THE EVALUATION MAP AND HOMOLOGY

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1. INTRODUCTION

There has been no serious study of the effects of the evaluation $\omega\colon X^X \to X$ on homotopy and homology groups. Perhaps this is due to the difficulty in calculating the homotopy and homology groups of X^X. However, ω and the generalized evaluation map $\hat{\omega}\colon X^X \times X \to X$ are "universal" for many problems. Thus each action of a group on X must factor through $\hat{\omega}$. Also, each boundary map in the homotopy exact sequence of a fibration must factor through ω_* [2]. Moreover, ω plays an important role in the study of evaluation maps of mapping spaces other than X^X. The generalized Whitehead product for suspensions is closely related to ω_* on homotopy groups [4].

Because of the various roles played by ω, information about ω will be valuable in the study of topology. This is especially true in cases where ω_* is trivial on some homotopy groups, for in these cases we can conclude that the transgression homomorphism in the homotopy exact sequence of a fibration is trivial. We shall show that the homology homomorphism ω_* is trivial for spaces whose homology groups satisfy a certain simple criterion.

In Section 2, we establish our notation and record some facts about the evaluation map. In Section 3, we study the effects of ω_* on the homology groups of X with Z_p or rational coefficients. Our main results tell us that $H_*(X; Z_p)$ is a nontrivial tensor product of two modules when $\omega_*(\lambda) \neq 0$, where λ denotes a primitive element in $H_*(X^X; Z_p)$, or when $\omega_*(\lambda)$ satisfies a certain condition. For many spaces, we can thus show that ω_* is zero in low dimensions.

Finally, in Section 4, we show that for suspensions ω_* is almost always zero. We use the generalized evaluation map $\hat{\omega}\colon X^X \times X \to X$ to study ω_*. We find that $\hat{\omega}_*$ for X is closely related to $\hat{\omega}_*$ for ΣX, even though ω_* for ΣX is almost always zero.

2. PRELIMINARIES

We shall let $L(X, Y; k)$ denote the space of maps homotopic to $k\colon X \to Y$ and furnished with the compact-open topology. We also denote $L(X, X; 1_X)$ by X^X.

Definition. The *generalized evaluation map* $\hat{\omega}$ is the map $\hat{\omega}\colon X^X \times X \to X$ given by $\hat{\omega}(f, x) = f(x)$. Let x_0 be a base point of X. Then the *evaluation map* $\omega\colon X^X \to X$ is defined by $\omega(f) = f(x_0)$. The *composition map* $\mu\colon X^X \times X^X \to X^X$ is given by $\mu(f, g) = f \circ g$.

Now ω is always continuous, and $\hat{\omega}$ and μ are continuous if X is locally compact. However, both $\hat{\omega}$ and μ carry singular simplices to singular simplices, and

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therefore both induce homomorphisms on homology groups (see [1, p. 274] for results that can be used to prove this).

The composition map μ makes X^X into an H-space. Let G be a ring with a unit. If x and y are in $H_*(X^X; G)$, we let xy denote $\mu_*(x \otimes y)$. The multiplication is associative (since composition is associative), but not necessarily commutative. With the diagonal map $\Delta: X^X \to X^X \times X^X$ inducing a co-algebra structure on $H_*(X^X; F)$, where F is a field, the group $H_*(X^X; F)$ becomes a Hopf algebra.

For $\lambda \in H_*(X^X; G)$ and $x \in H_*(X; G)$, we shall denote $\hat{\omega}_*(\lambda \otimes x)$ by $\lambda \cdot x$. Thus $H_*(X; G)$ is a left $H_*(X^X; G)$-module. The commutative diagram

$$\begin{array}{ccc} X^X \times X^X & \xrightarrow{\mu} & X^X \\ \downarrow{\scriptstyle 1\times\omega} & & \downarrow{\scriptstyle \omega} \\ X^X \times X & \xrightarrow{\hat{\omega}} & X \end{array}$$

gives us the equality $\alpha \cdot \omega_*(\beta) = \omega_*(\alpha\beta)$, where α and β are in $H_*(X^X; G)$. Thus $\omega_*: H_*(X^X; G) \to H_*(X; G)$ is an $H_*(X^X; G)$-module homomorphism. Finally, since $\omega = \hat{\omega} \mid (X^X \times x_0)$, we see that $\lambda \cdot 1 = \hat{\omega}_*(\lambda \otimes 1) = \omega_*(\lambda)$.

