

TRANSFER MAPS FOR FIBRATIONS AND DUALITY

J. C. Becker and D. H. Gottlieb

1. Introduction

In this paper we will describe a transfer construction for (Hurewicz) fibrations which is a generalization of that for fiber bundles studied in [4, 5]. We suppose given a commutative triangle

$$\begin{array}{ccc} E & \xrightarrow{f} & E \\ & \searrow p & \downarrow p \\ & & B \end{array}$$

where $p : E \rightarrow B$ is a fibration having fiber F a finite complex and base B a connected finite dimensional complex. With this data we show that there is an S -map, which we call a *transfer* map,

$$\tau(f) : B^+ \rightarrow E^+$$

having the property that

$$\tilde{H}^*(B^+) \xrightarrow{p^*} \tilde{H}^*(E^+) \xrightarrow{\tau(f)^*} \tilde{H}^*(B^+)$$

is multiplication by the Lefschetz number Λ of $f' : F \rightarrow F$, the restriction of f to the fiber. (Although f' is not unique we allow this abuse of language since Λ is independent of the choice of f' .)

The existence of $\tau(f)$ severely restricts the projection map of the fibration. For example

$$p_* : \{X; E^+\}_a \otimes Z[\Lambda^{-1}] \rightarrow \{X; B^+\}_a \otimes Z[\Lambda^{-1}]$$

is a split epimorphism for any (pointed) finite dimensional complex X .

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We will show that the boundary map $\omega : \Omega B \rightarrow F$ arising from the Puppe sequence of the fibration $p : E \rightarrow B$ is also restricted by the transfer. Precisely, we have

(1.1) THEOREM: Assume that F is connected. Then

$$\Lambda \omega_* : \{X; \Omega B\}_q \rightarrow \{X; F\}_q$$

is trivial for any finite dimensional complex X .

An independent method of extending the notion of transfer from fiber bundles to fibrations is given in [7]. The method which we describe here is intrinsic and has the advantage that many basic properties of the transfer are easily derived. A. Dold [9] has also independently defined the transfer, placing somewhat different restrictions on the projection p and fiber preserving map f .

The outline of the paper is as follows. In section 2 we give a homotopy characterization of the Lefschetz number of a map. Although an elementary fact it is the key point in defining the transfer. In section 3 we deal with some homotopy properties of ex -spaces and in section 4 with the duality theory of ex -spaces. This generalization of Spanier-Whitehead duality is purely formal except for the question of the existence of dual ex -spaces (theorem 4.2). In sections 5 thru 7 we define the transfer and establish its basic properties. In section 8 we prove theorem (1.1) mentioned above and describe some consequences of the theorem. In section 9 we consider smooth fiber bundles and in this case we give a more geometric description of the transfer.

2. The Lefschetz number

Suppose that F is a finite complex with base point and $f : F \rightarrow F$ is a base point preserving map. By the reduced Lefschetz number of f we mean

$$\bar{A}_f = \sum (-1)^i \text{tr} [f_* : \tilde{H}_i(F) \rightarrow \tilde{H}_i(F)].$$

Let $\mu : S^s \rightarrow F \wedge \hat{F}$ be a duality map in the sense of Spanier [15]. Then $F \wedge \hat{F}$ is $2s$ -self dual via the map

$$\vee \dagger : S^{2s} \xrightarrow{\mu \wedge \mu} (F \wedge \hat{F}) \wedge (F \wedge \hat{F}) \xrightarrow{\alpha} (F \wedge \hat{F}) \wedge (F \wedge \hat{F})$$

where $\alpha(x \wedge y \wedge x' \wedge y') = x' \wedge y \wedge x \wedge y'$. Denote this composite by ν and let $\hat{\mu} : F \wedge \hat{F} \rightarrow S^s$ be dual to μ relative to ν . The following lemma provides a homotopy description of the reduced Lefschetz number of f . It is the analogue for base point preserving maps of the Lefschetz fixed point theorem given by Dold [8, theorem (4.1)].

(2.1) LEMMA: The composite $S^s \xrightarrow{\mu} F \wedge \hat{F} \xrightarrow{f \wedge 1} F \wedge \hat{F} \xrightarrow{\hat{\mu}} S^s$ has degree \bar{A}_f .

PROOF: We have the following homotopy commutative diagram

$$(2.2) \quad \begin{array}{ccc} & & (F \wedge \hat{F}) \wedge (F \wedge \hat{F}) \\ & \nearrow \nu & \downarrow \hat{\mu} \wedge 1 \\ S^{2s} & & S^s \wedge (F \wedge \hat{F}) \\ & \searrow 1 \wedge \mu & \end{array}$$

Let Q denote the rational numbers and choose a generator $\gamma \in \tilde{H}_s(S^s; Q)$. Let $\{u_p\}$ be a basis for $\tilde{H}_*(F; Q)$ and $\{v_p\}$ a basis for $\tilde{H}_*(\hat{F}; Q)$. Let $d(u_p)$ and $d(v_p)$ denote respectively the dimension of u_p and v_p . Write

$$\mu_*(\gamma) = \sum_{ij} a_{ij} u_i \wedge v_j$$

and

$$\hat{\mu}_*(u_i \wedge v_j) = b_{ij} \gamma$$

Let $A = |a_{ij}|$, $B = |b_{ij}|$, and $D = |(-1)^{d(u_i)} \delta_{ij}|$ where δ_{ij} is the Kronecker symbol. By (2.2) we have

$$(1 \wedge \mu)_*(\gamma \wedge \gamma) = (\hat{\mu} \wedge 1)_* \nu_*(\gamma \wedge \gamma).$$

By expressing each side in terms of the basis elements $\gamma \wedge u_i \wedge v_j$ and equating coefficients, we obtain the relation $A = DAB^T A$. Since A is non-singular $AB^T = D$.

Now suppose that $f_*(u_i) = \sum_k c_{ik} u_k$. We have

$$\begin{aligned} \hat{\mu}_*(f \wedge 1)_* \mu_*(\gamma) &= \hat{\mu}_*(f \wedge 1)_* \left(\sum_{ij} a_{ij} u_i \wedge v_j \right) \\ &= \hat{\mu}_* \left(\sum_{i,j,k} a_{ij} c_{ik} u_k \wedge v_j \right) \\ &= \sum_{i,j,k} a_{ij} b_{kj} c_{ik} \gamma \\ &= \sum_{i,k} (-1)^{d(u_i)} \delta_{ik} c_{ii} \gamma \\ &= \sum_i (-1)^{d(u_i)} c_{ii} \gamma = \bar{A}_f \gamma \end{aligned}$$

This completes the proof.

3. Ex-spaces

Consider a trivial fibration $p : F \rightarrow *$ and a map $f : F \rightarrow F$, where F is a finite complex. In this case the transfer map we seek is to be of the form $\tau(f) : S^s \rightarrow S^s \wedge F^+$, for large s , and is to have the property that

$$S^s \xrightarrow{\tau(f)} S^s \wedge F^+ \xrightarrow{1 \wedge p^+} S^s \text{ have degree } \Lambda_f.$$

To construct $\tau(f)$ let $\mu : S^s \rightarrow F^+ \wedge \hat{F}$ be a duality map and take $\tau(f)$ to be composite

$$S^s \xrightarrow{\mu} F^+ \wedge \hat{F} \xrightarrow{(1, f^+) \wedge 1} F^+ \wedge F^+ \wedge \hat{F} \xrightarrow{1 \wedge \hat{\mu}} F^+ \wedge S^s \longrightarrow S^s \wedge F^+.$$

Then it is immediate from the preceding lemma that $p\tau(f)$ has degree Λ_f .

In order to define $\tau(f)$ in general we intend to carry out the above construction "fiberwise". This leads naturally to the consideration of ex-spaces and duality for ex-spaces. In this section we discuss some aspects of the homotopy theory of ex-spaces and in the following section we deal with duality proper.

