A History of Duality in Algebraic Topology

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§1. Introduction.

Duality in the general course of human affairs seems to be a juxtaposition of complementary or opposite concepts. This frequently leads to poetical sounding uses of language, both in the common language and in the precision of mathematical theorems. Thus the duality of Projective Geometry: Two points determine a line; two lines determine a point. Gergonne first introduced the word duality in mathematics in 1826. He defined it for Projective Geometry. By the time of Poincaré's note in the Comptes Rendus of 1893, duality was very much in vogue.

There are many dualities in algebraic topology. An informal survey of some topologists has revealed the following names of duality in current use. There are Poincaré, Alexander, Lefschetz, Pontrjagin, Spanier–Whitehead, Hodge, Vogell, Ranici, Whitney, Serre, Eckmann–Hilton, Atiyah, Brown–Comenetz: These are a few whose names reflect those of their discoverers. There are the dual categories, the duality between homology and cohomology and that between homotopy and cohomotopy. There is a duality between cup products and cap products, and between suspension and looping.

The remarkable things are: First, a great many of these seemingly separated dualities are intimately related; and Second, those workers who tried to extend or generalize various of these dualities were led to invent widely important notions such as infinite simplicial complexes or spectra; or they discovered remarkable new relationships among important classical concepts. Such a story demands a point of view. Fortunately, one has been provided by A. Dold and D. Puppe with their concept of *strong duality.* So we will jump forward to 1980 and explain their work in §2. Then the remainder of this work will describe in chronographical order the theorems and discoveries which led from Poincaré's first mention of Poincaré duality up to 1980 and the unifying concept of strong duality.

The main subject of this work is the history of strong duality from 1893 to 1980. First we describe strong duality, and then we proceed chronologically to discuss the evolution of strong duality.

In Section §2, we cover very briefly Eckmann–Hilton duality and then describe Dold and Puppe's approach, which illuminates the main subject of our paper, the development and unification of Poincaré, Alexander, Lefschetz, Spanier–Whitehead, homology–cohomology duality. This material is not as widely known as it should be, so we must give a somewhat technical sketch; but one indicator of the success of their point of view is that we now have a name for all those interelated dualities: namely, strong duality.

In §3 we take up the early days up to 1952. We give a modern statement of the Poincaré–Alexander–Lefschetz duality theorem; and then describe the origins of all the elements of the theorem. Our main sources are J.P. Dieudonné (1989), and William Massey's article in this volume.

Section §4 deals with Spanier–Whitehead duality; §5 with Atiyah duality, which clarifies S–duality for manifolds; and §6 outlines the story of how Poincaré–Alexander duality was extended to generalized cohomology and homology theories by means of S–duality.

In §7 we take up Umkehr maps. Here the Eckmann–Hilton type duality of reversing the direction of maps interfaces with the strong dualities arising from compactness. These Umkehr homomorphisms played a decisive role in the generalization of the Riemann–Roch theorem and the invention of K–theory. A special class of Umkehr maps, the Transfers of §8, apparently at first involving only reversing direction in the Eckmann–Hilton manner (and actually first discovered by Eckmann), enjoyed increasing generalization and unifications by means of various duality concepts until at last they inspired Dold and Puppe's categorical picture of strong duality.

With Dold and Puppe we must break off our narrative, as we are too close to recent times. We deeply regret we had neither the time, energy, knowledge, or space to do justice to the many results inspired by or reflecting upon duality. We especially regret the omissions of Eckmann–Hilton duality and Poincaré Duality spaces and surgery. See Hilton

(1980), Wall (1967) respectively.

§2. Categorical Points of View.

There are two major groupings of dualities in algebraic topology: *Strong duality* and *Eckmann–Hilton* duality. Strong duality was first employed by Poincaré (1893) in a note in which "Poincaré duality" was used without proof or formal statement. The various instances of strong duality (Poincaré, Lefschetz, Alexander, Spanier–Whitehead, Pontrjagin, cohomology–homology), seemingly quite different at first, are intimately related in a categorical way which was finally made clear only in 1980. Strong duality depends on finiteness and compactness. On the other hand, Eckmann–Hilton duality is a loose collection of useful dualities which arose from categorical points of view first put forward by Beno Eckmann and P.J. Hilton in Eckmann (1956).

Eckmann–Hilton duality was first announced in lectures by Beno Eckmann and also by Hilton, see Eckmann (1956), (1958). A very good description of how this duality works, and some eyewitness history is given in Hilton (1980). Instead of being a collection of theorems, Eckmann–Hilton duality is a principle for discovering interesting concepts, theorems, and questions. It is based on the dual category, that is, on the duality between the target and source of a morphism; and also on the duality between functors and their adjoints.

In fact it is a method wherein interesting definitions or theorems are given a description in terms of a diagram of maps, or in terms of functors. Then there is a dual way to express the diagram, or perhaps several different dual ways. These lead to new definitions or conjectures. Some, not all, of these definitions turn out to be very fruitful and some of the conjectures turn out to be important theorems.

A very important example of this duality is the notion of cofibration with its cofiber, C_f, written

$$A \xrightarrow{f} B \xrightarrow{j} C_f$$

which is dual to to the notion of fibration with its fibre F, written

$$F \xrightarrow{i} E \xrightarrow{f} B.$$

The word cofibration, and thus the point of view, probably first appeared in Eckmann (1958). (But see Hilton (1980), page 163 for a rival candidate.) Puppe (1958), obtained a sequence of cofibrations in which the third space is the cofiber of the previous map

$$A \xrightarrow{f} B \xrightarrow{j} C_f \longrightarrow SA \xrightarrow{Sf} SB \to SC_f \to \dots$$

This is known as the Puppe sequence.

The dual sequence for fibrations, in which every third space is the fiber of the previous fibration,

$$\dots \xrightarrow{\Omega i} \Omega E \xrightarrow{\Omega f} \Omega B \xrightarrow{\omega} F \xrightarrow{i} E \xrightarrow{f} B$$

is also frequently called the Puppe sequence, although it was first published by Yasutoshi Nomura (1960).

Eckmann–Hilton duality was conceived as a method based on a categorical point of view in the early 1950's. The challenge was to use the point of view to generate interesting results.

The interelated dualities of Poincaré, Alexander, Spanier–Whitehead, homology—cohomology, developed haphazardly from 1893 into the 1960's. They formed a collection of quite interesting results which were somehow intimately related. The challenge was to find a coherent categorical description of this phenomenon. It took a long time coming, perhaps because the challenge was not articulated. Then in a paper published in an obscure place with sketchy proofs, Albrecht Dold and Dieter Puppe (1980) laid down a framework for viewing duality. Even now, 17 years later, the name *strong duality* given to this type of duality, is not as current as it should be.

Dold and Puppe conceived their categorical description of strong duality by organizing a seminar to study a much more concrete problem: How exactly are the Becker–Gottlieb transfer and the Dold fixed point transfer related to each other? Using notation which invokes the duality of a number and its reciprocal, they define duality in a symmetric monoidal category $\mathcal{C}$ with multiplication $\otimes$ and a neutral object I. One has, therefore, natural equivalences

$$A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$$

$$I \otimes A \cong A \cong A \otimes I$$

$$\gamma \colon A \otimes B \cong B \otimes A.$$

(Here $\gamma \colon A \otimes B \to B \otimes A$ is a natural equivalence, whose choice is really part of the definition of the monoidal category $\mathcal{C}$. The explicit noting of γ is an important novel feature of Dold and Puppe's point of view).