If $\lambda \in H_n(X^X; G)$, we can regard λ as a homomorphism

$$\lambda: H_i(X; G) \to H_{i+n}(X; G)$$

given by $x \to \lambda \cdot x$.

3. THE EVALUATION MAP AND HOMOLOGY

In this section, we study the effect of an element $\lambda \in H_*(X^X; Z_p)$ on the homology $H_*(X; Z_p)$. First we introduce certain concepts and subspaces useful for our investigation. Next, we show that if λ is primitive and $\omega_*(\lambda) \neq 0$, then $H_*(X; Z_p)$ splits as a nontrivial tensor product of Z_p-modules (Theorem 1). Then we weaken the hypothesis on λ and show that we still get $H_*(X; Z_p)$ as a nontrivial tensor product of Z_p-modules (Theorem 2), although the splitting is not as nice as in Theorem 1. Then we briefly consider cohomology and prove that if the Euler-Poincaré number is not zero, then the lowest dimension for a nontrivial ω_* must be even (Theorem 3).

Let p represent either a prime number or ∞. We restrict ourselves to Z_p coefficients; by Z_∞ we denote rational coefficients.

Let us consider a fixed $\lambda \in H_n(X^X; Z_p)$. Suppose $x \in H_i(X; Z_p)$. We say that x has *depth* d if there exists an element $y \in H_{i-dn}(X; Z_p)$ such that $\lambda^d \cdot y = x$ and $\lambda^{d+1} \cdot z \neq x$ for every $z \in H_*(X; Z_p)$.

Now we shall define a collection of vector spaces $\{A_q^d\}$ with the properties

(a) $A_q^d \subseteq H_q(X; Z_p)$,

(b) every element of A_q^d has depth d,

(c) $H_q(X; Z_p) \cong A_q^q \oplus A_q^{q-1} \oplus \cdots \oplus A_q^0$,

(d) $\lambda(A_q^d) \supseteq A_{q+n}^{d+1}$, where $\lambda(A_q^d)$ denotes the image of A_q^d under λ.

First we set $A_0^0 = H_0(X; Z_p)$. Now suppose we have defined A_q^d. Let K be the subspace of A_q^d such that every $x \in K$ is mapped by λ onto an element $\lambda \cdot x$ of depth greater than $d + 1$. Then $A_q^d = K \oplus Q$, where Q is a subspace complementing K. We define $A_{q+n}^{d+1} = \lambda(Q)$, and we define A_{q+n}^0 by setting it equal to some subspace Q of $H_{q+n}(X; Z_p)$ such that

$$H_{q+n}(X; Z_p) \cong \lambda(H_q(X; Z_p)) \oplus Q .$$

We can easily verify that conditions (a), (b), (c), (d) are satisfied.

Now let N be a positive integer. Define a subspace M_N of $H_*(X; Z_p)$ by letting M_N be the direct sum of all the A_q^d with $d = 0, N, 2N, \cdots$. Let $[\lambda]$ be the subspace of $H_*(X^X; Z_p)$ generated by $1, \lambda, \lambda^2, \cdots$, and let $[\lambda]_N$ be the subspace of $[\lambda]$ generated by $1, \lambda, \cdots, \lambda^{N-1}$. Define a homomorphism $\psi\colon [\lambda]_N \otimes M_N \to H_*(X; Z_p)$ by setting $\psi(\lambda^i \otimes x) = \lambda^i \cdot x$; that is, let ψ be the restriction of ω_* to $[\lambda]_N \otimes M_N$.

