Maxwell's equations

Daniel Henry Gottlieb

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Abstract

We express Maxwell's equations as a single equation, first using the divergence of a special type of matrix field to obtain the four current, and then the divergence of a special matrix to obtain the Electromagnetic field. These two equations give rise to a remarkable dual set of equations in which the operators become the matrices and the vectors become the fields. The decoupling of the equations into the wave equation is very simple and natural. The divergence of the stress energy tensor gives the Lorentz Law in a very natural way. We compare this approach to the related descriptions of Maxwell's equations by biquaternions and Clifford algebras.

1 Introduction

Maxwell's equations have been expressed in many forms in the century and a half since their discovery. The original equations were 16 in number. The vector forms, written below, consist of 4 equations. The differential form versions consists of two equations; see [Misner, Thorne and Wheeler(1973); see equations 4.10, 4.11]. See also [Steven Parrott(1987) page 98 -100] The application of quaternions, and their complexification, the biquaternions, results in a version of one equation. William E. Baylis (1999) equation 3.8, is an example.

In this work, we obtain one Maxwell equation, (10), representing the electromagnetic field as a matrix and the divergence as a vector multiplying the field matrix. But we also obtain a remarkable dual formulation of Maxwell's equation, (15), wherein the operator is now the matrix and the field is now the vector. The relation of the four vector potential to the electromagnetic field has the same sort of duality; see equations (13) and (14).

These dual pairs of equations are proved equivalent by expanding the matrix multplication and checking the equality. Indeed, they were discovered this way. However, it is possible to ask if there is an explanation for this algebraic miracle. There is. It follows from the commutivity of certain types of matrices. This commutivity also implies the famous result that the divergence of the stress-energy tensor T is the electrom magnetic field F applied to the the current-density vector.

The commutivity itself, again discovered and proved by a brute force calculation, has an explanation arising from two natural representations of the biquaternions; namely the left and the right regular representations.

Why do matrices produce an interesting description of Maxwell's equations, when tensors are so much more flexible? It is because matrices are representations of linear transformations for a given choice of a basis. The basis is useful for calculation, but reformulating concepts and definitions in terms of the appropriate morphisms (in this case, linear transformations) almost always pays a dividend, as we topologists have discovered during the last century.

This paper arose out of an email from me to Vladimir Onoochin concerning questions about Maxwell's equations. I thank him for a very interesting correspondence.

2 Maxwell's Equations

We are not really concerned with physical units for our paper, however what we have written is compatible with natural Heavyside-Lorentz units where the speed of light $c = 1$ and the electric permittivity $\epsilon_0 = 1$. See Baylis(1999), section 1.1.

We will say that two vector fields **E** and **B** satisfies Maxwell's equations if

$$\begin{aligned} \nabla \times \mathbf{E} + \partial_\mathbf{t}\mathbf{B} &= \mathbf{0} && (1)\\ \nabla \times \mathbf{B} - \partial_\mathbf{t}\mathbf{E} &=: \mathbf{J} && (2)\\ \nabla \cdot \mathbf{E} &=: \rho && (3)\\ \nabla \cdot \mathbf{B} &= 0 && (4) \end{aligned}$$

Here =: means the right symbol is defined by the left side of the equation. Now we follow [Gottlieb (1998)] and [Gottlieb (2001)] and recall the notation for Lorentz transformations. Let M be Minkowski space with inner product $\langle\ ,\ \rangle$ of the form $- + ++$. Let e_0, e_1, e_2, e_3 be an orthonormal basis with e_0 a time-like vector. A linear operator $F : M \to M$ which is skew symmetric with respect to the inner product $\langle\ ,\ \rangle$ has a matrix representation, depending on the orthonormal basis, of the form

$$F = \begin{pmatrix} 0 & \vec{E}^t \\ \vec{E} & \times\vec{B} \end{pmatrix} \tag{5}$$

where $\times\vec{B}$ is a 3×3 matrix such that $(\times\vec{B})\vec{v} = \vec{v} \times \vec{B}$, the cross product of $\vec{v}$ with $\vec{B}$. That is

$$\begin{pmatrix} 0 & B_3 & -B_2 \\ -B_3 & 0 & B_1 \\ B_2 & -B_1 & 0 \end{pmatrix}$$

where **B** is given by (B_1, B_2, B_3).