An object DA is the *weak dual* of A if there is a natural bijection with respect to objects X of the sets of morphisms

$$\mathcal{C}(X \otimes A, I) \cong \mathcal{C}(X, DA).$$

We get a definite morphism $\varepsilon \colon DA \otimes A \to I$ called the *evaluation*. Now from this another morphism $\delta \colon A \to DDA$ can be constructed. If δ is an isomorphism (so the dual of the dual is the original object), then DA is called *reflexive*. If in addition $DA \otimes A$ is canonically self dual to itself, then DA is called a *strong dual* of A. Strong duals come equipped with *coevaluations* $\eta \colon I \to A \otimes DA$.

Now one important example is the category of R–modules for a commutative ring R. The unit I is R, and $DM = \mathrm{Hom}_R(M, R)$ is the weak dual. It is clear that every finitely generated projective module is strongly dualizable.

For stable homotopy, strong duality is Spanier–Whitehead duality, and for the stable category of spaces over a fixed base B, strong duals can be introduced in a fibrewise way.

Now the other half of Dold and Puppe's framework is to inquire how duality is preserved under functors. A *monoidal* functor $T \colon \mathcal{C} \to \mathcal{C}'$ between two monoidal categories is a functor together with transformations $TA \otimes TB \to T(A \otimes B)$. The dual D is an example of a contravariant monoidal functor. The homology functor from chain complexes to graded groups gives rise to a monoidal functor.

Unfortunately $T(A) \otimes T(B) \to T(A \otimes B)$ is not always an isomorphism, as the Künneth theorem shows. So we must say that A is *T–flat* if A is *strongly dualizable* and $T(A)$ is strongly dualizable to $T(DA)$. Subtleties arise here because sometimes we have to examine whether A is T–flat on an object-by-object basis.

Now the point of view given by these considerations is that strong duality takes place in various categories, and the duality theorems are expressed in terms of functors which carry strong duality from the objects of the source category to the objects of the target category. Thus the classical way to think about Poincaré duality, for example, is that it is an isomorphism between homology and cohomology of a closed manifold M, whereas the Dold–Puppe point of view is that a closed manifold and its Spanier–Whitehead dual are carried from the stable homotopy category by the homology functors to self dual graded rings, which means in this case the isomorphism between homology and cohomology comes from the Thom isomorphism and Atiyah duality.

Now strong duality seems to depend upon finiteness or compactness properties. In finite situations, one can define a notion of rank or trace. A particular triumph of the Dold–Puppe point of view is the notion of the *trace* of an endomorphism $f\colon A \to A$ of a strongly dualizable object.

$$\sigma f\colon I \xrightarrow{\eta} A \otimes DA \xrightarrow[\cong]{\gamma} DA \otimes A \xrightarrow{id \otimes f} DA \otimes A \xrightarrow{\varepsilon} I.$$

In the category of R–modules $\sigma f(1)$ is the usual *trace*. For chain complexes, $\sigma f(1)$ is the Lefschetz number, and for graded abelian R–modules, it is the Lefschetz number. Now if T is a monoidal functor and A is a T–flat object, then $T(\sigma f) = \sigma(Tf)$. For T the homology function, we get Hopf's result that the Lefschetz number of the chain complex is the Lefschetz number for homology. Some formal properties of trace are

$$\sigma(Df) = \sigma f \quad \text{(trace of } f = \text{ trace of } f \text{ transpose)}$$

$$\sigma(f \circ g) = \sigma(g \circ f) \quad \text{(trace preserved by commutation).}$$

Using these results, Dold and Puppe can prove the Lefschetz fixed point theorem.

§3. Poincaré–Alexander–Lefschetz duality.

In the text books of Spanier (1966), Dold (1972), Massey (1980), and Bredon (1993) we see, essentially, the final forms of the Poincaré–Alexander–Lefschetz duality theorem.

We choose the description of G. Bredon as our text, and then we shall detail the process by which the ideas necessary for the statement occurred, and the many important byproducts of the effort to understand duality.

Theorem *. *Let M^n be an n–manifold oriented by ϑ, and let $K \supset L$ be compact subsets of M. Then the cap product*

$$\cap\vartheta \colon \check{H}^p(K, L; G) \to H_{n-p}(M - L, M - K; G)$$

is an isomorphism, and it gives rise to an exact ladder relating the cohomology sequence of the pair (K, L) to the homology sequence of the pair $(M - L, M - K)$.

Henri Poincaré (1893) first mentions Poincaré duality in a note in Comptes Rendus. The goal was to "prove" that an odd dimensional closed oriented manifold has zero Euler–Poincaré number. Poincaré duality, expressed in terms of Betti numbers, is mentioned as if everyone should know it. The earlier note, Poincaré (1892), which is considered the first of Poincaré's papers on topology does not seem to mention duality. See Maja Bollinger (1972), Diedonné (1989), Henn and Puppe (1990).

In this note, Poincaré first mentions what we now call Poincaré duality. In the note, Poincaré "shows" that a closed odd dimensional oriented manifold has Euler-Poincaré number equal to zero. He says, "It is known" that the alternating sum of the Betti numbers is the same as the generalized Euler characteristic. On the other hand, he merely states the fact that the Betti numbers in complementary dimensions are equal, which is his version of Poincaré duality, with the same fanfare which one uses to state $1 + 1 = 2$, as if it were widely known. This proof is exactly what one would give today to a student knowledgeable about the Euler characteristic and Poincaré duality.

When Poincaré next refers to his duality, it is in his famous paper Analysis Situs, Poincaré (1895). There he states: "ce thèoréme n'a, je crois, jamais été énoncé; il était cependant connu de Plusieurs Personnes qui en ont même fait des applications", Bollinger (1972), see page 124.

Poincaré elected not to point out any of the several people who knew of it and have even made applications with it, including himself and his note. Poincaré (1893). So it is not surprising that the Editors of Poincaré's Oeuvres misplaced the note in Volume 11, instead of placing it in Volume 6 right before Analysis Situs; and that various scholars dealing with Poincaré duality may have, for this reason, omitted mentioning it in their works. According to Bollinger (1972), Poincaré concludes Analysis Situs with the very theorem he announced in his 1893 note: odd dimensional manifolds have zero Euler characteristics. In fact he gives two different proofs of it.

In Analysis Situs, Poincaré (1895), "Poincaré endeavored to prove his central theorem on homology, the famous duality theorem...", Dieudonné (1989), formulated in terms of Betti numbers. To do this, he invented the concept of intersection numbers, which were finally made precise in the work of Lefschetz (1926) and of De Rham (1931).

Now P. Heergard, in his dissertation in Danish in 1898, came up with a counterexample to Poincaré duality: A three dimensional manifold whose Betti numbers were $b_1 = 2$ and $b_2 = 1$!