LEMMA. *The homomorphism ψ is surjective.*

Proof. Let $x \in H_q(X; Z_p)$. Then x has a unique decomposition as a sum of elements $x = x_0 + x_1 + \cdots + x_d$, where $x_i \in A_q^i$ for each i. Each x_i has the form $\lambda^r \cdot y_i$, where $0 \leq r < N$ and $r \equiv i \pmod N$, and where y_i is some element in M_N. Thus $x_i = \psi(\lambda^r \otimes y_i)$, and therefore x must be in the image of ψ.

Let T be the map that interchanges the two factors of a product space; that is, define $T\colon A \times B \to B \times A$ by $T(a, b) = (b, a)$. Then T induces T_* on homology groups, and

$$T_*(\alpha \otimes x) = (-1)^{\dim \alpha \cdot \dim x} x \otimes \alpha .$$

We have the commutative diagram

$$
\begin{array}{ccc}
 & & (X^X \times X^X) \times (X \times X) \\
 & \overset{\Delta \times \Delta}{\nearrow} & \big\downarrow 1 \times T \times 1 \\
X^X \times X & \xrightarrow{\ \Delta\ } & (X^X \times X) \times (X^X \times X) \\
\big\downarrow \hat{\omega} & & \big\downarrow \hat{\omega} \times \hat{\omega} \\
X & \xrightarrow{\qquad \Delta \qquad} & X \times X ,
\end{array} \tag{1}
$$

where Δ stands for the diagonal map.

Now suppose that

$$\Delta_*(x) = \sum (x_i \otimes x_i') \quad \text{and} \quad \Delta_*(\lambda) = \sum (\lambda_i \otimes \lambda_i'),$$

where $x \in H_q(X; Z_p)$ and $\lambda \in H_n(X^X; Z_p)$. Then we see from the diagram (1) and the property of T that

$$\Delta_*(\lambda \cdot x) = \sum_{i,j} [((-1)^{b_j c_i} \lambda_i \cdot x_j) \otimes (\lambda_i' \cdot x_j')], \tag{2}$$

where $b_j = \dim x_j$ and $c_i = \dim \lambda_i'$.

Replacing X with X^X and $\hat{\omega}$ with μ, we have a similar diagram to the one above, and it gives us the well-known properties of Hopf algebras.

Now let us assume that λ is primitive, in other words, that

$$\Delta_*(\lambda) = (\lambda \otimes 1) + (1 \otimes \lambda).$$

If in addition λ has even dimension n, then

$$\Delta_*(\lambda^N) = \sum_{i=0}^{N} \left[\binom{N}{i} \lambda^i \otimes \lambda^{N-i} \right], \tag{3}$$

where $\binom{N}{i}$ is a binomial coefficient. On the other hand, if n is odd, then

$$\Delta_*(\lambda^2) = (\lambda^2 \otimes 1) + (1 \otimes \lambda^2).$$

Note that λ^2 has even dimension and is primitive, so that the previous formulas are relevant.

We shall let R stand for the smallest positive integer such that $\lambda^R = 0$. When such a number R exists, it must equal p^k for some k in case λ has even dimension, and it must equal $2p^k$ in case λ has odd dimension. Now let K stand for the smallest positive integer such that $\omega_*(\lambda^K) = 0$. Note that, since $\Delta_* \omega_* = (\omega_* \otimes \omega_*) \Delta_*$, we have the equation

$$\Delta_*(\lambda^N \cdot 1) = \sum_{i=0}^{N} \left[\binom{N}{i} (\lambda^i \cdot 1) \otimes (\lambda^{N-i} \cdot 1) \right]$$

for each N, if λ has even dimension, and the equation

$$\Delta_*(\lambda^2 \cdot 1) = (\lambda^2 \cdot 1 \otimes 1) + (1 \otimes \lambda^2 \cdot 1)$$

if λ is odd-dimensional. Then, just as before, $K = p^k$ or $K = 2p^k$ for some k, depending on whether λ is even- or odd-dimensional.

