THE RATLIFF–RUSH IDEALS IN A NOETHERIAN RING: A SURVEY[1]

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Let R be a Noetherian ring, and let I be a regular ideal in R. (By ring we mean a commutative ring with unity, and by a regular ideal we mean one that contains a nonzerodivisor.) The ideals of the form $(I^{n+1} :_R I^n) = \{x \in R \mid xI^n \subseteq I^{n+1}\}$ increase with n. The union of this family,

$$\widetilde{I} = \bigcup_{n=1}^{\infty} (I^{n+1} : I^n) = \{x \in R \ : \ xI^n \subseteq I^{n+1} \text{ for some } n\},$$

is an interesting ideal first studied by Ratliff and Rush [RR]. We call $\widetilde{I}$ the *Ratliff–Rush ideal associated to* I, and we say that I is a *Ratliff–Rush* ideal if $I = \widetilde{I}$.

[1]This article is partly based on Heinzer's presentation at the Colorado Springs conference and Lantz's talks at the Commutative Algebra Workshop, University of Missouri, Columbia, June 1991, and in the Purdue commutative algebra student seminar, July 1991.

In this mainly expository article, we survey some general properties of Ratliff–Rush ideals. Much of what we discuss here is taken from the recent articles [HLS] and [HJLS].

1. Several ways to realize Ratliff–Rush ideals. In [HLS] the behavior of the Ratliff-Rush property with respect to certain ideal- and ring-theoretic operations is considered and indications are given of how one might determine whether or not a given ideal is Ratliff–Rush.

There are a number of ways to think of $\widetilde{I}$. Given regular ideals I, J in a Noetherian ring R, it is possible that $I \neq J$ but $I^n = J^n$ for all $n >> 0$. Two of the very nice properties observed in [RR] about the Ratliff–Rush ideal $\widetilde{I}$ are:

Theorem 1. *Let I be a regular ideal in a Noetherian ring. Then:*

- *(1)* [RR, Theorem 2.1] *$\widetilde{I}$ is the unique largest ideal J of R with the property that $I^n = J^n$ for all $n >> 0$, i.e., $\widetilde{I}$ is the largest ideal sharing the same high powers with I; and*
- *(2)* [RR, Remark 2.3.2] *for all sufficiently large n, $I^n = \widetilde{I^n}$, i.e., I^n is a Ratliff–Rush ideal.*

Example. Let $R = k[x, y]$ be a polynomial ring in two variables over a field k, and let I be the ideal $(x^4, x^3y, xy^3, y^4)R$. Then $\widetilde{I} = (x, y)^4R = (I, x^2y^2)R$ since $I^2 = (x, y)^8R$. But $x^2y^2 \notin I$, so I is not a Ratliff–Rush ideal. Note that in this example I^n is Ratliff–Rush for each $n \geq 2$.

In the case of a domain, there is another way to approach the associated Ratliff–Rush ideal:

Theorem 2. [HLS, Fact 2.1] *If I is an ideal in a Noetherian domain R, then $\widetilde{I}$ is the intersection of the contractions to R of the extensions of I to* *the rings in its blowup* $\mathcal{B}(I) = \{R[I/x]_P : x \in R - 0, P \in \operatorname{Spec} R[I/x]\}$*:*

$$\widetilde{I} = \bigcap \{IS \cap R : S \in \mathcal{B}(I)\} \quad .$$

This is used in both [HLS] and [HJLS].

The passage from an ideal I to its associated Ratliff–Rush ideal $\widetilde{I}$ may be thought of as a weak "closure" operation on the set of regular ideals I in a Noetherian ring R. It is true that $I \subseteq \widetilde{I}$, $\widetilde{\widetilde{I}} = \widetilde{I}$, and if $I \subseteq J \subseteq \widetilde{I}$, then $\widetilde{J} = \widetilde{I}$. But it is not true in general that from $I \subseteq J$ it need follow that $\widetilde{I} \subseteq \widetilde{J}$, so we have refrained from calling $\widetilde{I}$ the Ratliff–Rush closure of I.

A natural question is: How common are Ratliff–Rush ideals? A principal regular ideal, indeed any ideal generated by a regular sequence, is Ratliff–Rush. An integrally closed ideal in a Noetherian domain is Ratliff–Rush.