When Poincaré examined Heergard's paper, he found that his notion of Betti numbers was not the same as that of Betti's, contrary to his belief. And worse, his "proof" worked just as well with this contradictory version. So, Poincaré came up with a new proof in two complements to Analysis Situs in 1899 and 1900, (see Poincaré (1895) pp. 290–390).

In this proof, Poincaré assumes his manifold is triangulated, finds an algorithm for calculating the Betti numbers, defines barycentric subdivision of the triangulation, shows that the Betti numbers do not change under subdivision, and defines the dual triangulation. He shows the dual triangulations have complementary Betti numbers, since the incidence matrices he uses to compute the Betti numbers are transposes of the incidence matrices in the complementary dimensions.

The incidence matrix algorithm also gave rise to torsion coefficients, which Poincaré introduced. Poincaré extended the duality theorem to the torsion coefficients. He did not use the fact that invariant factors of a transposed matrix are the same as those of the original matrix. Instead he showed the torsion coefficients were related by inventing the construction of the join.

Thus at 1900 we have theorem $*$ for the case $K = M$ and $L = \emptyset$ where M is a closed manifold (but with a finite triangulation structure attached), no homology groups and no cohomology. The concept of compactness was not made explicit until 1906 by Frechet. Frechet also introduced the abstract notion of metric space. In 1914, Hausdorff gave four axioms for neighborhoods, so topological spaces became a well–defined concept, and the standards of rigor became much higher in topology.

Another thread of duality begins with the duality of inside vs. outside. The intuitively obvious but difficult Jordan curve theorem, that a closed curve in a plane divides the plane into two parts was generalized by L.E.J. Brouwer (1911) (pages 489–494) for subsets of $\mathbb{R}^n$ homeomorphic to S^{n-1}.

J.W. Alexander (1922) attacked the Jordan–Brouwer separation theorem. He gave a new proof and generalized the result to what we call Alexander duality: $\dim \tilde{H}_p(X;\mathbb{Z}_2) = \dim \tilde{H}_{n-p-1}(S^n - X;\mathbb{Z}_2)$ where X is a subcomplex of S^n. In this work, Alexander introduced coefficient groups $\mathbb{Z}_2$, and considered the homology of non-finite complexes. This paper became the starting point of investigations of homology for more general spaces than merely finite complexes or open subsets of $\mathbb{R}^n$. On the other hand, it led Lefschetz (1926) to introduce the idea of relative homology, that is the homology of K mod L, which he wrote as $H_p(K, L)$ in his book, Lefschetz (1930). If K and $K - L$ are orientable combinatorial manifolds, Lefschetz proved that $b_p(K^*) = b_{n-p}(K, L)$ where K^* is the "complement" of L in K defined by dual cells which do not intersect L. Also he had the torsion coefficient relations, for indeed he showed the relevant incidence matrices were transposes of one another. By an argument which essentially is part of the exact sequence of the pair

(S^n, L), Lefschetz proved Alexander duality.

Two other events occurred during the 1920's which improved the duality theorem. The first event was the proof of the invariance of homology for different triangulations. J.W. Alexander (1915) and (1926) showed that homology was independent of the triangulation. But his first proof had difficulties, so a satisfactory proof must be dated around 1926. Thus Poincaré duality held for combinatorial manifolds, instead of merely manifolds with a specific triangulation. The question of which manifolds can be triangulated then becomes important in order to assess the domain of applicability of Poincaré duality. S. Cairns (1930) showed that C^1 manifolds were triangulable. Much later, it was discovered that there are some topological manifolds which cannot be triangulated.

The second event was the description of homology in terms of group theory. The homology group $H_*(X)$ was the cycles modulo the boundaries. This was factored into the free part, called the Betti group, and the torsion group. Thus $H_i \cong F_i \oplus T_i$, and Poincaré duality was expressed as $F_i \cong F_{n-i}$ and $T_i \cong T_{n-i-1}$. The Betti group persisted into the 1950's as a commonly used concept. We now know that the Betti groups were not really natural, that is the splitting of homology into free and torsion parts is not functorial, and the duality with cohomology demands homology groups instead of Betti and torsion groups. But in the absence of the concept of cohomology, there is no better way to express Poincaré duality than by means of isomorphisms of suitable Betti groups and torsion groups.

One can see this change clearly by comparing the two books Lefschetz (1930) and Lefschetz (1942). In "Topology," Lefschetz (1930), the absolute Poincaré duality theorem appears on page 140 stated solely in terms of Betti numbers and torsion coefficients. On page 203 of "Algebraic Topology," Lefschetz (1942), the theorem is headlined as the "Duality Theorem of Poincaré" and is stated in terms of Betti groups and torsion groups. Above it, on the same page is the untitled result that cohomology is isomorphic to homology in complimentary dimensions.

In the 1930's, seemingly different kinds of dualities were studied which led to cohomology. In the first section of his history of cohomology in this volume, William Massey discusses "The struggle to find more general and natural statements of the duality theorems of Poincaré and Alexander" (Massey (??)).

We will report very briefly on this story and, since Massey has told it so well, we will only mention the main points. We particularly want to note that Massey argues that Pontrjagin developed his duality between a discrete abelian group G and its compact group of continuous characters $\hat{G}$ to study Poincaré and Alexander duality, Pontrjagin (1934). Here Poincaré duality can be stated: $\tilde{H}_k(M;G)$ is Pontrjagin dual to $\tilde{H}_{n-k}(M;\hat{G})$. And Alexander duality can be stated: $\tilde{H}_k(X;G)$ is Pontrjagin dual to $\tilde{H}_{n-k-1}(S^n - X;\hat{G})$.

At the Moscow conference of 1935 both Kolmogoroff and Alexander announced the definition of cohomology, which they had discovered independently of one another. Both authors quickly published papers in which they both point out that for any finite complex K and any compact abelian group G, the homology group $H_r(K;G)$ and the cohomology group $H^r(K;\hat{G})$ are Pontrjagin dual to one another. Kolmogoroff explicitly notes the duality theorems in terms of cohomology being isomorphic to homology.

Alexander and Kolmogoroff suggested the possibility of a product structure in cohomology. E. Čech (1936) and H. Whitney (1937) and in (1938) provided the correct details. Čech made precise the cup product and proved its basic properties. He also defined the cap product. Using the cap product, Čech proves the Poincaré duality theorem for closed, oriented, combinatorial manifolds.

The fact that the cap product with the fundamental class of a manifold explicitly gives the duality isomorphism adds concreteness to Poincaré duality and has played a very important role in applications. For example, on the chain level, the cap product with the fundamental cycle gives a chain homotopy equivalence. This plays an important role in the study of Poincaré spaces. Or again, W.V.D. Hodge (1941) defined the Hodge duality between a p form α and an $n-p$ form $*\alpha$ on $\mathbb{R}^n$. In modern terms, de Rham cohomology, defined by the differential forms on a smooth oriented closed manifold M is isomorphic to real cohomology, and the Hodge dual $*$ can be regarded as the Poincaré duality isomorphism . Hodge duality shows up in the modern formulation of Maxwell's equations:

$$dF = 0, \qquad *d*F = j.$$

H. Whitney covered even more ground than Čech. He defined the induced homomorphism f^* of a map f and gave its relations with $\cup$ and $\cap$, introducing these symbols and the names cup and cap product.