THEOREM 1. *Let $\lambda \in H_n(X^X; Z_p)$ be primitive, and let* p *be a prime number or* ∞. *Then*

$$H_*(X; Z_p) \cong \omega_*[\lambda] \otimes M_K \quad \textit{as } Z_p\textit{-modules.}$$

Proof. Let K be the smallest integer such that $\omega_*(\lambda^K) = \lambda^K \cdot 1 = 0$. Then $\omega_*[\lambda]$ is clearly isomorphic to $[\lambda]_K$. In view of the Lemma, we need only show that ψ is one-to-one, in other words, that if $x \in M_K$, then $\lambda^{K-1} \cdot x = 0$ only when $x = 0$. We divide the proof into three cases: first, n is even or $p = 2$; second, both n and p are odd; and finally, $p = \infty$.

Let $\Delta_*(x) = \sum (x_i \otimes x_i')$. Then, in the first case, it follows from equations (2) and (3) that

$$\Delta_*(\lambda^N \cdot x) = \sum_{i,j} \left[\binom{N}{i} (\lambda^i \cdot x_j) \otimes (\lambda^{N-i} \cdot x_j') \right].$$

Now $K = p^k$ for some k, and x has depth equal to a multiple of p^k, say mp^k. Thus $x = \lambda^{mK} \cdot y$ for some $y \in H_*(X; Z_p)$. Now assume that $0 = \lambda^{K-1} \cdot x = \lambda^{(m+1)K-1} \cdot y$. Then

$$(4) \qquad \sum_{i,j} \left[\binom{mK + K - 1}{i} (\lambda^i \cdot y_j) \otimes (\lambda^{mK+K-1-i} \cdot y_j') \right] = 0 .$$

Now

$$\binom{mK + K - 1}{K - 1} = \binom{mp^k + p^k - 1}{p^k - 1} \not\equiv 0 \pmod p$$

(see [7, p. 5]). Also, $\lambda^{K-1} \cdot 1 \neq 0$. Thus the term

$$\binom{mK + K - 1}{K - 1} (\lambda^{K-1} \cdot 1) \otimes (\lambda^{mK} \cdot y)$$

is not zero, and it appears in equation (4). It must be cancelled by a linear combination of terms of the form $(\lambda^{K-1-j} \cdot z) \otimes (\lambda^{mK+j} \cdot z')$, where j is a positive integer and z and z' are elements of $H_*(X; Z_p)$. Thus $\lambda^{mK} \cdot y$ must be a linear combination of terms of the form $\lambda^{mK+j} \cdot z'$. Thus, for some u in $H_*(X; Z_p)$, we have the relation $x = \lambda^{mK} \cdot y = \lambda^{mK+1} \cdot u$. Hence x has depth greater than mK, and this contradicts our assumption.

Now assume that n and p are odd. Then

$$\Delta_*(\lambda^{2N+1}) = \sum_i \binom{N}{i} [(\lambda^{2i+1} \otimes \lambda^{2N-2i}) + (\lambda^{2i} \otimes \lambda^{2N-2i+1})] .$$

As before, we assume that $\lambda^{K-1} \cdot x = 0$, where $x \in M_K$. Then x has depth mK, and there exists a y such that $x = \lambda^{mK} \cdot y$. Now we apply Δ_* to both sides of the equation $0 = \lambda^{mK+K-1} \cdot y$. In the expansion of the right-hand side, we have the term

$$\binom{(m+1)K/2 - 1}{\frac{K}{2} - 1} (\lambda^{K-1} \cdot 1) \otimes (\lambda^{mK} \cdot y) .$$

Recall that in this case $K = 2p^k$ for some k. Thus the binomial coefficient is nonzero in Z_p, and also $\lambda^{K-1} \cdot 1 \neq 0$, so that the term above is not zero. Therefore, as in the other case, the term must cancel against a linear combination of terms of the form $\lambda^{K-1-j} \cdot z \otimes \lambda^{mK+j} \cdot z'$ with $j > 0$. Hence x must have depth greater than mK; this again contradicts our hypothesis and establishes the theorem for finite values of p.