The dual F^* of F is given by

$$F^* := \begin{pmatrix} 0 & -\vec{B}^t \\ -\vec{B} & \times\vec{E} \end{pmatrix} \tag{6}$$

We complexify F by $cF := F - iF^*$. Its matrix representations is

$$cF = \begin{pmatrix} 0 & \vec{A}^t \\ \vec{A} & \times(-i\vec{A}) \end{pmatrix} \text{where } \vec{A} = \vec{E} + i\vec{B} \tag{7}$$

Remark 1 *These complexified operators satisfy some remarkable Properties:*

a) $cF_1cF_2 + cF_2cF_1 = 2\langle\vec{A}_1, \vec{A}_2\rangle I$ *where* $\langle\ ,\ \rangle$ *denotes the complexification of the usual inner product of* $\mathbb{R}^3$*. Note,* $\langle\ ,\ \rangle$ *is not the Hermitian form, that is our inner product satisfies* $i\langle\vec{v}, \vec{w}\rangle = \langle i\vec{v}, \vec{w}\rangle = \langle\vec{v}, i\vec{w}\rangle$.

b) *The same property holds for the complex conjugates* $\bar{c}F_1$ *and* $\bar{c}F_2$*. The two types of matrices commute. That is* $cF_1\bar{c}F_2 = \bar{c}F_2cF_1$

c) $cF\bar{c}F =: 2T_F$ *where* T_F *is proportional to the stress-energy tensor of electromagnetic fields.*

Now $e^F = I + F + F^2/2! + F^3/3! + \ldots : M \to M$ *is a proper Lorentz transformation. It satisfies* $e^F = e^{cF/2}e^{\bar{c}F/2}$*. The algebraic properties of cF give rise to a simple expression for its exponential:*

$$e^{cF} = \cosh(\lambda_{cF})I + \frac{\sinh(\lambda_{cF})}{\lambda_{cF}}cF$$

where λ_{cF} *is an eigenvalue for* cF*. There are only, at most, two values for the eigenvalues of* cF*, namely* λ_{cF} *and* $-\lambda_{cF}$*.*

Now let us see how to describe Maxwell's field equations by means of these matrices. Now consider the 4×4 matrix F defined by equation (5). We multiply it by the 4×1 vector of operators $(-\partial_t, \nabla)^t$ to get

$$F\begin{pmatrix} -\partial_t \\ \nabla \end{pmatrix} = \begin{pmatrix} \rho \\ \mathbf{J} \end{pmatrix} \tag{8}$$

and

$$F^*\begin{pmatrix} -\partial_t \\ \nabla \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{0} \end{pmatrix} \tag{9}$$

See equation (23) for more details on this notation.

Equations 8 and 9 hold if and only if Maxwell's equations (1 - 4) hold. Using the definition of cF in equation (7) we get

$$cF\begin{pmatrix}-\partial_t\\ \nabla\end{pmatrix} = \begin{pmatrix}\rho\\ \mathbf{J}\end{pmatrix} \tag{10}$$

is true if and only if E and B satisfy Maxwell's equations.

