THE LAW OF VECTOR FIELDS

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1. Introduction.

When I was 13, the intellectual world began opening up to me. I remember discussing with my friends the list of Sciences which we were soon to be offered. There was Chemistry, Biology, Geology, Astronomy, and Physics. We knew what Chemistry was. It was the study of Chemicals. Biology was the study of life, Geology studied rocks and Astronomy the stars and planets. But what was Physics ?

Now, many years later, I can answer the question. Physics is that branch of Science which is described by Mathematics. Too naive ? Well any one line description of a subject must be naive, unless the subject is a one line subject. But this one liner has a great deal of truth to it and is in fact a point of view which provokes a lot of thought.

For example, we can use it as a tool to examine two famous statements about Science and Mathematics. The first one says that the other Sciences should eventually "mature" and become as mathematical as Physics. Whether one can eventually describe life mathematically or not is the key to whether biology will "mature" or not. I would guess that biology will never mature, but if it did then it would be Physics according to my definition. So the maturing process is replaced by a devouring process in which the "maturing" Science is in fact being eaten up. That seems to be the case for parts of Chemistry.

The second statement is really only a phrase. "The unreasonable effectiveness of Mathematics in Physics." Our one line point of view that Mathematics is the tool of Physics suggests that it is not unreasonable that Mathematics is effective in Physics. The success of Physics goes hand in hand with the effectiveness of Mathematics in Physics. And *yet* there is something unusual about the way Mathematics describes Physics. Something fantastically beautiful. This something is the existence of a few general laws or principles which imply mathematically most of the known facts of Physics. By that I mean the following. If we regard the objects of physical studies as real things, they have the dual attribute of not being well-defined in the mathematical sense. But since they are real we can try to measure them or measure their interactions. Then measurements give rise to numbers, and then to equations and then mathematical statements. These statements combine logically and produce derived statements which must agree with experiment. That is new experiments verify the mathematical relationships derived by logic. What is remarkable is the fact that there are a few general laws which imply most of these mathematical statements.

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What do I mean ? Look at the example of Newton's Law of Gravitation. Here is a mathematical statement of great simplicity which implies logically vast number of phenomena of incredible variety. For example, Galileo's observation that objects of different weights fall to the Earth with the same acceleration, or Kepler's laws governing the motion of the planets, or the daily movements of the tides are all due to the underlying notion of gravity. And this is established by deriving the the mathematical statements of these three classes of phenomena from Newton's Law of Gravity. The mathematical relationship of falling objects, planets, and tides is defined by the beautiful fact that they follow from Newton's Law by the same kind of argument. The derivation for falling objects is much simpler than the derivation of the tides, but the general method of proof is the same. This process of simple laws implying vast numbers of physical theorems repeats itself over and over in Physics, with Maxwell's Equations and Conservation of Energy and Momentum. To a mathematician, this web of relationships stemming from a few sources is beautiful. But is it unreasonable ? This is the question I ask. Are there mathematical Laws ? That is are there theorems which imply large numbers of known, important, and beautiful results by using the same kind of proofs ?

At thirteen I already knew what Mathematics was. Mathematics was the study of numbers. The future held many surprises for me, not the least of which was the fact I liked the subject. As my mathematical maturity grew I began to take part in the debate over the question: Is Mathematics invented or discovered ? I came to understand that while the basic objects are invented as objects of intrinsic interest, certain relationships seemed to be discovered, such as the Pythagorean theorem or the Euler-Poincare number. By discovered I mean that they are *there* in some real physical sense. The Euler-Poincare number is a simple concept in its definition, but it keeps appearing over and over again in one theorem after another and has so many different definitions. Surely it is a real thing. Finally I found myself agreeing with Morris Kline, Mathematics is a method. We deal with things that are well-defined, we know exactly what we are talking about even though they usually aren't real. The physicist on the other hand is dealing with things that are usually real but he doesn't completely know what he is talking about.