We shall work entirely in the category of compactly generated spaces [17]. Recall that an ex-space [13] $E = (E, B, p, \Delta)$ consists of maps $p : E \rightarrow B$ and $\Delta : B \rightarrow E$ such that $p\Delta = 1$. We assume throughout that B is a CW-complex and E has the homotopy type of a CW-complex. An ex-map $f : E \rightarrow E'$ is one which is both fiber and cross-section preserving, i.e. $p'f = p$ and $f\Delta = \Delta'$. The set of ex-homotopy classes of ex-maps from E to E' is denoted by $[E; E']$.

An ex-space E is an *ex-fibration* if there is a lifting function

$$\Gamma : E \times_B B^I \rightarrow E^I$$

with the property that $\Gamma(\Delta(b), \sigma) = \Delta\sigma$, when σ is a path in B beginning at b . We will also need the notion of a *well based* ex-space as in [13]. E is well based if there is a vertical retraction map $E \times I \rightarrow E \times \{0\} \cup \Delta(B) \times I$.

If $p : E \rightarrow B$ is a map we have an associated ex-space $\bar{E} = (\bar{E}, B, \bar{p}, \bar{\Delta})$ where \bar{E} is the disjoint union of E and B and \bar{p} and $\bar{\Delta}$ are the obvious maps. Observe that \bar{E} is well based, and if $p : E \rightarrow B$ is a fibration, \bar{E} is an ex-fibration.

If X is a pointed space we will also use X to denote the ex-space

$(X \times B, B, p, \Delta)$ where p is projection on the second factor and Δ is the cross section determined by the base point.

The fiberwise reduced product of ex-spaces E and E' is denoted by $E \wedge_B E'$. Let $r : E \times_B E' \rightarrow E \wedge_B E'$ denote the identification map. Because of the exponential law in the category of compactly generated spaces, $r \times 1 : (E \times_B E') \times Y \rightarrow (E \wedge_B E') \times Y$ is an identification for any space Y . From this it is easy to see that $r \times_B 1 : (E \times_B E') \times_B Y \rightarrow (E \wedge_B E') \times_B Y$ is an identification for any space Y over B . With this last observation it is easy to prove the following.

(3.1) LEMMA: *If E and E' are well based so is $E \wedge_B E'$. If E and E' are ex-fibrations so is $E \wedge_B E'$.*

(3.2) THEOREM: (Comparison theorem): *Let E and E' be ex-fibrations and suppose $g : E \rightarrow E'$ is such that its restriction to the fiber over b , $g_b : F_b \rightarrow F'_b$ is an n -equivalence, $b \in B$. Let X be a well based ex-space. Then $g_* : [X; E] \rightarrow [X; E']$ is injective if X is n -coconnected and surjective if X is $(n+1)$ -coconnected.*

The proof is the same as the proof given for bundles in [1; theorem 3.3]. For other versions of the comparison theorem, see Eggar [11; Theorem 3.9] and James [14; Theorem 3.2].

(3.3) COROLLARY: *Suppose that E and E' are well based ex-fibrations and $g : E \rightarrow E'$ is such that $g_b : F_b \rightarrow F'_b$ is a homotopy equivalence, $b \in B$. Then g is an ex-homotopy equivalence.*

Given $E = (E, B, p, \Delta)$ let $\Omega_B(E)$ denote the space of loops $\sigma : I \rightarrow E$ such that $\sigma(I) \subset F_b$ for some $b \in B$, and $\sigma(0) = \sigma(1) = \Delta(b)$. We have

$$\Omega(p) : \Omega_B(E) \rightarrow B \quad \text{and} \quad \Omega(\Delta) : B \rightarrow \Omega_B(E)$$

by $\Omega(p)(\sigma) = p(\sigma(0))$ and $\Omega(\Delta)(b) = \Delta(b) = \Delta(b)^*$ – the constant loop at $\Delta(b)$. If E is an ex-fibration so is $\Omega_B(E)$ as is easily checked.

There is the suspension map

$$(3.4) \quad \sigma : [E, E'] \rightarrow [S^1 \wedge_B E; S^1 \wedge_B E']$$

by $f \rightarrow 1 \wedge f$. By a standard argument involving the comparison theorem and the loop space $\Omega_B(S^1 \wedge E')$, we obtain the following suspension theorem (c.f. [1; Theorem 3.14] or [14; Theorem 4.3]).

(3.5) THEOREM: Suppose that E' is an ex-fibration such that each fiber F_b is $(n-1)$ -connected. Let E be well based. Then σ is injective if E is $(2n-1)$ -coconnected and surjective if E is $2n$ -coconnected.

Let

$$(3.6) \quad \{E; E'\}_q = \varinjlim_k [S^{k+q} \wedge E; S^k \wedge E']$$

with the natural abelian group structure. The cone over E is $C(E) = I \wedge_B E$ with 0 the base point of I .

Suppose that A is a subcomplex of B . Let $E_A = p^{-1}(A) \cup \Delta(B)$ regarded as an ex-space of B . Then, as in [13], we have an exact sequence

$$\cdots \rightarrow \{E \cup C(E_A); E'\}_q \rightarrow \{E; E'\}_q \rightarrow \{E_A; E'\}_q \rightarrow \cdots$$

Let E/E_A be the quotient of E obtained by identifying each fiber of E_A to its base point and let $c: E \cup C(E_A) \rightarrow E/E_A$ denote the natural map. Note that if E is well based so are E_A , E/E_A and $E \cup C(E_A)$.

(3.7) LEMMA: If E and E' are ex-fibrations and E is well based then $c^*: [E/E_A; E'] \rightarrow [E \cup C(E_A); E']$ is bijective.

A proof is given in section 10. Now if E and E' meet the requirements of the lemma we may replace $\{E \cup C(E_A); E'\}$ in the above sequence by $\{E/E_A; E'\}$ via c^* and so obtain an exact sequence

$$(3.8) \quad \cdots \rightarrow \{E/E_A; E'\}_q \rightarrow \{E; E'\}_q \rightarrow \{E_A; E'\}_q \rightarrow \cdots$$

4. Duality

In this section we will outline Spanier-Whitehead duality theory in the category of ex-spaces. Some aspects of this theory have been dealt with by K. Tsuchida [18]. We restrict ourselves to ex-spaces which are well based ex-fibrations having base B a finite dimensional complex and each fiber homotopy equivalent to a finite complex. Briefly, we will refer to such ex-spaces as ex-fibrations.

An ex-map $\mu: S^s \times B \rightarrow E \wedge_B \hat{E}$ is a *duality map* if for each $b \in B$ the restricted map $\mu_b: S^s \rightarrow F_b \wedge \hat{F}_b$ is a duality map in the usual sense.

Given such a duality map and ex-fibrations X and Y we have

$$(4.1) \quad D_\mu: \{X \wedge E; Y\}_q \rightarrow \{X; Y \wedge \hat{E}\}_{q+s}$$

defined by sending $f: S^{k+q} \wedge X \wedge E \rightarrow S^k \wedge Y$ to

$$S^{k+q+s} \wedge X \longrightarrow S^{k+q} \wedge X \wedge S^s \xrightarrow{1 \wedge \mu} S^{k+q} \wedge X \wedge E \wedge \hat{E} \xrightarrow{f \wedge 1} S^k \wedge Y \wedge \hat{E}$$

and

$$(4.2) \quad D^\mu: \{\hat{E} \wedge X; Y\}_q \rightarrow \{X; E \wedge Y\}_{q+s}$$

by sending $f: S^{k+q} \wedge \hat{E} \wedge X \rightarrow S^k \wedge Y$ to

$$S^{k+q+s} \wedge X \xrightarrow{1 \wedge \mu \wedge 1} S^{k+q} \wedge E \wedge \hat{E} \wedge X \longrightarrow E \wedge S^{k+q} \wedge \hat{E} \wedge X \xrightarrow{1 \wedge f} E \wedge S^k \wedge Y \longrightarrow S^k \wedge E \wedge Y.$$

(4.3) LEMMA: D_μ and D^μ are isomorphisms.