In fact, the version of Maxwell's equations involving the four vector fields **E**, **B**, **D**, **H** is equivalent to the matrix equation

$$\begin{pmatrix} 0 & \mathbf{D}+\mathbf{iB} \\ \mathbf{D}+\mathbf{iB} & \times(-i(\mathbf{E}+\mathbf{iH})) \end{pmatrix}\begin{pmatrix}-\partial_t\\ \nabla\end{pmatrix} = \begin{pmatrix}\rho\\ \mathbf{J}\end{pmatrix} \tag{11}$$

Remark 2 *Note the forms of the complex matrices in equation (7) and in equation (11): respectively*

$$\begin{pmatrix} 0 & \vec{A}^T \\ \vec{A} & \times(\pm i\vec{A}) \end{pmatrix} \text{ and } \begin{pmatrix} 0 & \vec{A}^T \\ \vec{A} & \times(\vec{C}) \end{pmatrix}$$

Now within the class of all matrices of the second form, only the first form has the property that its square is equal to a multiple of the identity.

Now let us consider a four vector field $\binom{\varphi}{\mathbf{A}}$. We define an associated **E** and **B** by the equations

$$\nabla\times\mathbf{A} =: \mathbf{B} \text{ and } -\partial_{\mathbf{t}}\mathbf{A} - \nabla\varphi =: \mathbf{E} \tag{12}$$

Then the **E** field and the **B** field defined above satisfy Maxwell's equations (1 - 4). We can describe the four vector field by means of a similar matrix equation. Let I denote the 4×4 identity matrix.

$$\left(\varphi I + \begin{pmatrix} 0 & \mathbf{A}^{\mathbf{t}} \\ \mathbf{A} & -i\times\mathbf{A} \end{pmatrix}\right)\begin{pmatrix}\partial_t\\ \nabla\end{pmatrix} = \begin{pmatrix}\partial_t\varphi + \nabla\cdot\mathbf{A}\\ -\mathbf{E}-\mathbf{iB}\end{pmatrix} \tag{13}$$

Now a remarkable duality holds. The following equation is also equivalent to the above equation.

$$\left(\partial_t I + \begin{pmatrix} 0 & \nabla^{\mathbf{t}} \\ \nabla & i\times\nabla \end{pmatrix}\right)\begin{pmatrix}\varphi\\ \mathbf{A}\end{pmatrix} = \begin{pmatrix}\partial_t\varphi + \nabla\cdot\mathbf{A}\\ -\mathbf{E}-\mathbf{iB}\end{pmatrix} \tag{14}$$

Similarly, equation (10) has a dual equation which holds if and only if Maxwell's equations are satisfied,

$$\left(\partial_t I - \begin{pmatrix} 0 & \nabla^{\mathbf{t}} \\ \nabla & i\times\nabla \end{pmatrix}\right)\begin{pmatrix} 0 \\ -\mathbf{E}-\mathbf{iB} \end{pmatrix} = \begin{pmatrix} \rho \\ \mathbf{J} \end{pmatrix} \tag{15}$$

These equations give rise to an interesting question:

Remark 3 *Can we give an explanation of the remarkable dualities between equations (10) and (15) and between equations (13) and (14) ? Yes! It is based on the fact that matrices of the form cF commute with the matrices of the form $\overline{c}F$. Hence the matrix on the left hand side of (13) commutes with the matrix on the left hand side of of (14). Thus the first columns of two products of the commuting matrix products must be equal. Since the vectors on the left hand side of (13) and 14) are the first columns of each of the two product matrices that commute, it follows that their products are equal. Exactly the same argument holds for equations (10) and (15).*

Now from part a) of **Remark** 1, we know that

$$\begin{pmatrix} 0 & \nabla^{\mathbf{t}} \\ \nabla & i\times\nabla \end{pmatrix}^2 = \nabla^2 I \tag{16}$$

Consistent with our notation we define

$$\begin{pmatrix} 0 & \nabla^{\mathbf{t}} \\ \nabla & i\times\nabla \end{pmatrix} =: \overline{c}\nabla \tag{17}$$

so that

$$(\partial_t I - \overline{c}\nabla)(\partial_t I + \overline{c}\nabla) = (\partial_t I + \overline{c}\nabla)(\partial_t I - \overline{c}\nabla) = (\partial_t^2 - \nabla^2)I \tag{18}$$