The Čech cohomology which appears on the left in our Theorem * was first defined by Steenrod (1936) in his thesis. It was, of course, the dual of Čech homology, and it was defined only for compact spaces. Dowker (1937) published a brief announcement for Čech cohomology defined for arbitrary spaces. Meanwhile, Alexander–Spanier homology was being developed by Alexander and Kolmogoroff and was modified and perfected by Spanier (1948). Then Hurewicz, Dugundji and Dowker (1948) showed that Čech cohomology and Alexander–Spanier cohomology were equivalent for a general class of spaces.

The history of singular homology theory actually begins way back in 1915 with Alexander (1915) in his first attempt to prove the topological invariance of topology. There were several attempts to implement the idea. But it was not until Eilenberg (1944) gave the correct definition that singular homology theory was satisfactorily defined.

Both singular homology and Čech–Alexander–Spanier cohomology are valid for pairs of topological spaces. So by 1948 we have the homology and cohomology groups used in Theorem $*$. Now the orientation class depends on the fact that $H_*(M, M-x) \simeq \tilde{H}_*(S^n)$. In 1947, Henri Cartan realized that Sheaf theory provided a mechanism to localize orientation. In the Séminaires Henri Cartan of 1950–1951, he defined the orientation sheet in the context of a generalized cochain complex of sheaves, and with this he could prove Poincaré and Alexander duality for C^0–manifolds [see Dieudonné (1989), p. 211].

The sheaf theory language of the proof was eliminated in mimeographed notes of Milnor in 1964. But there are versions of Poincaré and Alexander duality for homology and cohomology of sheaves, for local coefficients, for cohomology with compact supports; all of which play very important roles in various branches of mathematics besides topology.

Finally we discuss the book of Eilenberg–Steenrod, (1952), which does not mention Poincaré duality, and yet it sets in motion several ideas which improve the theorem. In Eilenberg–Steenrod (1945), the axioms for homology and cohomology theory were published. The proofs were deferred to the book. With the axioms, we see that for CW complexes at least, the particular versions of homology and cohomology do not really matter in the statement of Theorem $*$. Also, the axioms gave rise in the 1950's to the concept of generalized homology and cohomology, in which duality plays an even more important role.

It is clear that in their note of (1945), Eilenberg and Steenrod did not have all the details written down, because they stated that Čech homology satisfied the axioms. This is false and was rectified in their book. But the fact that Čech homology did not satisfy the axioms led to the devaluation of Čech homology, and one does not see it used today in algebraic topology. It is ironic that Čech's name is given to a cohomology theory he did not define, yet the fact that Čech first realized that the Poincaré duality isomorphism could be expressed by the cap product has all but been forgotten.

Eilenberg and Steenrod's book (1952) effected a revolution in mathematical notation. Perhaps not since Descartes' La géométrie has a book influenced how we write Mathematics. One knew they were looking at mathematics before 1600 because of the geometric diagrams with vertices and sides labeled by alphabetic letters. La géométrie in 1637 gave us nearly modern forms of equations, especially the notation of the exponent, i.e. a^3. The diagrams of Eilenberg–Steenrod not only made algebraic topology intelligible, but eventually swept out to other parts of mathematics, providing an efficient way to express complex, functorial relationships and giving us powerful methods of proofs by means of diagram chasing.

So, at last, we can talk about that quintessential diagram, the exact ladder, in the last part of Theorem $*$, its proof and utility depending upon the five–lemma of Eilenberg–Steenrod.

§4. Spanier–Whitehead duality.

In 1936, K. Borsuk (1936) showed that under certain conditions the set of homotopy classes of maps from a space X to a space Y, denoted by $[X;Y]$, could be given a natural abelian group structure. About a dozen years later E. Spanier (1949) returned to this idea and made a thorough investigation of these groups when Y is a sphere. He denoted $[X;S^n]$ by $\pi^n(X)$ and called it the n–th cohomotopy group of X. The group is defined when the dimension of X is less than $2n-1$. Spanier showed that the cohomotopy groups satisfied the Eilenberg–Steenrod axioms for cohomology to the extent that they could be formulated given that the groups are not defined for all n. He then went on to give a group-theoretic formulation of Hopf's classification of maps of an n-complex into S^n and Steenrod's classification of maps of an $(n+1)$–complex into S^n, noting in the introduction an apparent "duality" between Hopf's theorem for cohomotopy groups and Hurewicz's theorem for homotopy groups.

In order to bypass the difficulty that $[X;Y]$ is not always an abelian group, Spanier and J.H.C. Whitehead (1953) (1957) defined what they called the suspension category, giving birth to what is now called stable homotopy theory. They defined by means of suspension the S–group

$$\{X;Y\} = \lim[S^kX;S^kY].$$

They generalized Freudenthal's suspension theorem as follows: If Y is $(n-1)$–connected, the suspension map from $[X;Y]$ to $[SX;SY]$ is bijective if $\dim(X) < 2n-1$, and surjective if $\dim(X) = 2n-1$. Consequently, when the abelian group structure on $[X;Y]$ is defined, the natural inclusion from $[X;Y]$ to $\{X;Y\}$ is an isomorphism.

They also established exactness and excision properties for the S–groups. Defining $\{X;Y\}_q$ to be $\{S^qX;Y\}$ if $q>0$, and $\{X;S^{-q}Y\}$ if $q<0$, they pointed out that for fixed X

the Eilenberg–Steenrod homology axioms are satisfied whereas for fixed Y the cohomology axioms are satisfied (except of course for the dimension axiom). As with Spanier's paper on cohomotopy, their interest in these formal properties reflected the profound influence that the Eilenberg–Steenrod axiomatic approach was having upon the subject.

Shortly after introducing the S-category, Spanier and Whitehead (1955) developed their duality theory. Given a polyhedron X in S^n, an *n–dual* $D_n X$ of X is a polyhedron in $S^n - X$ which is an S–deformation retract of $S^n - X$ (i.e. some suspension of $D_n X$ is a deformation retract of the corresponding suspension of $S^n - X$). They defined for polyhedra X and Y in S^n a duality map

$$D_n\colon \{X;Y\} \to \{D_n Y; D_n X\}.$$

To do this they first consider the case where there are inclusions $i\colon X \to Y$ and $i'\colon D_n Y \to D_n X$. Then $D_n(\{i\})$ is defined to be $\{i'\}$. For a general S–map from X to Y they reduce to the case of an inclusion by means of the mapping cylinder construction. Eventually they show that this leads to a well defined isomorphism. They establish a number of basic properties of the duality map D_n including its relation with Alexander duality. In particular, they make precise the duality between the Hopf and Hurewicz maps which Spanier had noted earlier.