This follows from the corresponding fact for pointed spaces if all the ex-spaces involved are products. The proof in general is by induction over the skeleta of B using the exact sequence (3.8). The argument is standard and will be omitted.

If $\nu: S^s \times B \rightarrow X \wedge \hat{X}$ is a second duality map we have, as in the case of pointed spaces, an isomorphism

$$(4.4) \quad D(\mu, \nu): \{E, X\}_q \rightarrow \{\hat{X}; \hat{E}\}_q$$

defined so as to make the following diagram commutative

$$(4.5) \quad \begin{array}{ccc} \{E; X\}_q & \xrightarrow{D(\mu, \nu)} & \{\hat{X}; \hat{E}\}_q \\ \downarrow D_\mu & & \downarrow D^\nu \\ \{S^s \times B, X \wedge \hat{E}\}_{q+s} & & \end{array}$$

In particular, $f: E \rightarrow X$ is dual to $g: \hat{X} \rightarrow \hat{E}$ relative to μ and ν if and

only if the diagram

$$(4.6) \quad \begin{array}{ccc} S^s \times B & \xrightarrow{\mu} & E \wedge \hat{E} \\ \downarrow \nu & & \downarrow f \wedge 1 \\ X \wedge \hat{X} & \xrightarrow{1 \wedge g} & X \wedge \hat{E} \end{array}$$

is stably homotopy commutative.

(4.7) THEOREM: *If E is an ex-fibration there is an integer s , an ex-fibration \hat{E} , and a duality map $\mu : S^s \times B \rightarrow E \wedge \hat{E}$.*

A proof is given in section 11.

5. Transfer

Let \mathcal{F} denote the category of fibrations $p : E \rightarrow B$ such that B is a finite dimensional complex and each fiber is homotopy equivalent to a finite complex. We consider commutative triangles

$$\begin{array}{ccc} E & \xrightarrow{f} & E \\ \downarrow p & & \downarrow p \\ & B & \end{array}$$

where $p : E \rightarrow B$ is in \mathcal{F} . We will construct for such a triangle and for A a subcomplex of B a transfer map, which is an S -map

$$(5.1) \quad \tau(f) : B/A \rightarrow E/E_A.$$

Here $E_A = p^{-1}(A)$.

Consider the ex-space \bar{E} , the disjoint union of E and B . Since \bar{E} is an ex-fibration in the sense of section 4, there is an ex-fibration \hat{E} and a duality map

$$\mu : S^s \times B \rightarrow \bar{E} \wedge \hat{E}.$$

Analogous to the situation for pointed spaces (see section 2), $\bar{E} \wedge \hat{E}$ is canonically $2s$ -self dual. Let

$$\hat{\mu} : \bar{E} \wedge \hat{E} \rightarrow S^s \times B$$

be dual to μ . We have

$$S^s \times B \xrightarrow{\mu} \bar{E} \wedge \hat{E} \xrightarrow{(1, \bar{f}) \wedge 1} \bar{E} \wedge \bar{E} \wedge \hat{E} \xrightarrow{1 \wedge \hat{\mu}} \bar{E} \wedge S^s \longrightarrow S^s \wedge \bar{E}$$

which takes $S^s \times A \cup s_0 \times B$ into $S^s \times E_A \cup s_0 \times B$.

Identifying these subspaces to a point, the above map yields

$$\tau(f) : S^s \wedge B/A \rightarrow S^s \wedge E/E_A.$$

We will show now that the S -homotopy class of $\tau(f)$ is well defined. Firstly, if μ is replaced by a suspension, this has the effect of replacing $\tau(f)$ by its suspension. Suppose now that

$$(5.2) \quad \begin{array}{ccc} E_i & \xrightarrow{f_i} & E_i \\ \downarrow \rho_i & & \downarrow \rho_i \\ & B & \end{array}$$

$i = 1, 2$, are given and $h : E_1 \rightarrow E_2$ is a fiber homotopy equivalence such that $hf_1 = f_2h$. Let

$$\mu_i : S^s \times B \rightarrow \bar{E}_i \wedge \hat{E}_i, \quad i = 1, 2,$$

be duality maps and let $k : \hat{E}_2 \rightarrow \hat{E}_1$ be dual to \bar{h} . Then k is an ex-homotopy equivalence and we have commutativity relations

$$\begin{array}{ccc} S^s \times B & \begin{array}{l} \xrightarrow{\mu_1} \\ \xrightarrow{\mu_2} \end{array} & \begin{array}{l} \bar{E}_1 \wedge \hat{E}_1 \\ \downarrow \bar{h} \wedge k^{-1} \\ \bar{E}_2 \wedge \hat{E}_2 \end{array} \\ & & \begin{array}{l} \bar{E}_1 \wedge \hat{E}_1 \\ \downarrow \bar{h} \wedge k^{-1} \\ \bar{E}_2 \wedge \hat{E}_2 \end{array} \begin{array}{l} \xrightarrow{\hat{\mu}_1} \\ \xrightarrow{\hat{\mu}_2} \end{array} \\ & & S^s \times B \end{array}$$

where the second triangle is obtained by dualizing the first. The following diagram is then commutative.

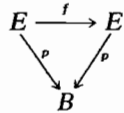
$$\begin{array}{ccccccc} & & \bar{E}_1 \wedge \hat{E}_1 & \xrightarrow{(1, \bar{f}_1) \wedge 1} & \bar{E}_1 \wedge \bar{E}_1 \wedge \hat{E}_1 & \xrightarrow{1 \wedge \hat{\mu}_1} & \bar{E}_1 \wedge S^s \\ & \nearrow \mu_1 & \downarrow \bar{h} \wedge k^{-1} & & \downarrow \bar{h} \wedge \bar{h} \wedge k^{-1} & & \downarrow \bar{h} \wedge 1 \\ S^s \times B & & \bar{E}_2 \wedge \hat{E}_2 & \xrightarrow{(1, \bar{f}_2) \wedge 1} & \bar{E}_2 \wedge \bar{E}_2 \wedge \hat{E}_1 & \xrightarrow{1 \wedge \hat{\mu}_2} & \bar{E}_2 \wedge S^s \\ & \searrow \mu_2 & & & & & \end{array}$$

Therefore $\bar{h}\tau(f_1) = \tau(f_2)$. Taking h to be the identity we see that $\tau(f)$ does not depend on the choice of duality map and moreover, $\tau(f)$ depends only on the fiber homotopy class of f .

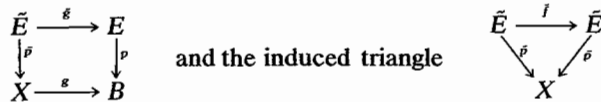
We also established the following functorial property

(5.3) With the data (5.2) if $h : E_1 \rightarrow E_2$ is a fiber homotopy equivalence such that hf_1 is fiber homotopic to f_2h then $h\tau(f_1) = \tau(f_2)$.

Now suppose we are given



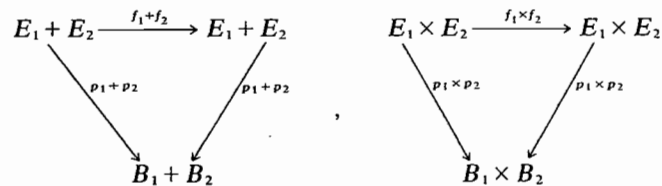
and a map $g : X \rightarrow B$. There is the pullback diagram



(5.4) We have $\tilde{g}\tau(\tilde{f}) = \tau(f)g$.