Now apply $(\partial_t I + \overline{c}\nabla)$ to equation (15) and obtain the wave equation and obtain the wave equation

$$(\partial_t^2 - \nabla^2)\begin{pmatrix} 0 \\ -\mathbf{E}-\mathbf{iB} \end{pmatrix} = \nabla\rho + \partial_t J - i\nabla\times J \tag{19}$$

implying the conservation of charge equation $\partial_t\rho + \nabla\cdot\mathbf{J} = \mathbf{0}$

On the other hand, if we apply $(\partial_t I - \overline{c}\nabla)$ to equation (14) we obtain, thanks to equation (15),

$$(\partial_t^2 - \nabla^2)\begin{pmatrix} \varphi \\ \mathbf{A} \end{pmatrix} = \begin{pmatrix} \rho \\ \mathbf{J} \end{pmatrix} + \text{vector depending on } \partial_t\varphi + \nabla\cdot\mathbf{A} \tag{20}$$

So if the covariant gauge condition is chosen, $\partial_t\varphi + \nabla\cdot\mathbf{A} = \mathbf{0}$, we have the wave equation

$$(\partial_t^2 - \nabla^2)\begin{pmatrix} \varphi \\ \mathbf{A} \end{pmatrix} = \begin{pmatrix} \rho \\ \mathbf{J} \end{pmatrix} \tag{21}$$

3 The electromagnetic stress-energy tensor

Not only is equation (10) equivalent to Maxwell's equations, but its complex conjugate below is also equivalent to Maxwell's equations, since the current is real.

$$\bar{c}F\begin{pmatrix} -\partial_t \\ \nabla \end{pmatrix} = \begin{pmatrix} \rho \\ \mathbf{J} \end{pmatrix} \tag{22}$$

Thus we have equations (10), its complex conjugate (22), and its dual (15), and the complex conjugate of (15) all being equivalent to the vector form of Maxwell's equations (1–4). Thus we not only reduce the number of Maxwell's equations from 4 to 1, we obtain 4 equivalent equations. Thus, by a talmudic argument, we can say we have reduced the four Maxwell's equation to 1/4 of an equation. These different forms of the equation interact with each other to produce new derivations of important results.

We have used the following notation of the divergence of a matrix field. What we mean by the notation

$$A\begin{pmatrix} -\partial_t \\ \nabla \end{pmatrix} \tag{23}$$

is that the column vector of operators multiplies into the matrix A of functions and then the operators are applied to the functions they are next to. In index notation we obtain a vector whose i–th row is $a_{ij}\partial_j := \partial_j(a_{ij})$.

Another way to achieve the same result is to take the differential of the matrix, dA. Here we obtain a matrix of differential one-forms whose (i,j)-th element is

$$da_{ij} = \partial_k(a_{ij})dx_k \tag{24}$$

Then we employ the differential geometry convention that $dx_i(\partial_j) = \delta_{ij}$. Thus $da_{ij}(\partial_j) = \partial_j(a_{ij})$. Thus our definitions leads us to

$$A\begin{pmatrix} -\partial_t \\ \nabla \end{pmatrix} = dA \cdot \begin{pmatrix} -\partial_t \\ \nabla \end{pmatrix} \tag{25}$$

Now the Leibniz rule gives us

$$d(AB) = (dA)B + A(dB) \tag{26}$$

This equation helps us to study the divergence of a matrix product. In particular, if dA and B commute, then the divergence of the product is B times the divergence of A plus A times the divergence of B.