If Mathematics is just a method which can be applied to any abstract well-defined situation that one choose to invent then what determines what is good Mathematics and what is not ? A difficult question which I will not get into. I only want to make the point that Mathematics is seemingly very malleable in a way that Physics is not. Physics must always refer to results given by nature. Yet the mathematical structure of Physics, the way so much is derived from a few principles or laws is unreasonably beautiful. Can we find laws and principles in Mathematics with the same wide implications as in Physics ? If we approach Mathematics as if it were Physics and ask questions with the same philosophy that the physicists have, would the structure of Mathematics begin to resemble that of Physics ? In other words, is the unreasonable mathematical beauty of Physics a property of nature, or only the consequence of a method of asking questions ?

To illustrate what I would call a law in Mathematics I will consider the following list of five famous theorems.

The Fundamental Theorem of Algebra
The Intermediate Value Theorem

The Gauss-Bonnet Theorem
The Poincare-Hopf Index Theorem
The Brouwer Fixed Point Theorem

Now suppose these five theorems implied some theorem, say Theorem A for definiteness. At first glance we would think that Theorem A had a sophisticated proof using roots of polynomials and curvature and vector fields and fixed points. But the value of A could diminish if it were discovered that A really only depended upon the Poincare-Hopf Index Theorem in an elementary way. Now consider the opposite situation. Suppose Theorem A implied the five famous theorems of the list. Then those results are related in the same way that the motion of planets, the daily rise and fall of tides, and falling objects are related. These are related because they follow from the Law of Gravity and the theorems in the list are related because they follow from Theorem A.

Now a very interesting thing happens. Contrary to the case where the list implies A, as the proof simplifies, the relationship of the theorems on the list becomes stronger. And if the proofs of each one of the theorems on the list have the same pattern their relationships are as closely bound as those of the physical consequences of the Law of Gravity.

In the case of a physical law one requires not only that the known phenomena are described but also that the law predicts new facts which then must be verified by experiment. So we should require of Theorem A that it should generalize the theorems it proves and in addition leads to new results of interest. This additional requirement does not seem to be too difficult to satisfy, for if a common proof can get all the list it seems natural that the method should result in unexpected results and generalizations. No, the difficulty won't be in the proofs or the generalizations, it would be in finding Theorem A, if it exists.

There is an important difference between the physical law and the mathematical law. The Law of Gravity is not known to be true. It derives its importance from the fact that so many physical laws can be derived from it. However it might happen that the Law of Gravity could be replaced by another law which implies all that the Law of Gravity implies and even more. In fact, that is what happened when Einstein's General Theory of Relativity replaced Newtonian Mechanics. On the other hand a mathematical law must be *true*. In fact if Theorem A were false, it would imply not only all the theorems on the list but all the theorems ever imagined. So the theorems on the list would stand in a relationship more eternal than that of tides and orbits and falling objects.

So how would we guess mathematical laws? I would think a good thing to try is to ask questions that physicists would ask. Every time that nature presents the physicist with a phenomenon he asks how is this derived from what we know? This is not so natural a question in mathematics, since every time a new result is discovered, the proof itself is the answer to the question. But the physicist when he is faced with a coincidence will try to find a theory which explains it. And so when Maxwell found that the constant in his equations which explained Electromagnetism turned out also to be the speed of light he proposed that light was Electromagnetic in origin.

So what is such a "physical question" in mathematics? Here is an example. The Euler-Poincare Number, or Euler Characteristic as it is frequently called, is a remarkable mathematical object. It appears in an unreasonably large number of theorems and arguments and contexts, many of which seem to have little relationship to each other except for the Euler-Poincare Number itself. It comes into fixed point theorems and curvature and in group theory and combinatorics. And the physical question is "why?". And the physical assumption is that there is a law which explains all these appearances, that they are not coincidental. So we should look for a theorem which derives all these myriad results containing the Euler-Poincare Number.

Here is another kind of physical question. An observer looking at Physics observes that in recent times the description of nature seems to be taking a Quantum Mechanical turn. If the observer is far enough away from Physics it seems to be that things that should be spread over space can be also thought of as being localized at discrete points and that integers seem to play some role in this process. So where are similar things in Mathematics? Of course in differential equations and in the spectra of operators in Hilbert Spaces. But where else?