This is easily checked.

We may form the sum and product of the triangles in (5.2) obtaining



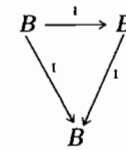
where $+$ denotes disjoint union.

$$(5.5) \quad \tau(f_1 + f_2) = \tau(f_1) \vee \tau(f_2) : (B_1/A_1) \vee (B_2/A_2) \rightarrow (E_1/E_{A_1}) \vee (E_2/E_{A_2})$$

$$(5.6) \quad \tau(f_1 \times f_2) = \tau(f_1) \wedge \tau(f_2) : (B_1/A_1) \wedge (B_2/A_2) \rightarrow (E_1/E_{A_1}) \wedge (E_2/E_{A_2}).$$

These properties follow from standard properties of duality maps as generalized to ex-spaces.

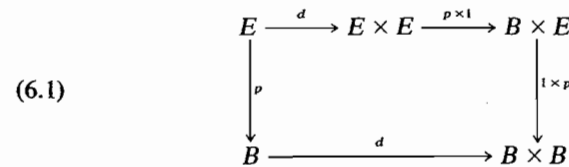
(5.7) For the triangle



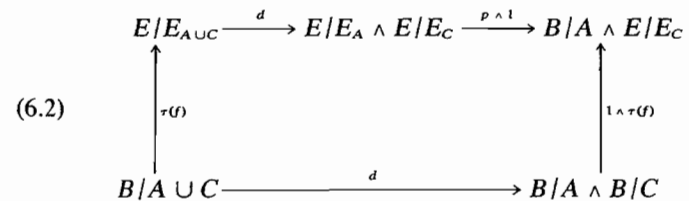
$\tau(1) : B/A \rightarrow B/A$ is the identity map.

6. Products

We consider now the multiplicative properties of the cohomology homomorphism induced by the transfer. We have a commutative diagram



where d is the diagonal map. From (5.3), (5.4), (5.6) and (5.7) we obtain, for subcomplexes A and C of B , a commutative diagram



Let M be a ring spectrum and N an M -module as in [19]. From the commutativity of the above diagram we obtain the formulas

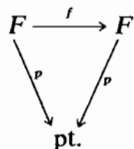
$$(6.3) \quad \tau(f)^*(p^*(x) \cup y) = x \cup \tau(f)^*(y),$$

$$x \in M^s(B/A), y \in N^t(E/E_C).$$

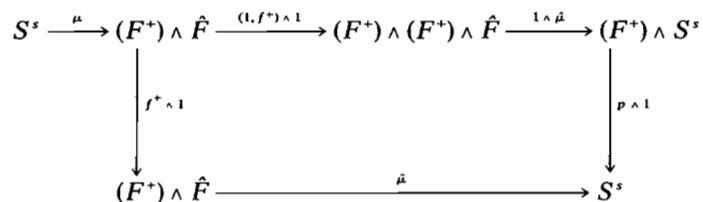
$$(6.4) \quad p_*(\tau(f)_*(x) \cap y) = x \cap \tau(f)_*(y),$$

$$x \in N_s(B/A \cup C), y \in M^t(E/E_C).$$

Now consider the triangle



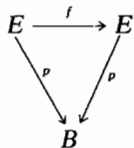
In the diagram



the composite $(1 \wedge \hat{\mu})(1, f^+) \wedge 1$ represents $\tau(f)$. Hence, by lemma (2.1) and the commutativity of the diagram we have (identifying pt.^+ with S^0).

(6.5) $p\tau(f): S^0 \rightarrow S^0$ has degree $\tilde{\Lambda}(f^+) = \Lambda(f)$ — the Lefschetz number of f .

We can now establish the fundamental property of the transfer. Consider



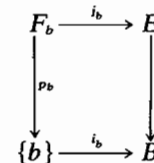
with $p: E \rightarrow B$ in \mathcal{F} . Let $f_b: F_b \rightarrow F_b$ denote the restriction of f to the fiber over $b \in B$ and let Λ denote the Lefschetz number of f_b . Let $H(\ ; \Gamma)$ denote singular theory with coefficients in the abelian group Γ .

(6.6) THEOREM: If B is connected the composite

$$\tilde{H}^*(B/A; \Gamma) \xrightarrow{p^*} \tilde{H}^*(E/E_A; \Gamma) \xrightarrow{\tau(f)^*} \tilde{H}^*(B/A; \Gamma)$$

is multiplication by Λ .

PROOF: Consider the inclusion



By (6.5), for $1 \in \tilde{H}^0(B^+; Z)$

$$i_b^* \tau(f)^* p^*(1) = \tau(f_b)^* p_b^*(1) = \Lambda.$$

Since $i_b^*: \tilde{H}^0(B^+; Z) \rightarrow \tilde{H}^0(\{b\}^+; Z)$ is an isomorphism,

$$\tau(f)^*(1) = \tau(f)^* p^*(1) = \Lambda$$

Applying (6.3), we have for $x \in H^*(B/A; \Gamma)$,

$$\begin{aligned} \tau(f)^* p^*(x) &= \tau(f)^*(p^*(x) \cup 1) \\ &= x \cup \tau(f)^*(1) = \Lambda x. \end{aligned}$$

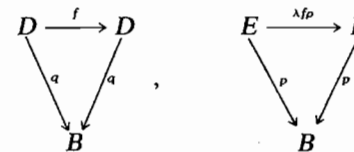
7. The retraction property

In this section we compare the transfer for a fibration with that of a retract up to homotopy.

Suppose that $p: E \rightarrow B$ and $q: D \rightarrow B$ are fibrations in \mathcal{F} and

$$D \xrightarrow{\lambda} E \xrightarrow{p} B$$

are fiber preserving maps such that $p\lambda \simeq 1$ over the identity. Then if $f: D \rightarrow D$ is a fiber preserving map we have triangles



(7.1) THEOREM: $\lambda\tau(f) = \tau(\lambda f\rho) : B/A \rightarrow E/E_\lambda$.

PROOF: Let

$$\begin{aligned} \mu_1 : S^s \times B &\rightarrow \bar{D} \wedge \hat{D} \\ \mu_2 : S^s \times B &\rightarrow \bar{E} \wedge \hat{E} \end{aligned}$$

be duality maps. Let $\hat{\lambda} : \hat{E} \rightarrow \hat{D}$ be dual to $\bar{\lambda} : \bar{D} \rightarrow \bar{E}$ relative to μ_1 and μ_2 , so that

$$(7.2) \quad \begin{array}{ccc} S^s \times B & \xrightarrow{\mu_1} & \bar{D} \wedge \hat{D} \\ \downarrow \mu_2 & & \downarrow \bar{\lambda} \wedge 1 \\ \bar{E} \wedge \hat{E} & \xrightarrow{1 \wedge \hat{\lambda}} & \bar{E} \wedge \hat{D} \end{array}$$

is commutative. Consider the diagram

$$(7.3) \quad \begin{array}{ccccc} & & \bar{D} \wedge \hat{D} & \xrightarrow{(1, \bar{f}) \wedge 1} & \bar{D} \wedge \bar{D} \wedge \hat{D} & \xrightarrow{1 \wedge \bar{\mu}_1} & \bar{D} \wedge S^s \\ & \nearrow \mu_1 & \downarrow \bar{\lambda} \wedge 1 & & \downarrow \bar{\lambda} \wedge 1 \wedge 1 & & \downarrow \bar{\lambda} \wedge 1 \\ S^s \times B & & \bar{E} \wedge \hat{D} & \xrightarrow{(1, f\rho) \wedge 1} & \bar{E} \wedge \bar{D} \wedge \hat{D} & & \\ & \searrow \mu_2 & \uparrow 1 \wedge \hat{\lambda} & & \uparrow 1 \wedge \bar{\rho} \wedge \hat{\lambda} & \searrow 1 \wedge \bar{\mu}_1 & \\ & & \bar{E} \wedge \hat{E} & \xrightarrow{(1, \bar{\lambda}f\rho) \wedge 1} & \bar{E} \wedge \bar{E} \wedge \hat{E} \text{ (A)} & & \\ & & \downarrow (1, \bar{\lambda}f\rho) \wedge 1 & & \downarrow 1 \wedge \bar{\lambda}f\rho \wedge 1 & & \\ & & \bar{E} \wedge \bar{E} \wedge \hat{E} & \xrightarrow{1 \wedge \bar{\mu}_2} & \bar{E} \wedge S^s & & \end{array}$$