In 1959, Spanier (1959) gave a new treatment of Spanier–Whitehead duality in which he shifted attention from the concept of a dual space to that of a duality map. The way in which this approach came about appears to be as follows: Let $F(X;S^n)$ denote the space of maps from X to S^n, $\omega\colon X \wedge F(X;S^n) \to S^n$ the evaluation map, and $\gamma \in H^n(S^n)$ a generator. John Moore (1956) had shown that slant product defines an isomorphism $\omega^*(\gamma)/\_\_ : H_q(F(X;S^n) \to H^{n-q}(X),\ q < 2(n - \dim(X))$. So the function space $F(X;S^n)$ appears to be $(n+1)$–dual to X at least through a range of dimensions. In order to remove the dimensional restriction, Spanier formed the spectrum $\mathbf{F}(X)$ whose n–th space is $F(X;S^n)$. The connecting maps $h\colon SF(X;S^n) \to F(X;S^{n+1})$ are given by $h(t,f)(x) = (t, f(x))$. He called $\mathbf{F}(X)$ the *functional dual* of X. Spectra had been introduced earlier by E. Lima (1959), a student of Spanier, in order to study duality for infinite complexes.

A spectrum $\mathbf{Y} = \{Y_n; \varepsilon_n\}$ is simply a sequence of spaces $Y_n, n \in \mathbb{Z}$, together with maps $\varepsilon_n: SY_n \to Y_{n+1}$. If X is a finite complex and $\mathbf{Y}$ is a spectrum,

$$\{X; \mathbf{Y}\} = \lim[S^n X; Y_n].$$

Let X^* be a deformation retract of $S^{n+1} - X$, hence an $(n+1)$–dual of X. To make precise the duality between X and $\mathbf{F}(X)$, Spanier wished to construct a weak equivalence of spectra $X^* \to S^n\mathbf{F}(X)$, where $S^n\mathbf{F}(X)$ is the n–fold suspension of $\mathbf{F}(X)$. It was apparently well known by this time that Alexander duality could be described by means of a slant product. Specifically, by removing a point of S^{n+1} which is not in X or X^*, one can regard X and X^* as subspaces of R^{n+1}. Define $\mu: X \times X^* \to S^n$ by $\mu(x, x^*) = (x - x^*)/|x - x^*|$. Its restriction to $X \vee X^*$ is null homotopic so it induces a map $\mu: X \wedge X^* \to S^n$. Then $\mu^*(\gamma)/\_\_: H_q(X) \to H^{n-q}(X^*)$ realizes the Alexander duality isomorphism. Spanier had used this description in an earlier paper (1959a) in which he studied the relation between infinite symmetric products and duality. Now the map $\mu: X \wedge X^* \to S^n$ defines a map $X^* \to F(X; S^n)$, and by virtue of the naturality of the slant product, the induced map of spectra $X^* \to S^n\mathbf{F}(X)$ is a weak equivalence.

Combining this weak equivalence with the exponential correspondence $[Z; F(X^*; S^n)] = [Z \wedge X^*; S^n]$, he showed that the map

$$R_\mu: \{Z; X\} \to \{Z \wedge X^*; S^n\}, \quad f \to \mu(f \wedge 1), \tag{1}$$

is an isomorphism. By symmetry,

$$L_\mu: \{Z; X^*\} \to \{X \wedge Z; S^n\}, \quad f \to \mu(1 \wedge f), \tag{2}$$

is an isomorphism.

Now, if Y is also a polyhedron in S^{n+1}, Y^* is a deformation retract of $S^{n+1} - Y$, and $\nu: Y^* \wedge Y \to S^n$ is the associated map, a duality isomorphism

$$D(\mu, \nu): \{X; Y\} \to \{Y^*; X^*\} \tag{3}$$

is defined in terms of the fundamental isomorphisms (1) and (2), by

$$\{X;Y\} \xrightarrow{R_\nu} \{X \wedge Y^*; S^n\} \xrightarrow{L_\mu^{-1}} \{Y^*; X^*\}.$$

The existence of the isomorphism R_μ of course does not depend on the geometric origins of X^* but only on the existence of the map μ. Thus, Spanier was led to define a *duality map* to be a map $\mu: X \wedge X^* \to S^n$ such that the slant product $\mu * (\gamma)/\_\_ : H_q(X) \to H^{n-q}(X^*)$ is an isomorphism. He then showed that the map $\tilde{\mu}: X^* \wedge X \to S^n$ obtained by composing μ with the interchange map $X^* \wedge X \to X \wedge X^*$ is also a duality map, from which it follows that L_μ is also an isomorphism. Given a second duality map $\nu: Y \wedge Y^* \to S^n$, he derived the duality isomorphism $D(\mu, \nu)$ as we have described above. In addition to being more general, the formal properties of the duality are much more readily established. Moreover, the theory gives a simple criterion for S–maps $f: X \to Y$ and $g: Y^* \to X^*$ to be dual. They are dual if and only if the diagram

$$\begin{array}{ccc}
X \wedge Y^* & \xrightarrow{f \wedge 1} & Y \wedge Y^* \\
{\scriptstyle 1\wedge g}\downarrow & & \downarrow{\scriptstyle \nu} \\
X \wedge X^* & \xrightarrow{\mu} & S^n
\end{array}$$

is homotopy commutative. (A comparison of the two approaches to duality revealed a minor notational problem: A geometric $(n+1)$–dual X^* gives rise to an n–duality map $X \wedge X^* \to S^n$. Spanier suggested that it would be more natural to call X^* an n–dual of X, which is the terminology that is now used.)

A few years later, Wall (1967) added an additional refinement which would prove useful in applications; particularly to surgery theory. He noted that the whole theory could be given a “dual” formulation. In this description, an n*–duality* is a map $\mu: S^n \to X \wedge X^*$ such that $\mu_*(\gamma)\backslash\_\_ : H^q(X) \to H_{n-q}(X^*)$ is an isomorphism, where γ is a generator of $H_n(S^n)$.

When Spanier’s textbook on algebraic topology appeared in 1966, it contained an exercise outlining a categorical formulation of Spanier–Whitehead duality which he attributed to P. Freyd and D. Husemoller. The fundamental isomorphisms (1) and (2) are easily generalized to isomorphisms

$$R_\mu\colon \{Z; E \wedge X\} \to \{Z \wedge X^*; E \wedge S^n\} \tag{3}$$

$$L_\mu\colon \{Z; X^* \wedge E\} \to \{X \wedge Z; S^n \wedge E\} \tag{4}$$

where E is an arbitrary CW–complex. In the categorical formulation one now defines μ to be a duality map if R_μ and L_μ are isomorphisms. (By a standard argument it is only necessary to assume that R_μ and L_μ are isomorphisms when Z and E are spheres.) One then shows by an induction over cells argument that for every finite CW–complex X there is an integer n and a finite CW–complex X^* for which there is a duality map $\mu\colon X \wedge X^* \to S^n$.

This formulation exhibits Spanier–Whitehead duality as an intrinsic property of the S–category quite independent of Alexander's duality theorem. The latter now emerges as the fundamental connection between this duality and geometry: If X is contained in S^{n+1} and X^* is a deformation retract of $S^{n+1} - X$ then X^* is a Spanier–Whitehead n–dual of X.