If $p = \infty$, we have rational coefficients, in which case $K = \infty$ and M_∞ consists of elements of depth zero. Applying the previous methods, we easily see that $\lambda x = 0$ implies x has depth greater than zero.

Remark. Compare this theorem with results of J. Milnor and J. C. Moore [5, Theorem 4.4]. Their hypothesis requires ω_* to be injective, and their conclusions concern tensor products of $H_*(X^X; Z_p)$-modules.

We may apply Theorem 1 in the following situation. Let α be in $\pi_i(X^X; 1_X)$; then $h_p(\alpha)$ in $H_i(X^X; Z_p)$ is primitive. Here we denote by h_p the composition

$$\pi_i(B) \xrightarrow{h} H_i(B; Z) \longrightarrow H_i(B; Z_p).$$

We define $G_i(X)$ to be the image of the homomorphism $\omega_*: \pi_i(X^X) \to \pi_i(X)$.

COROLLARY 1. *Suppose* $x \in G_i(X)$ *and* $h_p(x) \neq 0$. *Then* $H_*(X; Z_p) \cong A \otimes B$, *where* A *has one generator in dimensions* $0, i, 2i, \cdots, (p-1)i$ *if* i *is even, and one generator in dimensions* 0 *and* i *if* i *is odd.*

This corollary is a restatement of Theorems 4-1 and 4-4 of [3].

Suppose that in Theorem 1 we relax the condition that $\lambda \in H_*(X^X; Z_p)$ is primitive. We shall say that $x \in H_*(X; Z_p)$ is *decomposable* if x is the sum of terms of the form $\alpha \cdot y$, where $y \in H_i(X; Z_p)$ and $\alpha \in H_j(X^X; Z_p)$, and where α and y have positive dimension. That is, if $\tilde{A}$ is the subring of $H_*(X^X; Z_p)$ consisting of all elements of dimension greater than zero, and if $\tilde{B}$ is the subring of all elements of higher dimension in $H_*(X; Z_p)$, then the decomposable elements are $\hat{\omega}_*(\tilde{A} \otimes \tilde{B})$. An *indecomposable* element is one that is not decomposable.

THEOREM 2. *Suppose that* λ *is an element of* $H_n(X^X; Z_p)$ *and that* $\omega_*(\lambda)$ *is indecomposable and not zero. Then*

$$H_*(X; Z_p) \cong [\lambda]_p \otimes M_p \text{ as } Z_p\text{-modules if } n \text{ is even,}$$

and

$$H_*(X; Z_p) \cong [\lambda]_2 \otimes M_2 \text{ as } Z_p\text{-modules if } n \text{ is odd.}$$

Proof. First let n be even. Let $x \in M_p$. We must show that $\lambda^{p-1} \cdot x \neq 0$ if $x \in M_p$ and $x \neq 0$. We proceed as in the proof of Theorem 1.

To choose a basis for $H_*(X; Z_p)$, we begin by choosing a basis for the decomposable elements; then we add $\omega_*(\lambda)$, and then we fill out the basis with indecomposable elements. Let $y_1, \cdots, y_i, \cdots$ be the basis so chosen.

We claim that if $\lambda \cdot x = 0$, then x has depth $d \equiv -1 \pmod p$. To see this, suppose $x = \lambda^d y$. Then $0 = \lambda \cdot x = \lambda^{d+1} \cdot y$. Thus $\Delta_*(\lambda^{d+1} \cdot y) = 0$. Since

$$\Delta_*(y) = (y \otimes 1) + (1 \otimes y) + \sum_i (y_i \otimes y_i')$$

and

$$\Delta_*(\lambda) = (\lambda \otimes 1) + (1 \otimes \lambda) + \sum_j (\lambda_j \otimes \lambda_j'),$$