The commutativity of the triangle (A) follows from the commutativity of the diagram

$$\begin{array}{ccccc} & & \bar{D} \wedge \hat{D} & \xrightarrow{\bar{\mu}_1} & S^s \times B \\ & \nearrow \bar{\rho} \wedge \hat{\lambda} & \uparrow 1 \wedge \hat{\lambda} & & \uparrow \bar{\mu}_2 \\ \bar{E} \wedge \hat{E} & \xrightarrow{\bar{\rho} \wedge 1} & \bar{D} \wedge \hat{E} & \xrightarrow{\bar{\lambda} \wedge 1} & \bar{E} \wedge \hat{E} \end{array}$$

where the square is the dual of (7.2). The remaining commutativity relations in (7.3) are easily checked. The theorem follows by comparing the two outside paths in (7.3) from $S^s \times B$ to $\bar{E} \wedge S^s$.

8. Proof of theorem (1.1)

We begin with an observation concerning the transfer map when the base space is a suspension. Suppose that X is a finite dimensional complex with base point x_0 and we are given

$$\begin{array}{ccc} \bar{E} & \xrightarrow{f} & \bar{E} \\ \downarrow \bar{\rho} & & \downarrow \bar{\rho} \\ S(X) & & S(X) \end{array}$$

with $\bar{\rho} : \bar{E} \rightarrow S(X)$ in \mathcal{F} . Let F denote the fiber over x_0 and choose a base point $e_0 \in F$. Let

$$(8.1) \quad \Delta : S(X) \rightarrow \bar{E}/F$$

be defined by $\Delta(e^{2mit} \wedge x) = \bar{f}(e_0, \sigma(t, x))$ (1), where \bar{f} is a lifting function and $\sigma(t, x) : I \rightarrow S(X)$ is the path

$$\sigma(t, x)(\lambda) = e^{2mit\lambda} \wedge x.$$

Let Λ denote the Lefschetz number of $f' : F \rightarrow F$, the restriction of f to F .

(8.2) LEMMA: Assume that F is connected. Then $\Lambda\Delta$ is stably homotopic to $\tau(\bar{f}) : S(X) \rightarrow \bar{E}/F$.

PROOF: Let $C(X) = I \wedge X$ denote the reduced cone of X (with 0 the base point of I) and consider

$$\begin{array}{ccc} C(X) \times F & \xrightarrow{\psi} & \bar{E} \\ \downarrow \pi_1 & & \downarrow \bar{\rho} \\ C(X) & \xrightarrow{\rho} & S(X) \end{array}$$

where ρ is the natural identification and

$$\psi(t \wedge x, y) = \tilde{f}(y, \sigma(t, x))(1).$$

Then ψ is a homotopy equivalence on each fiber and the restriction of ψ to the fiber over x_0 is the identity. It follows that

$$\begin{array}{ccc} C(X) \times F & \xrightarrow{1 \times f'} & C(X) \times F \\ \downarrow \psi & & \downarrow \psi \\ \tilde{E} & \xrightarrow{f} & \tilde{E} \end{array}$$

is fiber homotopy commutative. Therefore, by (5.3), (5.4) and (5.6)

$\tau(\tilde{f})\rho = \psi\tau(1 \times f') = \psi(1 \wedge \tau(f')): C(X)/X \rightarrow \tilde{E}/F$, where $\tau(f'): S^0 \rightarrow F^+$ is associated with the trivial triangle

$$\begin{array}{ccc} F & \xrightarrow{f'} & F \\ & \searrow & \swarrow \\ & * & \end{array}$$

Let $i_0: S^0 \rightarrow F^+$ by $i_0(+1) = e_0$ and $i_0(-1) = +$. Since F is connected it is clear from the behaviour of the homomorphism in singular homology induced by $\tau(f')$ that $\tau(f') = \Lambda i_0$. Note that

$$(8.3) \quad \begin{array}{ccc} C(X)/X \wedge F^+ & \xrightarrow{\psi} & \tilde{E}/F \\ \uparrow 1 \wedge i_0 = i_0 & & \uparrow \Delta \\ C(X)/X & \xrightarrow{\rho} & S(X) \end{array}$$

is commutative. Therefore

$$\tau(\tilde{f})\rho = \psi(1 \wedge \tau(f')) = \psi(1 \wedge \Lambda i_0) = \Lambda \Delta \rho.$$

Since $\rho: C(X)/X \rightarrow S(X)$ is a homeomorphism, $\tau(\tilde{f}) = \Lambda \Delta$ and the proof is complete.

Now let

$$(8.4) \quad \begin{array}{ccc} E & \xrightarrow{f} & E \\ & \searrow \rho & \swarrow \rho \\ & B & \end{array}$$

be given where the fiber F of $p: E \rightarrow B$ is a finite complex and B is a complex (not necessarily finite dimensional). Choose base points $b_0 \in B$ and $e_0 \in F = p^{-1}(b_0)$ and let $\omega: \Omega B \rightarrow F$ denote the boundary map arising from the fibration $p: E \rightarrow B$.

If X is a finite dimensional complex with base point and $g: X \rightarrow \Omega B$ is a base point preserving map let

$$(8.5) \quad \begin{array}{ccc} \tilde{E} & \xrightarrow{f} & \tilde{E} \\ & \searrow \tilde{\rho} & \swarrow \tilde{\rho} \\ & S(X) & \end{array}$$

be induced by $S(X) \xrightarrow{S(g)} S(\Omega B) \xrightarrow{\epsilon} B$ where ϵ is the adjoint of the identity map. We will show now that

$$(8.6) \quad \begin{array}{ccc} \tilde{E}/F & \xrightarrow{k} & S(F) \\ \uparrow \Delta & & \uparrow S(\omega) \\ S(X) & \xrightarrow{S(g)} & S(\Omega B) \end{array}$$

is commutative, where Δ is as in (8.1) and k is from the Puppe sequence of the cofibration $F \rightarrow \tilde{E}$.

We have

$$(8.7) \quad \begin{array}{ccccc} \tilde{E}/F & \xrightarrow{k} & S(F) & & \\ \uparrow \psi & & \uparrow S(\psi) & \swarrow S(\omega) & \\ C(X) \times F/X \times F & \xrightarrow{k} & S(X \times F) & & S(\Omega B) \\ \uparrow j_0 & & \uparrow S(j_0) & \swarrow S(g) & \\ C(X)/X & \xrightarrow{k=\rho} & S(X) & & \end{array}$$

where k in each case is from the appropriate Puppe sequence and j_0 is the inclusion $y \rightarrow (y, e_0)$. The commutativity of the right hand triangle is by direct calculation. The commutativity of (8.6) now follows from the commutativity of (8.3) and (8.7).

We are now in a position to prove the theorem of the introduction.

(1.1) THEOREM: Assume that F is connected. Then

$$\Lambda\omega_* : \{X; \Omega B\}_q \rightarrow \{X; F\}_q$$

is trivial for any finite dimensional complex X .