Theorem 1 *Let T_F be the electromagentic stress energy tensor of the electromagnetic matrix field F. Then*

$$T_F \begin{pmatrix} -\partial_t \\ \nabla \end{pmatrix} = F \begin{pmatrix} \rho \\ \mathbf{J} \end{pmatrix} \tag{27}$$

Proof:

The first equation below follows from (25). The second equation follows from **Remark 1 c)**. The third equation follows from (26). The fourth equation follows from **Remark 1 b)**, which is the commutivity of matrices of the form cF with matrices of the form $\overline{c}G$, combined with the observation that $d(cF)$ has the same form as cF and similarly $d(\overline{c}F)$ has the same form as $\overline{c}F$. The fifth equation follows by linearity . The sixth equation follows from the form of Maxwell's equations found in equation (10) and its complex conjugate equation (22). Finally the seventh equation follows from the definition of cF and the fact that F is its real part.

$$T_F \begin{pmatrix} -\partial_t \\ \nabla \end{pmatrix} = d(T_F) \cdot \begin{pmatrix} -\partial_t \\ \nabla \end{pmatrix} = \frac{1}{2} d(cF\overline{c}F) \cdot \begin{pmatrix} -\partial_t \\ \nabla \end{pmatrix} = \frac{1}{2}(d(cF)\overline{c}F + cF d(\overline{c}F)) \cdot \begin{pmatrix} -\partial_t \\ \nabla \end{pmatrix} =$$

$$\frac{1}{2}(cF d(\overline{c}F) + cF d(\overline{c}F)) \cdot \begin{pmatrix} -\partial_t \\ \nabla \end{pmatrix} = \frac{1}{2} cF d(\overline{c}F) \cdot \begin{pmatrix} -\partial_t \\ \nabla \end{pmatrix} + \frac{1}{2}\overline{c}F d(cF) \cdot \begin{pmatrix} -\partial_t \\ \nabla \end{pmatrix} =$$

$$\frac{1}{2} cF \begin{pmatrix} \rho \\ \mathbf{J} \end{pmatrix} + \frac{1}{2}\overline{c}F \begin{pmatrix} \rho \\ \mathbf{J} \end{pmatrix} = F \begin{pmatrix} \rho \\ \mathbf{J} \end{pmatrix}$$

□

4 Biquaternions

We choose an orthogonal coordinate system (t, x, y, z) for Minkowski space $\mathbb{R}^{3,1}$. Let e_i be the corrosponding unit vectors (with respect to the Minkowski metric). We express $\vec{e}_1 = (1, 0, 0)^t$, $\vec{e}_2 = (0, 1, 0)^t$, $\vec{e}_3 = (0, 0, 1)^t$. So our standard choice of basis is given by $e_i = (0, \vec{e}_i)^t$ for $i \neq 0$.

Let cE_i be the matrix below where $\vec{A} = \vec{e}_i$ where $i = 1, 2, 3$.

$$cE_i = \begin{pmatrix} 0 & \vec{A}^T \\ \vec{A} & \times(-i\vec{A}) \end{pmatrix} \text{where } \vec{A} = \vec{E} + i\vec{B} = \vec{e}_i \tag{28}$$

Remark 4 *a) The set of sixteen matrices*

$$I, \quad cE_i, \quad \overline{c}E_i, \quad cE_i\overline{c}E_i$$

where $i = 1,2,3$, forms a basis for $M_4(\mathbb{C})$, the vector space of 4×4 complex matrices.

b) The square of each matrix in the basis is I.

c) Each matrix is Hermitian, and all the matrices, except for the identity I, have zero trace.

d) $cE_1 cE_2 = icE_3$ and $\bar{c}E_1 \bar{c}E_2 = -i\bar{c}E_3$.

e) cE_i and cE_j anti commute when $i \neq j$; and $\bar{c}E_i$ and $\bar{c}E_j$ anti commute when $i \neq j$. Also cE_i and $\bar{c}E_j$ commute for all i and j.

See Gottlieb(2001), Theorem 3.3.