As generalizations of the S–category arose, the appropriate formulation of Spanier–Whitehead duality soon followed. In 1970 at the International Congress in Nice, G. Segal (1970) introduced for a finite group G, the equivariant S–category whose objects are finite CW–complexes and whose morphisms are

$$\{X; Y\}_G = \lim [S^V X; S^V Y]_G,$$

the limit taken over all representations V of G. This marked the beginning of equivariant stable homotopy theory — a theory which has undergone rapid development in recent years. Duality was extended to this category by Wirthmüller (1970) and Dold and Puppe (1980).

A second generalization of S–theory involves the consideration of families of pointed spaces parametrized by a fixed space B. The homotopy theory of such spaces was developed by I.M. James (1971), and T. tom Dieck–K. Kamps– D. Puppe (1970), among others. There is the corresponding S–category based on fiberwise suspension, and duality in this category was derived in Becker–Gottlieb (1976).

§5. Atiyah's duality theorem.

Milnor and Spanier (1960) clarified the relation between Spanier–Whitehead duality and Poincaré duality on a closed differentiable manifold. Thom (1952) had introduced what is now called the Thom space M^α of a vector bundle α over M. It is defined to be $D(\alpha)/S(\alpha)$ where $D(\alpha)$,$S(\alpha)$ are respectively the unit disk and sphere bundles of α. The Milnor–Spanier theorem states that if M is a closed manifold embedded into euclidean space R^s with normal bundle ν and M^+ is M disjoint union with a point then

$$M^\nu \text{ is s-dual to } M^+.$$

Their proof was geometric; exhibiting M^ν as a deformation retract of the complement of M^+ in S^{s+1}. With the dual formulation of Spanier-Whitehead duality, which came later, a duality map

$$\mu: S^s \to M^+ \wedge M^\nu$$

is easily constructed. There is the Pontrjagin–Thom map $c: S^s \to M^\nu$ defined by embedding the normal disk bundle $D(\nu)$ into R^s as a tubular neighborhood and letting c collapse the complement of the interior of $D(\nu)$ to a point. Then μ is the composition of c with the "diagonal" map $M^\nu \to M^+ \wedge M^\nu$, $\vec{v}_x \mapsto x \wedge \vec{v}_x$. Lefschetz duality for $(S^s, D(\nu))$ implies that μ is a duality map. There is the commutativity relation

$$\begin{tikzcd} H^{k+s-n}(M^n) \arrow[r, "D"] & H_{n-k}(M+) \\ H^k(M^+) \arrow[u, "\phi_\mu"] \arrow[r, equal] & H^k(M^+) \arrow[u, "v\cap \_"'] \end{tikzcd}$$

where $D = \mu_*(\gamma)/\_$, $u \in H^{s-n}(M^n)$, ϕ_u is the Thom homomorphism, and $v = D(u)$. Since D is an isomorphism, $v \cap \_$ is an isomorphism precisely when ϕ_u is an isomorphism. This basic relation, which goes back to Thom, carries over to generalized homology–cohomology theories, and the Milnor–Spanier theorem implies that D remains an isomorphism. Thus, the question of the orientability of a manifold M is equivalent to that of the orientability of its normal bundle in the sense of Thom. The latter is usually studied as part of the general question of orientability of a vector bundle or spherical fibration with respect to a cohomology theory.

Atiyah (1961) generalized the Milnor–Spanier theorem to manifolds with boundary, and derived from it the following relation among Thom spaces: Let α and β be vector bundles over a closed manifold M such that $\alpha \oplus \beta \simeq M \times R^t$. Then if M is embedded in R^s with normal bundle ν,

$$M^\alpha \text{ is } (s+t)\text{–dual to } M^{\beta \oplus \nu}.$$

This relationship, which was also obtained by R. Bott and A. Shapiro (unpublished), is now known as Atiyah duality. It provides a fundamental connection between duality theory and the theory of differentiable manifolds. Atiyah gave two applications of this relation. The first was to extend the work of I.M. James reducing the question of the existence of vector fields on spheres to a homotopy question about what he called stunted projective spaces. This reduction was later used by Adams in his celebrated paper (1962), in which he obtained a complete solution of the problem. The second application was to show that the stable normal bundle of a closed manifold is an invariant of homotopy type. This had been conjectured by Milnor and Spanier (1960).

§6. Generalized Homology and Cohomology Theories.

The concept of a generalized homology or cohomology theory emerged over a period of roughly seven years from 1955 to 1962. Along with the examples — stable homotopy–cohomotopy (Spanier and Whitehead (1957)), K–theory (Atiyah and Hirzebruch (1962a)), bordism and cobordism (Atiyah (1962b), Conner and Floyd (1964)) — the search for a satisfactory duality theory guided the development of the subject.

If $\mathbf{E} = \{E_n; e_n\}$ is a spectrum, a generalized cohomology theory $H^*(\ ;\mathbf{E})$ is defined on the category of (pointed) finite CW-complexes by

$$H^q(X;\mathbf{E}) = \lim [S^{n-q}X; E_n].$$

E.H. Brown (1961) showed that every generalized cohomology theory defined on the category of finite CW-complexes, and having countable coefficient group, arises from a spectrum in this way. It was well known that the cohomology theories which existed up to this time had such a description, the first of which (singular cohomology) goes back to Eilenberg–MacLane (1943).

A year later G W. Whitehead (1962) undertook a comprehensive study of generalized homology–cohomology theories from a homotopy point of view. He defined the generalized homology groups of X with coefficients in the spectrum $\mathbf{E}$ by

$$H_q(X;\mathbf{E}) = \overrightarrow{\lim}[S^{n+q}; E_n \wedge X].$$

As motivation for this definition, he cited D.M. Kan's (1958) theory of adjoint functors. He had also shown in an earlier paper (1956) that his definition gave the correct answer when $\mathbf{E}$ is an Eilenberg–MacLane spectrum.

After laying out the theory of products, Whitehead proved general Poincaré and Alexander duality theorems. ¿From the fact that $H^*(\ \ ;\mathbf{E})$ and $H_*(\ \ ;\mathbf{E})$ are related by Alexander duality, he established conclusively that his definition of homology was the correct one. It is a consequence of Spanier–Whitehead duality that every cohomology theory H^* determines a "formal dual" homology theory H_*: Given a space X, choose a duality map $X \wedge X^* \to S^n$ and define $H_q(X) = H^{n-q}(X^*)$. This eventually leads to a homology theory H_*. Now, the fact that $H^*(\ \ ;\mathbf{E})$ and $H*(\ \ ;\mathbf{E})$ are related by Alexander duality implies that $H_*(\ \ ;\mathbf{E})$ is the formal dual of $H^*(\ \ ;\mathbf{E})$ as desired.

Whitehead's fundamental paper gave a complete and satisfactory generalization of Poincaré and Alexander duality to arbitrary homology–cohomology theories.

§7. Umkehr maps.

An Umkehr map is a map related to an original map which reverses the arrow, that is the source of the original map becomes the target of the Umkehr map. The name has not solidified yet; sometimes Umkehr maps are called wrong way maps, or Gysin maps, or even transfer maps.