Consider a vector field defined on a manifold or Euclidean space. This vector field has singular points, that is zeroes or undefined points. Associated to each singular point is an integer called the index. If we consider the vector field as evolving in time, these singular points seem to move in paths. Sometimes singular points collide and sometimes one will split into several. The index is preserved in these interactions. So if one "particle" splits into two the sum of the indices of the new particles equals the index of the old particle. Or if two collide their indices add up to the index of the new particle. No particle can magically appear or disappear unless it has index zero.

We say that the index of a vector field is the sum of the indices of all the singular points. We will introduce an equation involving the index of a vector field and the Euler-Poincare Number. It is an equation which gives the index as the difference of the Euler-Poincare number and the index of a vector field defined on a smaller dimensional space. So it is like an inductive definition of index. Hence the Quantum like properties of the index follow from the equation. The equation implies the list of theorem mentioned above. So Theorem A really exists! The proofs are easy and all of the same pattern, so the method of proof leads to generalized theorems and completely new results. Many new results. All related to onec another by their similar proofs. Results involving the Euler-Poincare Number. The theorem explains in part why the Euler-Poincare Number is so frequently seen. For these reasons I have called it the Law of Vector Fields.

Let M be a compact manifold with or without boundary ∂M. Let V be a continuous vector field on M with no zeros on ∂M. Then

$$Ind(V) + Ind(\partial_- V) = \chi(M). \tag{1}$$

Here $\chi(M)$ denotes the Euler-Poincare number of M, and $\partial_- V$ is a vector field defined on part of ∂M as follows. Let $\partial_- M$ be the subset of ∂M containing all m so that $V(m)$ points inward. Then $\partial_- V$ is the vector field on $\partial_- M$ given by first restricting V to ∂M, and then projecting $V \mid \partial M$, using the outward pointing vector field N, onto its component field tangent to ∂M. This is denoted ∂V. Then $\partial_- V = \partial V \mid \partial_- M$.

What is the general method of proof? First select a relevant vector field V. Then determine $IndV$ and $Ind\partial_{-}V$. Thus if we restrict ourselves to closed manifolds we see that $Ind\partial_{-}V$ is always zero since the boundary is empty. Hence the index of a vector field on a closed manifold equals the Euler-Poincare Number. This is the Poincare-Hopf Index Theorem.

If we have a map from $\mathbb{R}^n \to \mathbb{R}^n$ it determines a vector field on $\mathbb{R}^n$ by assigning to each point x the vector based at x and parallel to $f(x)$. Then the index of a zero of this vector field is the degree of the map of the boundary of a neighborhood into $\mathbb{R}^n - 0$ given by restriction of f. The map on the boundary can be homotopied to a certain map into the unit sphere. Then $Ind\,\partial_{-}V$ is equal to the coincidence number of this certain map and the Gauss Map. If we apply this to maps from intervals of the real line to the real line we get the Intermediate Value Theorem. If we apply it to polynomial maps of the Complex Numbers we get the Fundamental Theorem of Algebra.

If we have a map $f : M \to \mathbb{R}^n$ where M is an n-manifold in $\mathbb{R}^n$, we consider the vector field on M which assigns to each point m the tangent vector $m - f(m)$. The zeroes of this vector field are also the fixed points of f and the index of a zero of this vector field is the fixed point index of f. A zero of $\partial_{-}V$ which points inward is a coincidence point of the Gauss Map and the "Gauss Map" of the vector field. Now consider a map of the Unit Disk to itself. The vector field defined above always sticks outside so the $Ind\partial_{-}V$ term is zero. Thus the index of the vector field is one and so the map must have a fixed point. This is the Brouwer Fixed Point Theorem. It has a vast generalization for bodies in space. If f is a map of a body into Euclidean space so that the vector field is never tangent to the boundary, then the field either points inward or outward on each component of the boundary. We see that $Ind\,\partial_{-}V$ is equal to the sum of the Euler-Poincare Numbers pointing inward. Thus if the difference of this number and $\chi(M)$ is not zero, there must be a fixed point. For even dimensional M that implies that $\chi(M)$ is zero if f has a fixed point.