we have the term $(d+1)(\lambda \cdot 1) \otimes (\lambda^d \cdot y)$ in the expansion of $\Delta_*(\lambda^{d+1} \cdot y)$. This term must cancel with a linear combination of terms of the form $y_i \otimes (\lambda^{d+1} \cdot y_i')$ and $(\lambda_j \cdot y_i) \otimes (\lambda_j' \cdot y_i')$. The expansions of the terms $\lambda_j \cdot y_i$ in terms of the basis $y_1, \cdots, y_i, \cdots$ do not contain the basis element $\omega_*(\lambda) = \lambda \cdot 1$. Thus terms of the type $(\lambda_j \cdot y_i) \otimes (\lambda_j' \cdot y_i')$ do not cancel with $(d+1)(\omega_*(\lambda)) \otimes (\lambda^d \cdot y)$. Thus only a linear combination of terms of type $y_i \otimes (\lambda^{d+1} \cdot y_i')$ cancels $\omega_*(\lambda) \otimes (\lambda^d \cdot y)$. Hence $\lambda^d \cdot y = x$ is equal to a linear combination of terms of the type $\lambda^{d+1} \cdot y_i'$. Thus $x = \lambda^{d+1} \cdot u$ for some u, and hence x has depth greater than d; this contradicts our assumption. Only when $d + 1 \equiv 0 \pmod p$ do we escape a contradiction. Thus we have shown that $\lambda \cdot x = 0$ only if $d \equiv -1 \pmod p$.

If x has depth $d \not\equiv -1 \pmod p$, then $\lambda \cdot x$ has depth $d+1$. (For otherwise, $\lambda \cdot x = \lambda \cdot v$, where v has depth greater than d. Thus $x - v \neq 0$ and $x - v$ has depth d. Now $\lambda \cdot (x - v) = \lambda \cdot x - \lambda \cdot v = 0$; by the preceding paragraph, this implies that $d \equiv -1 \pmod p$, and this is a contradiction.)

Now suppose $x \in M_p$. Then x has depth $d \equiv 0 \pmod p$, and hence $\lambda \cdot x$, which is not zero, has depth $d \equiv 1 \pmod p$. Therefore $\lambda^2 \cdot x$ has depth $d \equiv 2 \pmod p$ and is not zero, and this process continues until $\lambda^{p-1} \cdot x \neq 0$. Thus $\psi(\lambda^{p-1} \otimes x) \neq 0$; hence ψ is one-to-one, and hence, by the Lemma, ψ is an isomorphism. Thus, if n is even, the theorem is proved.

In the case n is odd, the proof runs similarly. First we show that the relation $\lambda \cdot x = 0$ implies $d \equiv 1 \pmod 2$. The remainder of the proof is identical with that above.

Let us consider the homomorphisms induced on cohomology by ω and $\hat{\omega}$. We shall assume that $H^*(X; Z_p)$ and $H^*(X^X; Z_p)$ are of finite type, in other words, that the i^{th} cohomology groups are finitely generated, for each i. This condition occurs frequently (for example, when $\pi_*(X)$ is of finite type and X is strongly simple, that is, n-simple for all n). We can use the Federer spectral sequence to show that $\pi_*(X^X)$ is of finite type; then Theorem 20 on page 510 of [6] tells us that the homology is of finite type.

Consider the homomorphism

$$\hat{\omega}^*: H^*(X; Z_p) \to H^*(X^X; Z_p) \otimes H^*(X; Z_p).$$

If $x \in H^*(X; Z_p)$, it is easily seen that $\hat{\omega}^*(x) = (1 \otimes x) + (\omega^*(x) \otimes 1) + \sum (\lambda_i \otimes x_i)$.

Now recall that $\mu: X^X \times X^X \to X^X$ is the composition map. We see that $\hat{\omega} \circ (1 \times \omega) = \omega \circ \mu$. Thus we have the formula

$$\mu^*(\omega^*(x)) = (1 \otimes \omega^*(x)) + (\omega^*(x) \otimes 1) + \sum (\lambda_i \otimes \omega^*(x_i)).$$

We shall use the theorem of Milnor and Moore [5, Proposition 4.21] that tells us that a primitive decomposable element α in a connected Hopf algebra over Z_p with associative commutative multiplication has the form $\alpha = \beta^p$.