PROOF: ΩB has the homotopy type of a CW-complex Y . If $\phi : Y \rightarrow \Omega B$ is a homotopy equivalence it is sufficient, to prove the theorem, to show that $\Lambda\omega_*\{g\} = 0$ if X is a finite dimensional subcomplex of Y and $g : X \rightarrow \Omega B$ is the inclusion followed by ϕ . By the commutativity of (8.6) and Lemma (8.2)

$$\Lambda\omega_*\{g\} = \Lambda\{k\Delta\} = \{k\tau(f)\}.$$

We have a commutative diagram

$$\begin{array}{ccccc} & & & \bar{E} & \\ & & & \downarrow i & \\ \bar{E}^+ & \xrightarrow{c'} & \bar{E}/F & \xrightarrow{k} & S(F) \\ \uparrow \tau(f) & & \uparrow \tau(f) & & \\ S(X)^+ & \xrightarrow{c} & S(X) & & \end{array}$$

where $c, c', c'',$ and j are quotient maps. Since $\{kj\} = 0$ we have $\{k\tau(f)c\} = c^*\{k\tau(f)\} = 0$. Since c^* is monomorphic, $\{k\tau(f)\} = 0$ and the proof is complete.

REMARKS: (1) The map ω frequently appears in other forms, hence theorem (1.1) applies for (a) coset maps $\rho : G \rightarrow G/H$, or more generally for (b) maps $\omega : M \rightarrow X$ which factor through the evaluation map $\mathcal{H}(X) \rightarrow X$ where $\mathcal{H}(X)$ is the space of homotopy equivalences of X , or for (c) fibre inclusions of principal bundles. Theorem (1.1) states that $\Lambda\omega_* = 0$ and $\Lambda\omega^* = 0$ for all homology and cohomology theories on the category of finite dimensional complexes. This is an extension of two results of [7], wherein theorem (1.1) was proved only for singular cohomology and for homotopy groups in the stable range. See also [5].

(8.8) COROLLARY: Let $\alpha \in \pi_i(S^{2n})$. Then $[\alpha, \iota_{2n}] = 0$ implies that $2\{\alpha\} = 0$, where $\{\alpha\}$ denotes the stable homotopy element represented by

α and $[\alpha, \iota_{2n}]$ is the Whitehead product of α with the generator of $\pi_{2n}(S^{2n})$.

PROOF: The fact that $[\alpha, \iota_{2n}] = 0$ implies there is a map $F : S^i \times S^{2n} \rightarrow S^{2n}$ such that F restricted to $* \times S^{2n}$ is the identity and F restricted to $S^i \times *$ represents α . Taking adjoints, we see that α factors through $\omega : M \rightarrow S^{2n}$ where M is the space of degree one maps on S^{2n} , and ω is the evaluation map given by evaluation at the base point. Thus (1.1) may be applied to α in view of the remark. In this case $\Lambda = \chi(S^{2n}) = 2$.

Let G be a compact connected Lie group, H a closed subgroup of G , and $\rho : G \rightarrow G/H$ the projection.

(8.9) COROLLARY: As an S -map $\chi(G/H)\rho : G \rightarrow G/H$ is trivial, where $\chi(G/H)$ is the Euler characteristic of G/H .

In particular, if N is the normalizer of a maximal torus in G then $\rho : G \rightarrow G/N$ is stably trivial since $\chi(G/N) = 1$ [6, 12]. On the other hand, it is interesting to note that $\rho_* : \pi_k(G) \rightarrow \pi_k(G/N)$ is an isomorphism for $k > 2$.

9. Smooth fiber bundles

In the case of a smooth fiber bundle $p : E \rightarrow B$ a more geometric description can be given for the transfer associated with a fiber preserving map. We assume that B and F are closed manifolds.

Let $\tilde{p} : E \rightarrow B \times R^s$ be a fiber preserving embedding. Its normal bundle β is inverse to the bundle α of tangents along the fiber and we have an isomorphism $\alpha \oplus \beta \cong R^s$ associated with the embedding. Let

$$c : B^+ \wedge S^s \rightarrow E$$

denote the Pontryagin-Thom map of this trivialization.

The diagonal inclusion into the fiber square, $d : E \rightarrow E^2$, has normal bundle α so that we have

$$c' : (E^2)^{\pi_1^*(\beta)} \longrightarrow E^{\alpha \oplus \beta} = E^+ \wedge S^s$$

where $\pi_1 : E^2 \rightarrow E$ is projection onto the first factor.

If $f: E \rightarrow E$ is a fiber preserving map let

$$(\tilde{1}, \tilde{f}): E^\beta \rightarrow (E^2)^{\pi_1^*(\beta)}$$

send v_e to $(e, f(e), v_e)$.

(9.1) PROPOSITION: $\tau(f): B^+ \rightarrow E^+$ is represented by

$$B^+ \wedge S^s \xrightarrow{c} E^\beta \xrightarrow{(\tilde{1}, \tilde{f})} (E^2)^{\pi_1^*(\beta)} \xrightarrow{c'} E.$$

First observe that the S -map determined by $c'(\tilde{1}, \tilde{f})c$ is independent of the choice of embedding \tilde{p} and of the tabular neighborhood maps used in constructing c and c' .

Now, for $\tilde{p}: E \rightarrow B \times R^s$, let β denote the normal bundle, let $S(\beta)$ denote the total space of the unit sphere bundle, and let \hat{E} denote the quotient of $D(\beta)$ obtained by identifying each fiber of $S(\beta)$ to a point. We regard \hat{E} as an ex-space of B by $\hat{p}: \hat{E} \rightarrow B$ and $\hat{\Delta}: B \rightarrow \hat{E}$ where $\hat{p}([v_e]) = p(e)$ and $\hat{\Delta}(p(e)) = [v_e]$, where $v_e \in S(\beta)$. Then \hat{p} is the projection of a fiber bundle whose fiber over b is the Thom space $F_b^{\nu_b}$, where ν_b is the normal bundle of the embedding $F_b \rightarrow \{b\} \times R^s$.

Choose a fiber preserving tubular neighborhood $D(\beta) \subset B \times R^s$ and let $\theta: B \times S^s \rightarrow \hat{E}$ denote the associated Pontryagin-Thom map. Let

$$(9.2) \quad \mu: B \times S^s \rightarrow \bar{E} \wedge_B \hat{E}$$

be the composite $B \times S^s \xrightarrow{\theta} \hat{E} \xrightarrow{d} \bar{E} \wedge_B \hat{E}$, where $d(v_e) = e \wedge v_e$. By Atiyah's duality theorem [1] μ is a duality map.

The diagonal embedding $E \rightarrow E \times_B D(\beta)$ has normal bundle $\alpha \oplus \beta = E \times R^s$. Choosing a fiber preserving tubular neighborhood $E \times D^s \subset E \times_B D(\beta)$, we obtain

$$(9.3) \quad \theta': \bar{E} \wedge_B \hat{E} \rightarrow \bar{E} \wedge_B S^s.$$

Let $\theta'': \bar{E} \wedge_B \hat{E} \rightarrow B \times S^s$ denote θ' followed by the projection $\bar{E} \wedge_B S^s \rightarrow B \times S^s$.

(9.4) LEMMA: θ'' is dual to μ .

PROOF: We must show that

$$\begin{array}{ccc} B \times S^s \wedge S^s & \xrightarrow{\nu} & (\bar{E} \wedge_B \hat{E}) \wedge_B (\bar{E} \wedge_B \hat{E}) \\ & \searrow \mu \wedge 1 & \downarrow 1 \wedge \theta'' \\ & & (\bar{E} \wedge_B \hat{E}) \wedge S^s \end{array}$$

is homotopy commutative. Since isotopic embeddings determine homotopic duality maps, the duality map determined by

$$E \xrightarrow{(\tilde{p}, 0)} B \times R^s \times R^s$$

is homotopic to that determined by

$$E \xrightarrow{(\tilde{p}, \tilde{p})} B \times R^s \times R^s.$$

The former duality map is $\mu \wedge 1$ whereas, using the factorisation

$$E \xrightarrow{d} E^2 \xrightarrow{\tilde{p} \times \tilde{p}} B \times R^s \times R^s,$$

the latter is easily seen to be homotopic to $(1 \wedge \theta'')\nu$.