But on the other hand: We observe in [HLS] that, even in the monoid rings $k[[t^a, t^b, \dots]]$, where k is a field and $a, b, \dots$ are positive integers with greatest common divisor one, the family of Ratliff–Rush ideals has the following properties: Products of Ratliff–Rush ideals, even a power of a Ratliff–Rush ideal or a principal multiple of a Ratliff–Rush ideal, need not be Ratliff–Rush [HLS, (1.11)]. Even in a polynomial ring in two variables over a field, a power of a Ratliff–Rush ideal need not be Ratliff–Rush [HJLS, Example 6.1 (E3)]. K. N. Raghavan has shown the existence of an example of an ideal generated by a system of parameters in a two-dimensional local domain (of course, not a Cohen–Macaulay ring) which is not Ratliff–Rush [HJLS, Example 1.2].

In [HLS], we studied mainly the case of a one-dimensional local domain (R, M). In this context, the intersection of the rings in the blowup of a nonzero ideal is a ring between the domain and its integral closure, so to study Ratliff–Rush ideals, we could study such intermediate rings. To begin with, Ratliff and Rush remarked that if every ideal in a domain is either principal or integrally closed, then each ideal is Ratliff–Rush. We turned this statement somewhat inside out:

Theorem 3. [HLS, Theorem 2.8] *Let R be a one-dimensional local domain. Then every Ratliff–Rush ideal is either principal or integrally closed iff there are no rings properly between R and its integral closure.*

Then we displayed a one-dimensional local domain in which every Ratliff–Rush ideal is either principal or integrally closed, but in which there are nonzero ideals that are not Ratliff–Rush [HLS, Example 2.10(ii)].

A concept related to passing between an ideal and its associated Ratliff–Rush ideal is that of a reduction: for ideals $J \subseteq I$, J is a *reduction* of I if $JI^n = I^{n+1}$ for all $n >> 0$ [NR]. The *reduction number* of I with respect to the reduction J is the smallest n for which $JI^n = I^{n+1}$. In this situation, it follows that $J^kI^n = I^{n+k}$ for all $k \geq 0$. If J is a reduction of I, then I is integral over J, and the Rees rings $R[Jt] \subseteq R[It]$ have the property that $R[It]$ is integral as an extension ring of $R[Jt]$.

A regular ideal is a reduction of its associated Ratliff–Rush ideal. But in general, the condition on ideals $J \subseteq I$ that $\widetilde{J} = \widetilde{I}$, i.e., $J^n = I^n$ for $n >> 0$, is stronger than that J is a reduction of I. For example, if $R = k[x, y]$ is a polynomial ring in two variables over a field k and $J = (x^2, y^2)R$, then J is a reduction of $I = (x^2, xy, y^2)R$, but $J = \widetilde{J}$ is properly contained in $I = \widetilde{I}$.

If (R, M) is a local ring and I is an M-primary ideal, then for all sufficiently large n, the length $\lambda(R/I^n)$ is a polynomial in n of degree the dimension of R, called the *Hilbert polynomial* of I and denoted P_I. The integral closure I' of I is the largest ideal of which I is a reduction. The

Hilbert polynomials P_I and $P_{I'}$ have the same highest degree coefficient, i.e., the same multiplicity; while P_I and $P_{\widetilde{I}}$ are the same polynomial, i.e., all the coefficients are the same. Indeed, $\widetilde{I}$ is the largest ideal having the same Hilbert polynomial as I.

Example. If k is a field and R is the subring $k[[t^3, t^4, t^5]]$ of the formal power series ring $k[[t]]$, and if $I = (t^3, t^4)R$ and $J = t^3R$, then I is properly contained in $M = (t^3, t^4, t^5)R$, and $\widetilde{I} = M$ since $I^2 = M^2$; we have $P_I(n) = P_M(n) = 3n - 2$. On the other hand, for the reduction J of I and of M, we have $P_J(n) = 3n$.

We return to the topic of Hilbert polynomials in Section 3.

We would like to indicate why it is true that all high powers of a proper regular ideal I are Ratliff–Rush ideals. There is a nice presentation of this in [Mc, Chapter VIII]. If $(I^{n+1} : I) = I^n$ for all sufficiently large n, then $(I^{n+h} : I^h) = I^n$ for all positive integers h. For we have $(I^{n+2} : I^2) = ((I^{n+2} : I) : I)$, etc. Let $x \in I$ be a nonzerodivisor. Using the Artin–Rees lemma on the descending chain $I^n \cap xR$, and the equality $x(I^n : x) = I^n \cap xR$, it follows that for large n one has $(I^{n+1} : I) = I^n$. This in turn implies that for large m and any h, one has $(I^{m(h+1)} : I^{mh}) = I^m$, which means $I^m = \widetilde{I^m}$.