Now from the above theorem, we see that the 4 elements $\{I, cE_i\}$ form a basis for the biquaternions. Baylis(1999) chooses to denote his basis for the biquaternions, which he views as the real Clifford algebra on three generators, with a basis $\mathbf{e_i}$ where $i = 0,1,2,3$ where $\mathbf{e_0}$ is the multiplicative identity and $\mathbf{e_i^2} = \mathbf{1}$, and the $\mathbf{e_i}$ anti commute for $i \neq 0$. Baylis chooses the orientation $\mathbf{e_1 e_2} = \mathbf{ie_3}$. Thus our representation of Baylis's biquaternions is given by $\mathbf{e_0} \mapsto I$ and $\mathbf{e}_i \mapsto cE_i$.

Let us denote the biquaternions given above by P. The alternative choice where the biquaternions satisfy the relation $\mathbf{e_1 e_2} = -\mathbf{ie_3}$ will be denoted $\overline{P}$ and is represented by $\mathbf{e}_i \mapsto \bar{c}E_i$. The symbol P stands for paravectors, which is Baylis' name for the space of biquaternions. The name paravectors stems from Baylis' underlying Clifford algebra approach.

Given a paravector $A \in P$, we define *left multiplication by A* as

$$L_A : P \to P : X \mapsto AX$$

and *right multiplication by A* as

$$R_A : P \to P : X \mapsto XA$$

Lemma 2 *Both left multiplication and right multiplication by A are linear transformations, and they commute: that is $L_A R_B = R_B L_A$*

Now given the basis $\{\mathbf{e_i}\}$, the linear transformations can be represented by matrices muliplying the coordinate vectors from the left.

Theorem 3 *L_A is represented by the matrix $aI + cF$ corresponding to A. Also, R_A is represented by the transposed matrix $(aI + cF)^t$. If $A \in \overline{P}$, then A is represented by $aI + \bar{c}F$ and so R_A is represented by the transposed $(aI + \bar{c}F)^t$.*

proof: Consider L_1, by which we mean left multiplication by $\mathbf{e_1}$. So the basis elements are transformed by L_1: $\mathbf{e_0} \mapsto \mathbf{e_1 e_0} = \mathbf{e_1}$, $\mathbf{e_0} \mapsto \mathbf{e_1 e_1} = \mathbf{e_0}$, $\mathbf{e_0} \mapsto \mathbf{e_1 e_2} = \mathbf{ie_3}$, $\mathbf{e_0} \mapsto \mathbf{e_1 e_3} = -\mathbf{ie_2}$. This corresponds to the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

This matrix is cE_1. Right multiplication by $\mathbf{e_1}$ gives us the matrix $\overline{c}E_1$. Note that this is the transpose of cE_1. In the same way we see that left multiplication by $\mathbf{e_i}$ gives rise to the matrix cE_i and right multiplication by $\mathbf{e_i}$ gives rise to $cE_i^t = \overline{c}E_i$.

Now every element $A \in P$ is a unique linear combination $a_i e_i$, so the representation of A is the same linear combination of the cE_i : Namily, $a_i cE_i$. Thus L_A is represented by $a_i cE_i$. Also R_A is represented by $(a_i cE_i)^t = a_i \overline{c}E_i$ $\square$

Baylis(1999) in his textbook, describes an algebraic system in which the paravectors are uniquely written as linear combinations $a_i\mathbf{e_i}$. The choice of the basis implicitly defines an isomorphism

$$\Phi : P \to \mathbb{C}^4 : X = a_i\mathbf{e_i} \mapsto (a_0, a_1, a_2, a_3)^t \tag{29}$$

Whereas if v is a nonnull vector in complex Minkowski space M the evaluation map is an isomorphism

$$\Theta_v : P \to \mathbb{C}^4 : X = a_i cE_i \mapsto Xv \tag{30}$$

proof: See Gottlieb(1999), Theorem 6.9.

Note that if $v = e_0$, then $\Phi = \Theta_{\mathbf{e_0}}$. This follows since $a_i cE_i \mapsto a_i cE_i e_0 = a_i e_i = (a_0, a_1, a_2, a_3)^t$.