THEOREM 3. *Suppose* $H_*(X; Z)$ *is finitely generated and* $H_*(X^X; Z)$ *has finite type. Suppose* $\chi(X) \neq 0$. *Let* n *be the smallest positive dimension such that* $\omega_*: H_n(X^X; Z_p) \to H_n(X; Z_p)$ *is nontrivial. Then* n *is even.*

Proof. Assume n is odd. Suppose $\lambda \in H_n(X^X; Z_p)$ and $\omega_*(\lambda) \neq 0$. Together with the hypothesis that $\chi(X) \neq 0$, Theorem 1 implies that λ is not primitive.

Let us consider ω^* in cohomology. For positive dimensions less than n, ω^* is trivial, by duality. For dimension n, there exists an $x \in H^n(X; Z_p)$ such that $\omega^*(x)$ is not zero and is decomposable. Now

$$\begin{aligned}\mu^*(\omega^*(x)) &= (1 \otimes \omega^*(x)) + (\omega^*(x) \otimes 1) + \sum (\lambda_i \otimes \omega^*(x_i)) \\ &= (1 \otimes \omega^*(x)) + (\omega^*(x) \otimes 1),\end{aligned}$$

since $\omega^*(x_i) = 0$ by hypothesis. Thus $\omega^*(x)$ is primitive and decomposable, and hence $\omega^*(x) = \beta^p$, by the theorem of Milnor and Moore. Since n is odd, n/p must be odd and p must be odd. Thus β is an odd-dimensional cohomology class, so that

$\beta^2 = 0$. Therefore $\beta^p = 0$, and this contradicts the fact that $\omega^*(x) = 0$. Thus n must be even.

4. THE EVALUATION MAP AND SUSPENSIONS

If X is in an H-group, then $X^X = X_0^X \times X$ (where X_0^X is the subspace of X^X that preserves base points), and the evaluation map is the projection onto X. Thus ω_* is surjective. On the opposite extreme we have the suspensions ΣX. Here, for the most part, ω_* is trivial. However, the homology homomorphism $\hat{\omega}_*$ for appropriate ΣX is closely related to the homology homomorphism $\hat{\omega}_*$ for appropriate X.

Let $\Sigma\colon \tilde{H}_n(X; Z_p) \xrightarrow{\cong} \tilde{H}_{n+1}(\Sigma X; Z_p)$ be the suspension isomorphism. Let CX be the cone over X, and let $C_+ X$ and $C_- X$ be the upper and lower hemispheres of ΣX. Then we have the diagram

$$\tilde{H}_n(X) \xleftarrow[\partial]{\cong} H_{n+1}(C_+ X, X) \xrightarrow[i_*]{\cong} H_{n+1}(\Sigma X, C_- X) \xleftarrow[j_*]{\cong} H_{n+1}(\Sigma X, *),$$

where i and j are inclusions and ∂ is the boundary homomorphism. Now $\Sigma = j_*^{-1} i_* \partial^{-1}$.

We define $S\colon X^X \to \Sigma X^{\Sigma X}$ by letting $S(f)\colon \Sigma X \to \Sigma X$ be the suspension of the map $f\colon X \to X$. Then we have a commutative diagram

$$\begin{array}{ccc}
H_*(X^X) \otimes H_*(X) & \xrightarrow{\hat{\omega}_*} & H_*(X) \\
\uparrow {\scriptstyle 1\otimes\partial} & & \uparrow {\scriptstyle \partial} \\
H_*(X^X) \otimes H_*(C_+ X, X) & \longrightarrow & H_*(CX, X) \\
\downarrow {\scriptstyle 1\otimes i_*} & & \downarrow {\scriptstyle i_*} \\
H_*(X^X) \otimes H_*(\Sigma X, C_- X) & \longrightarrow & H_*(\Sigma X, C_- X) \\
\uparrow {\scriptstyle 1\otimes j_*} & & \uparrow {\scriptstyle j_*} \\
H_*(X^X) \otimes H_*(\Sigma X, *) & \longrightarrow & H_*(\Sigma X, *)\ .
\end{array}$$