2. The associated graded ring. Given an ideal I in a commutative ring R, an interesting ring construction is the associated graded ring of I in R:

$$\mathrm{G}(I) = R/I \oplus I/I^2 \oplus \cdots \oplus I^n/I^{n+1} \oplus \cdots .$$

This ring has a natural grading by the nonnegative integers and is presented as a homomorphic image of the Rees ring of I or of the extended Rees ring of I as follows:

$$\mathrm{G}(I) \cong R[It]/IR[It] \cong R[t^{-1}, It]/(t^{-1})R[t^{-1}, It] \quad .$$

The existence of zero-divisors of a certain form in $\mathrm{G}(I)$ is related to whether I and the powers of I are Ratliff–Rush ideals: Let $\mathrm{G}(I)^+$ denote the homogeneous ideal of $\mathrm{G}(I)$ generated by the elements of positive degree $I/I^2 \oplus I^2/I^3 \oplus \cdots$. An element $a \in R - I$ is in $\widetilde{I}$ iff the image a^* of a in R/I annihilates some power of $\mathrm{G}(I)^+$. Thus $I = \widetilde{I}$ iff there fails to exist such an element. Using that $\widetilde{I^2} = \bigcup(I^{2n+2} : I^{2n})$, we see that if $I = \widetilde{I}$, then I^2 is properly contained in $\widetilde{I^2}$ iff there exists $a \in I - I^2$ such that a^* in I/I^2 annihilates some power of $\mathrm{G}(I)^+$.

These are illustrations of the general:

Fact 4. [HLS,(1.2)] *There exists a nonzerodivisor in* $\mathrm{G}(I)^+$ *iff* $I^n = \widetilde{I^n}$ *for all positive integers* n *(i.e., all the powers of* I *are Ratliff–Rush ideals).*

Another way to phrase this is that $\widetilde{I}$ is the preimage in R of the annihilator in R/I (regarded as the degree-0 piece of $\mathrm{G}(I)$) of $(\mathrm{G}(I)^+)^n$ for sufficiently large n. Interpreting this in terms of graded local cohomology, $H^0_{\mathrm{G}(I)^+}(\mathrm{G}(I))_0 = \widetilde{I}/I$, and more generally, $H^0_{\mathrm{G}(I)^+}(\mathrm{G}(I))_n = (\widetilde{I^{n+1}} \cap I^n)/I^{n+1}$; so it follows that the first n for which I^{n+1} is not Ratliff–Rush (if there is such an n) is the first n for which $H^0_{\mathrm{G}(I)^+}(\mathrm{G}(I))_n$ is nonzero. In particular, all powers of I are Ratliff–Rush ideals iff $\mathrm{grade}(\mathrm{G}(I)^+) > 0$. Cf. [HJLS, (1.3)]

Returning to the question of how common Ratliff–Rush ideals are, we remark that if an M-primary ideal I in a Cohen–Macaulay local ring (R, M) has reduction number at most one (i.e., if there exists a reduction J of I

which is generated by a system of parameters and which is such that the equation $JI = I^2$ holds), then $\mathrm{G}(I)$ is Cohen–Macaulay [V], so if an M-primary ideal has reduction number at most one, then all its powers are Ratliff–Rush.

3. Ratliff–Rush ideals and Hilbert Polynomials. Ratliff and Rush showed that $\widetilde{I}$ is the largest ideal J for which $J^n = I^n$ for sufficiently large n; so if I is primary for the maximal ideal in a local ring, $\widetilde{I}$ is the largest ideal containing I and having the same Hilbert polynomial. This shows us that the "coefficient ideals" introduced in [Sh1] are all Ratliff–Rush ideals:

Definition. Let I be an M-primary ideal in a quasi-unmixed local ring (R, M) of dimension d. Write the Hilbert polynomial of I in the form:

$$P_I(n) = e_0(I)\binom{n+d-1}{d} - e_1(I)\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(I) \quad ,$$

so that the coefficients $e_j(I)$ are integers. Then, for each m in $\{0, \ldots, d\}$, the *e_m-ideal associated to I*, denoted $I_{\{m\}}$, is the unique largest ideal J containing I for which $e_j(J) = e_j(I)$ for $j = 0, \ldots, m$.

In particular, $I_{\{0\}}$ is the integral closure I' of I and $I_{\{d\}}$ is the Ratliff–Rush ideal $\widetilde{I}$ associated to I.