Now Baylis(1999) exposes Electromagnetism in the language of Clifford algebra on three generators, which is isomorphic to the biquaternions. We therefore want to adopt his notation as much as possible, while thinking of our matrices replacing his abstract symbols. Our goal is not to advance a new method of calculation so much as to use the matrices to understand the underlying geometry which arises out of an electromagnetic field. Our point of departure, explained in Gottlieb(1999), is to consider the linear transformation $F : M \to M$ which arises in the Lorentz law. Choosing an orthonormal basis $\{e_i\}$ with respect to the Minkowski metric $<,>$, we consider those linear transformations $F : M \to M$ which are anti symmetric with respect to the Minkowski metric. That is

$$\overline{F} = -F \tag{31}$$

where $\overline{F}$ is defined by the equation

$$< \overline{F}v, w > = < v, Fw > \tag{32}$$

That is, $\overline{F}$ is the adjoint of F with respect to the Minkowski metric.

The matrix form which these skew symmetric operators take is that of the matrix in equation (5). Now our point of view is the following. The Minkowski metric is taken to be the primary object, based on the work of Robb(1936), who produced a protocol as to how the metric could be measured by means of light rays. Also we take linear transformations as primary, since they are the morphisms of the category of vector spaces and linear transformations. This point of view is suggested by the successes of algebraic topology. We do not hesitate to employ other inner products on M or the convenience of tensors, but we will never change the underlying sign of the metric to match the sign suggested by the wave equation.

For example, the Minkowski metric $<,>$ is taken to be defined independently of a choice of basis. Having chosen an orthonormal basis, we of course represent F by a matrix of type (5). We can also define based on this choice of a basis, the Euclidean metric $<,>_C$ and the Hermitian metric $<,>_H$. Now the adjoints of F are defined by $< F^t v, w >_C = < v, Fw >_C$ and $< F^\dagger v, w >_H = < v, Fw >_H$. On the matrix level, t is the transpose and $\dagger$ is the complex conjugate transpose. The adjoint $\overline{(aI + cF)} = aI - cF$.

Now Clifford algebras have certain involutions which are defined ad hoc and are very useful in Clifford algebras. These are *Clifford conjugation (or spatial reversal)* denoted by $p \mapsto \overline{p}$ and *hermitian conjugation (or reversion)* denoted by $p \mapsto p^\dagger$

Now starting from a real linear transformation, the definition of cF is seems just a clever trick that is very useful. If you take a point of view that the biquaternions are the key concept, then the complexification trick is explained. Similarly, the Clifford conjugation and hermitian conjugation are tricks from the point of view of Clifford algebra, but from the matrix point of view they correspond to the Minkowski adjoint and the Hermitian adjoint respectively.

Another important involution is *complex conjugation* denoted by $p \mapsto p^{\overline{c}}$ The bar is reserved for Clifford conjugation, and the * is reserved for the Hodge dual , see equation (6). Now the space of 4×4 matrices is a tensor product of P and $\overline{P}$. Now transpose $t : P \to \overline{P}$ and complex conjugation $\overline{c} : P \to \overline{P}$. The composition of complex conjugation and transpose is the Hermitian adjoint $\dagger : P \to P$. Both the Minkowski adjoint and the Hermitian adjoints reverse the orders of multiplication because they are both adjoints. Thus $\overline{AB} = \overline{B}\ \overline{A}$ and $(AB)^\dagger = B^\dagger A^\dagger$. Complex conjugation is important on the underlying vector space $\mathbb{C}^4$ where it is given in our new notation by $(a_i e_i)^{\overline{c}} = a_i^{\overline{c}} e_i$. Using the isomorphism Φ, see (29), it can be grafted onto the biquaternions as $(a_i \mathbf{e_i})^{\overline{c}} = a_i^{\overline{c}} \mathbf{e_i}$. This definition is not used in Baylis(1999) probably because it has no simple relation to the ring structure of the biquaternions. However, the natural extension of complex conjugate to matrices, given by $(a_{ij})^{\overline{c}} = (a_{ij}^{\overline{c}})$, is important on the matrix level. For our matrix description of biquaternions, we get

$$(aI + cF)^{\overline{c}} = a^{\overline{c}} I + \overline{c} F \tag{33}$$

If we apply transpose to this equation we get

$$(aI + cF)^\dagger = a^{\overline{c}} I + cF^\dagger \tag{34}$$

The problem for a strictly biquaternion approach is that the right side of (33) is no longer in the the biquaternions P, and although $\overline{c}$ does preserve order of products, it also is not linear over the complex numbers.

