining) condition:

> *$L(s, \tau' \times \tau)$ has a pole at $s = 0$ if and only if $R_{\mathbf{G}}(f_\tau, f') \neq 0$ for some f_τ and f', or equivalently, if and only if τ' comes by twisted endoscopy from τ (Definition 5.1).*

As will be discussed in the next few paragraphs, this seems to be in complete agreement with definitions given in [**20**].

We must now study the singular contributions, and we therefore continue with our assumption $n \geq 2m$ and let

$$R_{\mathrm{sing}}(f_\tau, f') = \sum_i |W(\mathbf{T}_i)|^{-1} \lim_{\omega_i \to T_i} \operatorname*{Res}_{s=0} \int_{T_i' \backslash \omega_i} \psi_{\mathcal{A}}(s, \gamma)\, |D(\gamma)|\, d\gamma$$

so that the residue of the intertwining operator $A(s, \tau' \otimes \tau, w_0)$ at $s = 0$ can be written as

$$cR_{\mathbf{G}}(f_\tau, f') + R_{\text{sing}}(f_\tau, f'),$$

where $c = (2n \log q)^{-1}$. Observe that we have suppressed the dependence of R_{sing} on L and L'.

Our goal in what follows is to separate out the poles coming from the two L–functions $L(s, \tau' \times \tau)$ and $L(s, \tau', \wedge^2 \rho_n)$ by means of the non-vanishing of $R_{\mathbf{G}}$ and R_{sing}, respectively, when $n \geq 2m$.

By Lemma 4.5, $R_{\text{sing}}(f_\tau, f') = 0$ if and only if the process of taking the residue and integration can be interchanged. On the other hand, the theory of L–functions discussed earlier demands the existence of poles coming from those of $L(s, \tau', \wedge^2 \rho_n)$ at $s = 0$. Such poles depend on τ' alone, and cannot be reflected in the non-vanishing of $R_{\mathbf{G}}$. Consequently, one must expect $R_{\text{sing}} \neq 0$ in general, as we will discuss below. Therefore, the fact that interchanging the process of taking the residue at $s = 0$ and integration must be checked is not just an analytic impediment, but rather a fact which reflects deep arithmetic connections which govern the problem.

Now, let us assume that τ' comes from $SO_{n+1}(F)$ as explained in [**20**] (Proposition 5.1 and Theorem 7.6 of [**20**]), or in other words, $L(s, \tau', \wedge^2 \rho_n)$ has a pole at $s = 0$. Then, by the simplicity of the pole of $A(s, \tau' \otimes \tau, w_0)$ at $s = 0$, $L(s, \tau' \times \tau)$ must be holomorphic there. (See below and the discussion after Definition 5.1.) Moreover, assume that τ is such that $R_{\mathbf{G}}(f_\tau, f') = 0$ for all f_τ and f'. We expect this to be true for all τ anyway, if τ' comes from $SO_{n+1}(F)$. Then, by the theory of L–functions [**19**] as explained before

$$L(s, \tau' \times \tau)^{-1} L(2s, \tau', \wedge^2 \rho_n)^{-1} A(s, \tau' \otimes \tau, w_0)$$

must be non-zero and holomorphic. This implies that $R_{\text{sing}}(f_\tau, f') \neq 0$ for some data. We therefore have:

Proposition 5.2. *Suppose $n \geq 2m$. Assume that τ' comes from $SO_{n+1}(F)$. Then $R_{\text{sing}} \neq 0$ for any irreducible unitary supercuspidal representation τ of $G(m)$ from which τ' does not come by twisted endoscopy (Definition 5.1). (We* *expect this to be the case for every irreducible unitary supercuspidal representation* τ *of* $G(m)$ *if* τ' *comes from* $SO_{n+1}(F)$.)

When $n < 2m$, the term $R_{\mathbf{G}} = 0$, and therefore the control of the poles of both L-functions lies within R_{sing} which now constitutes the whole residue. Further analysis of the term R_{sing} is now necessary to distinguish the two L-functions.

We conclude the paper by stating a result about $L(s, \tau' \times \tau)$ for any even n and any m with no further assumption on the relation between τ and τ'. We may, and will, assume that $\tau' \simeq \widetilde{\tau}'$.

Proposition 5.3.

(a) *Suppose* τ' *comes from* $SO_{n+1}(F)$ *(non-vanishing condition (5.2) of [**20**]) or equivalently* $L(s, \tau', \wedge^2 \rho_n)$ *has a pole at* $s = 0$. *Then* $L(s, \tau' \times \tau)$ *is holomorphic at* $s = 0$.

(b) *If* τ' *does not come from* $SO_{n+1}(F)$, *then* $L(s, \tau' \times \tau)$ *has a pole at* $s = 0$ *if and only if* $cR_{\mathbf{G}} + R_{\text{sing}} \neq 0$ *for some data.*

Proof. The proposition is a consequence of the simplicity of the poles of $A(s, \tau' \otimes \tau, w_0)$, Theorem 4.8, and the holomorphy and non-vanishing of

$$L(s, \tau' \times \tau)^{-1} L(s, \tau', \wedge^2 \rho_n)^{-1} A(s, \tau' \otimes \tau, w_0).$$

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Department of Mathematics, Purdue University, West Lafayette, IN 47907