Theorem 5. [HJLS, Corollary 3.12] *If (R, M) is a two-dimensional quasi-unmixed local domain and if I is an M-primary ideal, then high powers of I are e_1-ideals if and only if $\mathcal{B}(I)$ is Cohen–Macaulay (i.e., all the rings in the blowup of I are Cohen–Macaulay).*

More generally, it is shown in [HJLS] that, if (R, M) is a two-dimensional quasi-unmixed analytically unramified local domain and I is an M-primary ideal, then if the model $\mathcal{B}(I)^{(1)}$ is constructed so that its affine pieces are the localizations at the height-one primes of the affine pieces of $\mathcal{B}(I)$, then $I_{\{1\}}$ is the contraction of the extension of I to $\mathcal{B}(I)^{(1)}$. We also show that the other coefficient ideals associated to I are the contractions of the extensions of I to other models.

During the talk at the Colorado Springs Conference, Larry Levy asked whether a Ratliff–Rush M-primary ideal of a local ring (R, M) has the property that $\lambda(R/I^n)$ is given by the Hilbert polynomial of I for *all* positive integers n? It can be seen that this is not true in general. For example, if k is a field and (R, M) is the one-dimensional local domain $k[[t^3, t^4]]$, then M and even all its powers are Ratliff–Rush ideals, but the Hilbert polynomial $P_M(n) = 3n - 3$ of M does not satisfy $P_M(1) = \lambda(R/M)$.

Then after the talk at the Colorado Springs Conference, Tom Marley asked if a converse to Larry Levy's question is true, i.e., if I is an M-primary ideal in a local ring (R, M) and if $P_I(n) = \lambda(R/I^n)$ for all positive n, does it follow that I is a Ratliff–Rush ideal? It is well known that if I is an M-primary ideal with this property in a one-dimensional Cohen–Macaulay local ring (R, M), then I is a stable ideal (cf. Section 4), so I and all its powers are Ratliff–Rush [L, Corollary 1.6]. But it is noted in [HJLS, Example 6.1, (E1)] that an example of Sally in [Sy2, Section 5] shows that this need not be true for an M-primary ideal of a 2-dimensional regular local ring.

4. Every nonzero ideal Ratliff–Rush. Motivated by a comment of Ratliff and Rush, we classify in [HLS, Section 3] the Noetherian domains in which every nonzero ideal is Ratliff–Rush.

It is not hard to see that a ring in which every nonzero ideal is Ratliff–

Rush is a one-dimensional domain, and that it is enough to look locally. So we consider a one-dimensional local domain (R, M). Such an domain is called *stable* iff each of its nonzero ideals is stable, that is, has a principal reduction and reduction number at most one. A stable ideal is Ratliff-Rush, so in a stable domain all ideals are Ratliff-Rush. The converse also holds:

Theorem 6. [HLS, Theorem 3.9] *If every nonzero ideal in a one-dimensional local domain is Ratliff–Rush, then the domain is stable.*

And these conditions are almost equivalent to the condition that every module between the domain and its integral closure is a ring. (Roger Wiegand has pointed out to us that there is one exceptional case, i.e., the case in which (R, M) is a one-dimensional local domain with integral closure R' such that R/M is the field with two elements and R'/MR' is the direct sum of three copies of the field with two elements.)

Work of Sally and Vasconcelos in [SV1] and [SV2] shows that if the multiplicity of a one-dimensional local domain is two, then the domain is stable; but they also give an example of a stable domain of multiplicity three. Using a result of Rush in [R], we show in [HLS] that, although that example could be generalized somewhat, many of the properties of that example are properties of every stable local domain with multiplicity greater than two. In particular:

Theorem 7. [HLS, Corollary 3.11] *If (R, M) is a stable local domain of multiplicity greater than two, then:*

(1) the integral closure R' of R is local,

(2) the residue field of R' is isomorphic to R/M under the canonical map of R' onto its residue field,

(3) R' is not finitely generated as an R-algebra, and

(4) for each R-subalgebra S of R', the square of the (unique) maximal ideal of S is contained in MS.

Moreover:

(5) (Huneke [HLS, Proposition 3.14]*) the maximal ideal M' of R' is the extension of M to R'.*

5. A question on coefficient ideals. The paper [HJLS] considers the Ratliff–Rush and "coefficient ideal" properties of M-primary ideals, especially in two-dimensional local rings (Cohen–Macaulay or even regular).