The first appearance of Umkehr maps occurred in Heinz Hopf (1930). For a map $f\colon M \to N$ between two combinatorial manifolds of the same dimension, f induces a homomorphism $f_*\colon H_*(M;Q) \to H_*(N;Q)$. Now intersection theory gave rise to a ring structure, called the intersection ring, due to Lefschetz. The map f_* is not a ring homomorphism, however Hopf managed to define a "wrong way homomorphism" which did preserve the ring structure when the manifolds had the same dimension. He called it the "Umkehr homomorphismus" from $H_*(N;Q) \to H_*(M;Q)$.

With the invention of cohomology, Hans Freudenthal (1937) could explain the Umkehr homomorphism in terms of Poincaré duality. In modern notation the Umkehr homomorphism is what we call the Poincaré duality map

$$f_! = D_M \circ f^* \circ D_N^{-1}$$

where D_M denotes the Poincaré isomorphism from cohomology to homology. This idea begs to be generalized to manifolds of different dimensions. It was finally done by Hopf's student Gysin (1941) in his dissertation. Because of this, Umkehr maps are frequently called Gysin maps.

Integration along the fibre was introduced by A. Lichnerowicz (1948). Suppose $F \to E \xrightarrow{\pi} B$ is a fibration. If it is a fibre bundle with F and E and B all C^∞ manifolds, then using de Rham cohomology, an i–form on E can be integrated over each fibre to give an $(i-n)$–form on B, where n is the dimension of F. This gives a cochain map on the de Rham cochain complex of forms, and thus we get an Umkehr map $\pi^!\colon H^i(E;\mathbb{Q}) \to H^{i-n}(B;\mathbb{Q})$ called *integration along the fibre.*

Chern and Spanier (1950) extended integration along the fibre to more general fibre bundles where $H_n(F) \cong \mathbb{Z}$ for the top dimension n. With the Serre spectral sequence in Serre (1951), integration along the fibre could be defined for any oriented fibration whose fibre has a top nonzero homology group $H_n(F; V) \cong V$ and V is any field of coefficients. Then reading along the top line of the E^∞ and E^2 terms gives integration along the fibre both for homology and cohomology.

In the late 1950's and early 1960's, Umkehr maps played an important role in the generalization of the Riemann–Roch theorem and in the Atiyah–Singer index theorem.

Dieudonné (1989), gives a very lively account of the Riemann–Roch theorem. "In the late 1950's the growing usefulness of categorical notions gradually convinced mathematicians that morphisms rather than objects had to be emphasized in many situations. It was that trend that led Grothendieck to believe that the Riemann–Roch–Hirzebruch formula ... is only a special case of a relative Riemann–Roch relation dealing with a morphism $f: X \to Y$ of smooth projective varieties; the relation ... would then be the case in which Y is reduced to a single point. The problem was thus to replace both sides [of the relation] by meaningful generalizations when X and Y are arbitrary." On one side of the Riemann–Roch–Hirzebruch equation he needs to replace "integration" by an Umkehr map. So Grothendieck introduced the Poincaré duality map $f^!: H^*(X) \to H^*(Y)$, which is the dual to the map $f_!$ on homology defined by Gysin (1941). For the other side of the equation, he had to invent K–theory.

Atiyah and Hirzebruch (1959) extended K–theory, and hence the Riemann–Roch theorem and the K–theory Umkehr map.

An important set of notes was produced by J.M. Boardman (1966). In it he develops his very influential ideas on spectra. Chapter V was devoted to "Duality and Thom Spectra." In section 6, entitled Transfer Homomorphisms, Boardman collected together eight constructions and called them transfers. (By now, the word "transfer" is usually taken to mean an Umkehr map which does not shift dimension.) These Umkehr homomorphisms all satisfied seven equations and turned out to agree on generalized homology and cohomology in the situations in which their definitions were valid. The seven relations forms a very tricky but useful calculus. In particular the Umkehr is functorial. A particularly useful relation gives $f^!(f^*(\alpha)) = \alpha \cup f^!(1)$. That is: Composition of the Umkehr homomorphism with the induced homomorphism results in multiplication by a fixed element $f^!(1)$. This turns out to be very important in the development of the transfer.

Boardman, as a graduate student, had discovered a simple Umkehr map for bordism theory, and he asked several people for more examples and got quite a few leads. Although the constructions were quite-different looking, the homomorphisms agreed on the intersections of their domains of validity.

Boardman's thesis (Cambridge 1964) problem was a computational problem and he was collecting as many tools as he could. What he published of his thesis used a slick proof, and his Umkehr maps were not mentioned. Fortunately, he collected his material in Chapter V, section 6, of his notes, where they played a seminal role in our next topic — Transfers.

§8. Transfers.

Duality has influenced and been influenced by almost every subarea of algebraic topology. Even if we concentrate only on the key theorems of strong duality, we find the number of topics too vast to describe. We choose the subject of Transfers to illustrate the action and reaction of strong duality on a particular subject. Therefore we limit our discussion of the transfer to its origins and its complex relation to strong duality. We choose Transfers, even though there are more important topics we could have considered and even though the subject is still developing, because we know that area well and have worked in it and so we can provide information which is not readily available in the published record. In addition to our own recollections, we benefited from conversations and correspondence with J.M. Boardman, A. Dold, D.S. Kahn and S.B. Priddy.

The transfer began as a group-theory construction, which produced a homomorphism from a group G made abelian to a subgroup H of G of finite index made abelian. Thus $G^{ab} \to H^{ab}$, [see Marshall Hall (1959), p. 201].

It was Beno Eckmann who realized that this was a special case of a construction made for covering spaces and when applied to $K(\pi,1)$'s gave the group-theory result. He gave the name *transfer* to the homomorphism in cohomology for a covering space with finite fibre, $\tau\colon H^i(\tilde{X}) \to H^i(X)$, whose composition with the projection homomorphism p^* is multiplication by the number of elements in the fibre.

The relationship to group theory came about as follows. W. Hurewicz (1936) recognized that aspherical spaces were classified up to homotopy by their fundamental group. Now we call these spaces $K(\pi,1)$, where π is the group. If H is a subgroup of π, then $K(H,1)$ is a covering space of $K(\pi,1)$. Then the group cohomology of π is the cohomology of the $K(\pi,1)$. Eckmann (1953) noted that the group theory transfer was in fact the dual of his transfer homomorphism for $H_1(K(\pi,1))$, which is the group π made abelian. One can see part of the idea for transfer in Eckmann (1945).

The emphasis on describing the transfer as a homomorphism was probably part of the general movement inspired by the Eilenberg–Steenrod axioms which indicated that the morphisms were as important as the objects. The (covering space) transfer was also used by Conner and Floyd and S.D. Liao in the study of finite transformation groups in the early 1950's.

By the early 1970's, covering transfers were in the air. An interesting account of the transfer up to this time is given in J.F. Adams (1978) book in §4.1. Mainly, the thrust was the ad hoc construction of covering space transfers for different cohomology theories. These could all be united by a construction of the transfer as an S–map. Various S–map constructions were made independently by 1971 by Dan Kahn, Jim Becker, and F.W. Roush (1971). None of these were published since the discoverers did not believe they were important.