Proposition (9.1) is now a consequence of the following commutative diagram.

$$\begin{array}{ccccccc} B \times S^s & \xrightarrow{\mu} & \bar{E} \wedge_B \hat{E} & \xrightarrow{(\tilde{1}, \tilde{f}) \wedge 1} & \bar{E} \wedge_B \bar{E} \wedge_B \hat{E} & \xrightarrow{1 \wedge \hat{\mu}} & \bar{E} \wedge S^s \\ & \searrow \theta & \uparrow d & & \uparrow d' & \nearrow \theta' & \downarrow \\ & & \hat{E}^\beta & \xrightarrow{g} & \bar{E} \wedge \hat{E} & & \\ & & \downarrow & & \downarrow & & \\ B^+ \wedge S^s & \xrightarrow{c} & E^\beta & \xrightarrow{(\tilde{1}, \tilde{f})} & (E^2)^{\pi_1^*(\beta)} & \xrightarrow{c'} & E^+ \wedge S^s \end{array}$$

Here $d'(e' \wedge v_e) = e \wedge e' \wedge v_e$, $g(v_e) = f(e) \wedge v_e$, and $h(e' \wedge v_e) = (e, e', v_e)$. The unlabeled arrows denote the natural identification map. The commutativity of the upper right hand triangle follows from the fact that $\hat{\mu} = \theta''$.

REMARK It follows from Proposition (9.1) and the retraction property (7.1) that the two methods of constructing the transfer which are outlined in [3], do in fact lead to the same map.

10. Proof of (3.7)

Suppose that $E = (E, B, p, \Delta)$ and $E' = (E', B', p', \Delta')$ are ex-spaces. If $h: E \rightarrow E'$ and $f: B \rightarrow B'$ are such that $p'h = fp$ and $h\Delta = \Delta'f$ we will say that h is a map over f . If E' is an ex-fibration then we obtain from the special nature of the lifting function for E' the following covering homotopy property.

(10.1) Given $F: B \times I \rightarrow B'$ and a map $h: E \rightarrow E'$ over F_0 there is $H: E \times I \rightarrow E'$ such that $H_0 = h$ and H_t is a map over F_t , $0 \leq t \leq 1$.

Suppose that E is an ex-space of B and $A \subset B$ is a subcomplex. As before, let $c: E \cup C(E_A) \rightarrow E/E_A$ denote the natural map. Let $\lambda: B \times I \rightarrow B \times \{1\} \cup A \times I$ be a retraction map and let $F: B \times I \rightarrow B$ denote λ followed by projection onto B .

(10.2) LEMMA: Suppose that E is an ex-fibration. There is $q: E/E_A \rightarrow E \cup C(E_A)$ over F_0 and homotopies $H: E/E_A \times I \rightarrow E/E_A$ and $K: E \cup C(E_A) \times I \rightarrow E \cup C(E_A)$ such that

- (a) $H_0 = cq$, $H_1 = 1$, and H_t is over F_t , $0 \leq t \leq 1$.
- (b) $K_0 = qc$, $K_1 = 1$, and K_t is over F_t , $0 \leq t \leq 1$.

PROOF: Consider

$$\begin{array}{ccc} E \times \{1\} & \xrightarrow{i_1} & E \times 1 \\ \downarrow p \times 1 & & \downarrow p \times 1 \\ B \times I & \xrightarrow{\lambda} & B \times I. \end{array}$$

Applying (10.1) for the ex-fibration $E \times I$ there is $M: E \times I \rightarrow E \times I$ such that $M_1 = i_1$ and M_t is over λ_t , $0 \leq t \leq 1$. Then we actually have

$$M: E \times I \rightarrow E \times \{1\} \cup E_A \times I.$$

Let q' denote

$$E \xrightarrow{M_0} E \times \{1\} \cup E_A \times I \longrightarrow E \cup C(E_A).$$

Since $M_0(E_A) \subset E_A \times \{0\}$ we have $q'(E_A) \subset \Delta(B)$. Let $q: E/E_A \rightarrow E \cup C(E_A)$ denote the collapse of q' . Then q is a map over F_0 .

To construct H , the map

$$E \times I \xrightarrow{M} E \times \{1\} \cup E_A \times I \longrightarrow E \cup C(E_A) \xrightarrow{c} E/E_A$$

has a quotient $H: E/E_A \times I \rightarrow E/E_A$ which is the desired map.

To construct K let $N: E \times I \rightarrow E \cup C(E_A)$ denote the composite

$$E \times I \xrightarrow{M} E \times \{1\} \cup E_A \times I \longrightarrow E \cup C(E_A).$$

Define $K: E \cup C(E_A) \times I \rightarrow E \cup C(E_A)$ by $K(e, t) = N(e, t)$, if $e \in E$, and $K([e, s], t) = s * N(e, t)$, if $e \in E_A$, where $s * [e', t] = [e', st]$ for $[e', t] \in C(E_A)$. This completes the proof.

We will now prove (3.7) which asserts that

$$c^*: [E/E_A; E'] \rightarrow [E \cup C(E_A); E']$$

is bijective provided E and E' are ex-fibrations.

To show that c^* is onto let $\theta: E \cup C(E_A) \rightarrow E'$ be an ex-map and consider

$$\begin{array}{ccc} E/E_A \times \{0\} & \xrightarrow{\theta q} & E' \\ \downarrow p \times 1 & & \downarrow p' \\ B \times I & \xrightarrow{F} & B. \end{array}$$

There is $N: E/E_A \times I \rightarrow E'$ over F such that $N_0 = \theta q$. Let $\psi = N_1: E/E_A \rightarrow E'$. Then ψ is an ex-map and we will show that $c^*([\psi]) = [\theta]$. We have homotopies

$$E \cup C(E_A) \times I \xrightarrow{c \times 1} E/E_A \times I \xrightarrow{N} E'$$

over F , and

$$E \cup C(E_A) \times I \xrightarrow{K} E \cup C(E_A) \xrightarrow{\theta} E'$$

also over F . Then

$$(\theta K^{-1}) \circ (N(c \times 1)): E \cup C(E_A) \times I \rightarrow E'$$

is a homotopy from θ to ψc over $F^{-1} \circ F$. By a standard argument involving the covering homotopy property (10.1) we see that θ is ex-homotopic to ψc .

To show that c^* is one-one let $\psi_0, \psi_1: E/E_A \rightarrow E'$ be such that $\psi_0 c$ is ex-homotopic to $\psi_1 c$ by $p: E \cup C(E_A) \times I \rightarrow E'$ say. We have

$$\begin{array}{ccc} E/E_A \times (I \times \{0\} \cup I \times I) & \xrightarrow{Q} & E' \\ \downarrow p \times 1 \times 1 & & \downarrow p' \\ B \times I \times I & \xrightarrow{\tilde{F}} & B \end{array}$$

where $\tilde{F}(b, t, \lambda) = F(b, \lambda)$ and $Q(e, t, 0) = p(q(e), t)$, $Q(e, 0, \lambda) = \psi_0 H(e, \lambda)$, and $Q(e, 1, \lambda) = \psi_1 H(e, \lambda)$. There is then $R: E/E_A \times I \times I \rightarrow E'$ over \tilde{F} which extends Q . Then $S: E/E_A \times I \rightarrow E'$ by $S(e, t) = R(e, t, 1)$ is an ex-homotopy from ψ_0 to ψ_1 .