The horizontal homomorphisms are induced by the map $X^X \times \Sigma X \to \Sigma X$ given by $(f, x) \to Sf(x)$. Thus we have the commutative diagram

$$\begin{array}{ccc}
H_*(X^X) \otimes \tilde{H}_*(X) & \xrightarrow{\hat{\omega}_*} & \tilde{H}_*(X) \\
\downarrow {\scriptstyle S_*\otimes\Sigma} & & \downarrow {\scriptstyle \Sigma} \\
H_*(\Sigma X^{\Sigma X}) \otimes H_*(\Sigma X) & \xrightarrow{\hat{\omega}_*} & H_*(\Sigma X)\ .
\end{array}$$

This proves the following theorem.

THEOREM 4. $S_*(\lambda) \cdot (\Sigma x) = \Sigma(\lambda \cdot x)$ *for* $x \in \tilde{H}_*(X; Z_p)$.

COROLLARY 2. *The kernel of* S_* *is contained in the annihilator of* $\tilde{H}_*(X; Z_p)$.

The corollary follows from the fact that Σ is an isomorphism.

Although the homomorphism $\hat{\omega}_*$ on $\tilde{H}_*(X)$ is closely related to $\hat{\omega}_*$ on $\tilde{H}_*(\Sigma X)$, the next theorem shows that ω_* is almost always trivial.

THEOREM 5. *If* $\omega_*: H_n(\Sigma X^{\Sigma X}; Z_p) \to H_n(\Sigma X; Z_p)$ *is nontrivial, then* ΣX *is a rational homology* n*-sphere and a* Z_p*-homology* n*-sphere; and if* p *is odd, then* n *must be odd.*

Proof. Let n be the smallest positive dimension such that there exists a $\lambda \in H_n(\Sigma X^{\Sigma X}; Z_p)$ satisfying the condition $\omega_*(\lambda) \neq 0$. Now suppose

$$\Delta_*(\lambda) = (\lambda \otimes 1) + (1 \otimes \lambda) + \Sigma (\lambda_i \otimes \lambda_i').$$

By noting that $\Delta_*(x) = (1 \otimes x) + (x \otimes 1)$ for each $x \in \tilde{H}_*(\Sigma X; Z_p)$, we obtain the equation

$$\Delta_*(\lambda \cdot x) = (\lambda \cdot x \otimes 1) + (1 \otimes \lambda \cdot x) + (\lambda \cdot 1 \otimes x) \pm (x \otimes \lambda \cdot 1)$$
$$+ \sum_i ((\lambda_i \cdot 1 \otimes \lambda_i' \cdot x) + (\lambda_i \cdot x \otimes \lambda_i' \cdot 1)).$$

But $\lambda_i \cdot 1 = \omega_*(\lambda_i) = 0$, since $\dim \lambda_i < \dim \lambda$. Thus

$$\Delta_*(\lambda \cdot x) = (\lambda \cdot x \otimes 1) + (1 \otimes \lambda \cdot x) + (\omega_*(\lambda) \otimes x) \pm (x \otimes \omega_*(\lambda)).$$

Hence $\lambda \cdot x$ is not primitive or zero unless $(\omega_*(\lambda) \otimes x) \pm (x \otimes \omega_*(\lambda)) = 0$. This can only occur if $\omega_*(\lambda) = kx$ for some $k \in Z_p$, where either $p = 2$, or n is odd (use Theorem 3). Thus ΣX is a homology Z_p-sphere and hence must be a rational homology sphere.

In the case of homotopy groups, the homomorphism $\omega_*: \pi_*(\Sigma X^{\Sigma X}) \to \pi_*(\Sigma X)$ is usually not trivial. It is related to the Hopf construction, and it is found in a long exact sequence where the following homomorphism is the generalized Whitehead product with $1_{\Sigma X}$. This is shown by George Lang in his thesis [4].

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