J.F. Adams (1978) wrote in his book on page 104–105: "Transfer came to the attention of the general topological public when Kahn and Priddy (1972) published their well-known paper in 1972 ... Kahn and Priddy wrote: 'the existence of the transfer seems to be well-known, but we know of no published account.' The topological world thus learned that all well-informed persons were supposed to know about transfer, although hardly anyone did unless they were lucky ... the rapid spread of a general conviction that the transfer was very good business owed much to the fact they solved a problem of some standing in homotopy theory."

The transfer for finite covering fibrations has a spectacular generalization to fibrations. If $F \xrightarrow{i} E \xrightarrow{p} B$ is a Hurewicz fibration with compact fibre F (and very mild conditions on B), there is an S–map $\tau\colon B^+ \to E^+$ which induces homomorphisms τ_* and τ^* on ordinary homology and cohomology respectively, such that $p_* \circ \tau_*$ and $\tau^* \circ p^*$ are both multiplication by the Euler–Poincaré characteristic $\chi(F)$.

The only hints that transfers could exist for fibrations came from an early consequence of the Leray–Serre spectral sequence: For a field of coefficients, if i^* maps onto the top dimensional cohomology group of F, then p^* must be injective in cohomology with the same coefficients. Now Borel (1956) observed that if F were a closed smooth oriented manifold and $F \xrightarrow{i} E \xrightarrow{p} B$ were a smooth oriented fibre bundle, then i^* mapped onto the top cohomology for $\mathbb{Z}_p$ coefficients whenever p did not divide $\chi(F)$. Thus for such p, the projection p^* is injective. This result would follow immediately if there were a transfer for fibrations.

Also, dual to the projection p is the transgression $\omega\colon \Omega B \to F$ from the Nomura–Puppe Sequence mentioned in §2. It had recently been discovered, Gottlieb (1972), that $\chi(M)\omega^* = 0$ for M a smooth manifold. This result was true for cohomology with *any* coefficients. A proof depended on the same fact about i^* which was central to Borel's result.

These considerations led to the question of the existence of a transfer for fibrations in the Fall of 1972. Boardman's Umkehr map calculus immediately gave a transfer in Borel's special situation for singular cohomology:

$$\tau(\cdot) := p^!(\cdot \cup \chi)$$

where $p^!$ is the spectral sequence version of integration along the fibre and χ is the Euler class of the bundle of tangents along the fibre. By the Spring of 1973 the transfer theorem had been extended to fibre bundles whose fibre was a manifold with boundary, Gottlieb (1975), and the transfer existed as an S–map for fibre bundles whose structure group was a compact Lie group, Becker–Gottlieb (1975).

In the announcement Becker–Casson–Gottlieb (1975) all the conditions on the fibration were essentially removed (as long as F was homotopic to a finite complex). The transfer was an S–map which satisfied in singular homology and cohomology the following relations.

$$p_* \circ \tau_* = \text{ multiplication by } \chi(F)$$
$$\tau^* \circ p^* = \text{ multiplication by } \chi(F)$$

In fact, if the fibration is equipped with a fibre-preserving map, then there is a transfer which satisfies the above conditions with a suitable Lefschetz number replacing the Euler characteristic.

A consequence of the generalization of the transfer was the generalization of the *transgression theorem* $\chi(F)\omega^* = 0$, which now holds in all cohomology theories and in all homology theories, for essentially any fibration whose fibre is homotopy equivalent to a finite complex. In addition, for fibre-preserving maps, the Euler characteristic can be replaced by the same Lefschetz number as in the transfer theorem (see Becker–Gottlieb (1976)).

To extend the transfer theorem to Hurewicz fibrations and to construct the transfer as an S–map, the use of strong duality became vital. The two methods indicated in the announcement were explained in detail in Casson–Gottlieb (1977) and in Becker–Gottlieb (1976). In the first method, the role of integration along the fibre was played by the Poincaré duality map. Topological maps on Thom complexes induced the Poincaré duality map on the homology level of the underlying manifolds, and the cup product could be induced by a map between Thom complexes as well. These considerations yielded the transfer as an S–map in the smooth fibre bundle case where everything was a smooth oriented manifold. The fact that every Hurewicz fibration was a fibrewise retract of these smooth oriented fibre bundles was proved by a series of tricks, and the retraction of the smooth transfer resulted in a transfer for the general case.

On the other hand, the second method depended upon the existence of S–duality in the category of ex–spaces over a space B. This allowed the construction of the transfer via a chain of duality maps and a diagonal map.

Meanwhile Albrecht Dold (1974a) was conducting a deep study of fixed point theory. He studied the index of parameterized families of maps. He was influenced by R.J. Knill (1971). As a by-product to defining the index for a parameterized family of maps, Dold discovered a transfer. He did not regard it as important until Puppe told him about the transfer for fibrations. He sent a note to the Comptes Rendus, Dold (1974b), and expanded the paper in Dold (1976), calling the transfer the fixed point transfer. Nowadays it is also called the Dold transfer or the Becker–Gottlieb–Dold transfer as well.

Dold and Puppe organized a seminar on the transfer for fibrations. As a result of this they realized, Dold–Puppe (1980), that the observation that the degree of the map $S^n \xrightarrow{\mu} DX \wedge X \to X \wedge DX \xrightarrow{\mu} S^n$ equals $\chi(X)$ leads to the categorical definition of trace given here in §2 of this paper. The categorical definition of transfer is given by the map

$$\tau_f \colon I \xrightarrow{\eta} A \otimes DA \xrightarrow{\gamma} DA \otimes A \xrightarrow{Df \otimes \Delta} DA \otimes A \otimes A \xrightarrow{\varepsilon \otimes id} I \otimes A = A.$$

Thus in the category of ex–spaces over B, which they call Stab_B, they note that if p is a well–sectioned Hurewicz fibration whose fibre has the stable homotopy type of a finite CW complex, then the fibrewise dual in Becker–Gottlieb (1976) results in the transfer for fibrations for Stab_B, while the fact that fibrewise ENR also have strong duals, leads to the Dold transfers, Dold–Puppe (1980).

In addition, they can prove the Lefschetz fixed point theorem as a consequence of their point of view. So, many important theorems which seemed to be independent, and which seemed to have little to do with duality, can be shown to be consequences of Dold and Puppe's concept of strong duality for monoidal categories.

Postscript: Dold–Puppe (1980) explicitly remarked that the condition on the fibre in the transfer theorem could be relaxed from being homotopically equivalent to a finite complex to merely being S–equivalent to a finite complex, and that this was implicitly proved in

Becker–Gottlieb (1976). This advance permits the observation that the transfer exists in a purely group-theoretic setting where fibration is replaced by surjective homomorphism and the condition on the fibre is replaced by the condition that the kernel has finitely generated homology, Gottlieb (1983). Thus the transfer is returned back to group theory with all the topological conditions removed in a vastly more general situation. But its construction is not at all group–theoretic.

We also note that there is quite recent work by Dwyer (1996) which relaxes the hypothesis on the fibre to the case where the fiber of the map satisfies only a homological finiteness condition relative to some spectrum.

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