11. Proof of (4.7)

Let $E = (E, B, p, \Delta)$ be an ex-fibration in the sense of section 4. Let $\Sigma(E)$ denote the unreduced fiberwise suspension of E and $\Sigma(p): \Sigma(E) \rightarrow B$ the projection. There is the "south pole" cross section $\delta: B \rightarrow \Sigma(E)$ given by $\delta(b) = [e, 0]$ where $p(e) = b$ and we let $\Sigma_0(E) = (\Sigma(E), B, \Sigma(p), \delta)$. It is easy to check that $\Sigma_0(E)$ is an ex-fibration as in section 4. The quotient map $\pi: \Sigma_0(E) \rightarrow S^1 \wedge E$ is an ex-map and is a homotopy equivalence on each fiber. Hence by the comparison theorem (3.2), π is an ex-homotopy equivalence. Note that if $p: E \rightarrow B$ is a fibration with fiber a finite complex (not necessarily equipped with a cross section) we may still form $\Sigma_0(E)$ which is an ex-fibration.

Now let E be an ex-fibration. To construct an ex-fibration \hat{E} and a duality map

$$\mu: S^s \times B \rightarrow E \wedge \hat{E}$$

we proceed by induction over the skeleta of B . Let B have dimension n and let A denote the $(n-1)$ -skeleton of B . Assume there is an ex-fibration $D(E|A)$, with fiber \hat{F} say, and a duality map

$$(11.1) \quad \omega: S^s \times A \rightarrow E|A \wedge D(E|A).$$

Let B be obtained from A by adjoining cells via $\lambda_j: S^{n-1} \rightarrow A, j \in J$, and let $\bar{\lambda}_j: D^n \rightarrow B$ denote the characteristic map. Let $\bar{\psi}_j: F \times D^m \rightarrow \lambda_j^*(E)$ be an ex-fiber homotopy equivalence and $\psi_j: F \times S^{n-1} \rightarrow \lambda_j^*(E)$ its restriction.

We have a duality map

$$(11.2) \quad \eta_j: S^s \times S^{n-1} \rightarrow \lambda_j^*(E) \wedge \lambda_j^*(D(E|A))$$

induced by ω . Choose a duality map

$$(11.3) \quad \nu: S^s \times S^{n-1} \rightarrow (F \times S^{n-1}) \wedge (\hat{F} \times S^{n-1})$$

and let $\phi_j: \lambda_j^*(D(E|A)) \rightarrow \hat{F} \times S^{n-1}$ be dual to ψ_j relative to η_j and ν . Then ϕ_j is an ex-homotopy equivalence and we have a homotopy commutative triangle.

$$(11.4) \quad \begin{array}{ccc} & & \lambda_j^*(E) \wedge \lambda_j^*(D(E|A)) \\ & \nearrow \eta_j & \uparrow \phi_j \wedge \phi_j^{-1} \\ S^s \times S^{n-1} & & \\ & \searrow \nu & (F \times S^{n-1}) \wedge (\hat{F} \times S^{n-1}) \end{array}$$

Let $\theta_j: \hat{F} \times S^{n-1} \rightarrow D(E|A)$ denote the composite

$$\hat{F} \times S^{n-1} \xrightarrow{\phi_j^{-1}} \lambda_j^*(D(E|A)) \xrightarrow{\bar{\lambda}_j} D(E|A),$$

and let X' be obtained by adjoining, for each $j \in J, \hat{F} \times D^n$ to $D(E|A)$ via θ_j . We have an ex-space (X', B, p', Δ') where p' and Δ' are the obvious maps. Moreover, by results of Dold and Thom [10 (2.2) and (2.10)], $p': X' \rightarrow B$ is a quasifibration. We replace p' by a fibration in the usual way obtaining a commutative square

$$(11.5) \quad \begin{array}{ccc} X' & \xrightarrow{\alpha} & X \\ \downarrow p' & \uparrow \Delta' & \downarrow p \\ B & \xrightarrow{\quad} & B \end{array}$$

where $X = \{(x, \sigma) \in X' \times B^1 | p'(x) = \sigma(0)\}$, $p(x, \sigma) = \sigma(1)$, $\Delta(b) = (\Delta'(b), b^*)$ where b^* denotes the constant path at b , and $\alpha(x) = (x, p'(x)^*)$.

Now $p: X \rightarrow B$ is a fibration with each fiber $p^{-1}(b)$ of the weak homotopy type of \hat{F} . Since X (being homotopy equivalent to X') has the homotopy type of a CW-complex, it follows from [16 Proposition

0] that $p^{-1}(b)$ has the homotopy type of a CW-complex, hence is homotopy equivalent to \hat{F} .

Let $\hat{E} = \Sigma_0(X)$. Then \hat{E} is an ex-fibration as in section 4. We will show now that there is a duality map

$$\mu : S^{s+1} \times B \rightarrow E \wedge \hat{E}.$$

From (11.4) we obtain the following homotopy commutative diagram.

$$(11.6) \quad \begin{array}{ccc} S^1 \wedge \lambda_j^*(E) \wedge \lambda_j^*(D(E|A)) \rightarrow E \wedge S^1 \wedge D(E|A) & \xrightarrow{1 \wedge \pi^{-1}} & E \wedge \Sigma_0(D(E|A)) \\ \uparrow 1 \wedge \eta_j & & \downarrow \tilde{\lambda}_j \psi_j \wedge \Sigma(\alpha_j) \\ S^{s+1} \times S^{n-1} & \xrightarrow{\tilde{\lambda}_j \psi_j \wedge 1 \wedge \theta_j} & \\ \downarrow 1 \wedge \nu & & \\ S^1 \wedge (F \times S^{n-1}) \wedge (\hat{F} \times S^{n-1}) \rightarrow (F \times S^{n-1}) \wedge ((S^1 \wedge \hat{F}) \times S^{n-1}) & & \\ & \xrightarrow{1 \wedge \pi^{-1}} & (F \times S^{n-1}) \wedge (\Sigma_0(\hat{F}) \times S^{n-1}) \end{array}$$

Let ω' denote the duality map

$$\begin{aligned} S^{s+1} \times A &\xrightarrow{1 \wedge \omega} S^1 \wedge E|A \wedge D(E|A) \rightarrow E|A \wedge S^1 \wedge D(E|A) \\ &\xrightarrow{1 \wedge \pi^{-1}} E|A \wedge \Sigma_0(D(E|A)) \xrightarrow{1 \wedge \Sigma(\alpha)} E|A \wedge \hat{E}|A, \end{aligned}$$

and let ν' denote the duality map

$$\begin{aligned} S^{s+1} \times S^{n-1} &\xrightarrow{1 \wedge \nu} S^1 \wedge (F \times S^{n-1}) \wedge (\hat{F} \times S^{n-1}) \\ &\rightarrow (F \times S^{n-1}) \wedge ((S^1 \wedge \hat{F}) \times S^{n-1}) \xrightarrow{1 \wedge \pi^{-1}} (F \times S^{n-1}) \wedge (\Sigma_0(\hat{F}) \times S^{n-1}) \end{aligned}$$

We have from (11.6) a homotopy commutative diagram

$$\begin{array}{ccc} & A \times S^{s+1} & \\ \lambda_j \times 1 \nearrow & & \searrow \omega' \\ S^{s+1} \times S^{n-1} & & E \wedge \hat{E} \\ \downarrow \nu' & & \uparrow \tilde{\lambda}_j \psi_j \wedge \Sigma(\alpha_j) \\ & (F \times S^{n-1}) \wedge (\Sigma_0(\hat{F}) \times S^{n-1}) & \end{array}$$

It follows now that $\omega'(\lambda_j \times 1)$ has an extension over $S^{s+1} \times D^n$, for each $j \in J$, and therefore ω' has an extension $\mu : S^{s+1} \times B \rightarrow E \wedge \hat{E}$, which is clearly a duality map.

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