The composition of Hermitian conjugation with Clifford conjugation leads to an product order preserving automorphism $+ : P \to P : (aI + cF) \mapsto (aI + cF)^+ = a^{\bar{c}}I - cF^\dagger$. This automorphism is called the grade automorphism.

Our Hodge dual, see (6), eventually differs with that of Baylis. We have two choice for the Hodge dual, either the matrix in (6) or its negative. With our definition $cF := F - iF*$ we agree with Baylis' choice of orientation $e_1e_2 = ie_3$ via the left regular representation. Then extending the definition of dual to cF we use $cF^* = (F - iF^*)^* := F^* - iF^{**} = F^* + iF = i(F - iF^*) = icF$. So in P it looks like the definition should be multiplication by i. If we used the alternative definition of $(\bar{c}F)^* = -i\bar{c}F^*$ this gives multiplication by $-i$. But P agrees with the orientation $e_1e_2e_3 = i$. Baylis chooses to define the Hodge dual by $p^* := pe_3e_2e_1$, which is effectively multiplying by $-i$.

5 Biparavectors

Let $cE_0 := I$, and let $\bar{c}E_0 := I$ Then $\{cE_i \otimes cE_j\}$ forms a vectorspace basis for $P \otimes P$. Similarly, $\{cE_i \otimes \bar{c}E_j\}$ forms a vectorspace basis for $P \otimes \overline{P}$. Now we will call any element in $P \otimes P$ a *biparavector*. So a biparavector is a sum of 16 terms give by $a_{ij}cE_i \otimes cE_j$. Now a bivector gives rise to a linear transformation $T : P \to P : X \mapsto a_{ij}cE_iXcE_j$ In fact, every linear transformation can be represented this way as a biparavector. Now by **Theorem 3** the linear transformation given by the biparavector $\{a_{ij}cE_i \otimes cE_j\}$ is represented by the matrix $\{a_{ij}cE_i\bar{c}E_j\}$

Theorem 4 *Let $A = ac_{ij}E_i \otimes cE_j$ be a biparavector representing a linear transformation M, of $\mathbb{C}^4$, repersented by the matrix again called M.*
a) Then $M = a_{ij}cE_i\bar{c}E_j$ where $a_{ij} = \frac{1}{4}trace(McE_i\bar{c}E_j)$.
b) Let $x \in \mathbb{C}^4$ correspond to $X \in P$ by $Xe_0 = x$. Then $Mx = A(X)e_0 := a_{ij}cE_iXcE_je_0$

proof: follows from remark 4. The trace from part a) follows from **Remark 4, c)**. The trace of the product of two matrices $tr(AB) = tr(BA)$ gives rise to an innerproduct on the vector space of matrices. The the above cited remark shows that the basis in question is orthnormal with respect to the inner product given by $\frac{1}{4}trace$.

Corollary 5 *a) The stress-energy tensor $T_F = \frac{1}{2}cF\bar{c}F$, so the equivalent biparavector is $cF \otimes cF^\dagger$*
b) A real proper Lorentz transformation $e^F = e^{\frac{1}{2}cF}e^{\frac{1}{2}\bar{c}F}$ So the equivalent biparavector is $e^{\frac{1}{2}cF} \otimes e^{\frac{1}{2}cF^\dagger}$

proof: The key point is that $\bar{c}F^t = cF^\dagger$

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