Suppose (R, M) is 2-dimensional and Cohen–Macaulay. Narita [Nr] has shown that for any M-primary ideal I, the constant term e_2 of the Hilbert polynomial of I is nonnegative. It is easy to see that this implies that:

Fact 8. [HJLS, Proposition 3.3] *If I is a Ratliff–Rush M-primary ideal and if $e_2(I) = 0$, then I is a first coefficient ideal, i.e., an e_1-ideal.*

The converse is not true in general in a 2-dimensional Cohen–Macaulay local domain (R, M), for it can happen for example that $e_2(M) > 0$ [HJLS, Example 3.5]. We would like, however, to raise the following:

Question. Let I be a Ratliff–Rush M-primary ideal in a two-dimensional regular local ring (R, M), and write

$$P_I(n) = e_0 \binom{n+1}{2} - e_1 n + e_2 \quad .$$

If $e_2 > 0$, must there be an ideal J containing I for which

$$P_J(n) = e_0 \binom{n+1}{2} - e_1 n + f_2$$

where $f_2 < e_2$? In other words, if I is an e_1-ideal, does it follow that the constant term $e_2(I)$ of the Hilbert polynomial of I is 0?

It follows from [Sh1, Theorem 4] that if I is an M-primary ideal in a 2-dimensional Cohen–Macaulay local ring (R, M), then all the powers of I are e_1-ideals iff $\mathrm{G}(I)$ is unmixed. It is shown in [H1, Theorem 2.1] that if I has reduction number one, then $e_2 = 0$; and it is observed in several places that if I is an M-primary ideal in a 2-dimensional regular local ring (R, M), then I has reduction number at most one iff the associated graded ring $\mathrm{G}(I)$ is Cohen–Macaulay, or equivalently iff the Rees algebra $R[It]$ is Cohen–Macaulay [HM, Propostion 2.6], [JV, Theorem 4.1], [Sh2, Corollary 4(f)]. So the question of whether an M-primary ideal I in a 2-dimensional regular local ring (R, M) can have the property that all its powers are e_1-ideals and also have $e_2(I) > 0$ is equivalent to asking whether there can exist such an I for which $\mathrm{G}(I)$ is unmixed and not Cohen–Macaulay.

6. The Ratliff–Rush concept for modules. We close with some comments about possible extensions of the Ratliff–Rush construction to modules.

If E is a module over a commutative ring R, then to each ideal I of R one can associate the submodule of E,

$$\widetilde{I}_E = \bigcup_{n=1}^{\infty} (I^{n+1}E : I^n) = \{ \ a \in E \ : I^n a \subseteq I^{n+1}E \text{ for some } n \ \}.$$

If $E = R$ and I is a regular ideal in R, then the definition reduces to that of the usual Ratliff–Rush ideal associated to I in R. In general, we have that $\widetilde{I}_E$ is a submodule of E and $IE \subseteq \widetilde{I}_E$. Perhaps with certain hypotheses on E and I, it might be of interest to consider those ideals I of R that are Ratliff–Rush with respect to E, where I is tentatively defined to be *Ratliff–Rush with respect to* E if $IE = \widetilde{I}_E$.

Assume that R is a Noetherian ring and E is a finitely generated R-module. In considering the Ratliff–Rush concept on E, there are some natural connections that can be made with the graded ring $\mathrm{G}(I) = R/I \oplus I/I^2 \oplus \dots$ and the graded $\mathrm{G}(I)$-module

$$\mathrm{G}(I) \otimes E = E/IE \oplus IE/I^2E \oplus I^2E/I^3E \oplus \dots \qquad .$$

For example, an element $a \in E - IE$ is in $\widetilde{I}_E$ iff the image a^* of a in E/IE is annihilated by some power of $\mathrm{G}(I)^+$. Thus, $IE = \widetilde{I}_E$ iff there fails to exist such an element a in $E - IE$. Moreover, in analogy with the material in Section 2, if $IE = \widetilde{I}_E$, then I^2E is properly contained in $\widetilde{I^2}_E$ iff there exists $a \in IE - I^2E$ such that a^* in IE/I^2E is annihilated by some power of $\mathrm{G}(I)^+$, and in general one has:

Fact 9. *There exists an element in* $\mathrm{G}(I)^+$ *that is a nonzerodivisor on the module* $\mathrm{G}(I) \otimes E$ *iff* $I^nE = \widetilde{I^n}_E$ *for all positive integers* n *(i.e., all the powers of* I *are Ratliff–Rush with respect to* E*).*

Question. What conditions ensure that all suitably high powers of I are Ratliff–Rush with respect to E

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