Suppose that $f : M \to \mathbb{R}^n$ is a smooth map with no critical points on the boundary of the n-dimensional manifold M. Let V be a vector field on $\mathbb{R}^n$. Then there is a pullback vector field $f^*(V)$ on M. The vector field law gives the following equation.

$$Ind\, f^*V = \sum_i w_i v_i + (\chi(M) - deg\hat{N}). \tag{2}$$

Here w_i is the winding number of f restricted to ∂M about the ith zero of V, and v_i is the index of the ith zero. Also $\hat{N}$ is the Gauss Map. If f is an immersion we can pullback a V with no zeroes and $Indf^*(V) = 0$. Thus the degree of the Gauss Map equals $\chi(M)$. This is the essence of the Gauss-Bonnet Theorem. Equation (2) simplifies in odd dimension since the Gauss Map of any immersion of a closed oriented manifold equals one half of the Euler-Poincare Number. This is an old fact of H. Hopf, which incidently follows from the Vector Field Law itself.

If we let M denote a body in $\mathbb{R}^n$ and p a point not on the boundary of M, we can define a vector field on M by letting the base of the vector be the point m and the head of the vector be the point p. Then the index of this vector field V is zero if p is not in M and is $(-1)^{n-1}$ if p is inside M. On the other hand the indices of $\partial_- V$ are related to the centers of curvature of the boundary of M at the zero. In fact if m is a zero of $\partial_- V$ then in the generic situation the index at m is ± 1 where the sign depends on the parity of the number of the centers of curvature between p and m along the normal line to ∂M at m. Thus we get an equation relating the Euler-Poincare Number and the curvature of the boundary of a body. This equation has infinite variations when we replace the point p by higher dimensional planes and V by a vector field arising from the projection of M into this plane.

The fact that curvature can be calculated by observing the flip flops of the sign of the index of a chosen test vector field at a point of a manifold imbedded in Euclidean space leads one to conjecture that perhaps the intrinsic curvature has a vector field which permits its calculation. If this is so we have the interesting possibility that curvature, which is the soul of the Theory of General Relativity, has a natural quantization mathematically in terms the index of vector fields.

There is more than I can write about here. Interesting vector fields arise when a Lie Group acts on a manifold, when a smooth real valued mapping has a gradient, when a system of charged particles satisfies Coulomb's Law. We can add the Borsuk Ulam Theorem and the Jordan Separation Theorem to the list of theorems proved by "Theorem A", using the same simple method. I conjecture that there are more theorems to be added to the list beyond those which have occurred to me. So I will finish by indulging in a few speculations. Our friends the physicists seem ready to conjecture at the slightist provocation, so maybe we should make a few "physical" conjectures.

Why does the Vector Field Law have so many consequences? I guess that it does because calculations on the boundary determines the quantity in the interior. It is like a integral Stokes Theorem in this respect and Stokes Theorem itself has law like properties. But in addition it is based on a very simple concept. Pointing inside or equivalently, entering inside. The concept of entering inside thus gives rise to the index and the attendant quantization. Since it is such a simple and common thing, pointing inside, it might be expected to occur everywhere. Think of a subset of space. As a set we have the concept of inside and outside. When we add Topology to the subset we get the notion of boundary, and if we add a little bit of Geometry we can consider direction and motion and we get the idea of moving inside or outside.

Is Mathematics unreasonably effective in Physics? I would guess not. First because I predict that there are other laws in Mathematics as sweeping as the Vector Field Law. As we think about which concepts have numerous consequences we will probably discover that a good definition, say like that of function, also acts as a law. Mathematics is flexible enough so that theorems can be made into definitions and conversely. And second, because maybe Physics doesn't all flow out of a few mathematical laws. It seems to as far as my amateur education can tell, and great men such as Richard Feynman advocate the idea, but just recently I have seen some books which argue that this is just an illusion. I hope not because Physics as Feynman teaches it is a beautiful mathematical subject. But even if Physics were not really as mathematically beautiful as it seems, if we could teach Mathematics as Feynman teaches Physics it would be a wonderful thing.

Finally I predict that half the occurrences of the Euler-Poincare Number in mathematics will be be explained by the Vector Field Law. And the Vector Field Law, based as it is on the physically intuitive ideas of dimension and of pointing inside and outside, which gives a mathematical picture in which continuous fields of vectors give rise to singular points which behave very much like particles under the evolution of the field, particles whose quantum number is the index, this Law of Vector Fields, I predict, has absolutely nothing to do with Physics!

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