IDEAL THEORY IN TWO-DIMENSIONAL REGULAR LOCAL DOMAINS AND BIRATIONAL EXTENSIONS

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0. Introduction.

(0.1) Let $(R, \mathbf{m})$ be a two-dimensional regular local domain with infinite residue field $R/\mathbf{m}$. Associated to an $\mathbf{m}$-primary ideal I in R is its Hilbert polynomial

$$P_I(n) = e_0(I)\binom{n+1}{2} - e_1(I)n + e_2(I) ,$$

the integer-valued polynomial giving the length of the R-module R/I^n for sufficiently large positive integers n. The coefficient e_0 is well known to be a positive integer, the *multiplicity* of I, and in our context, the coefficients e_1 and e_2 are known to be nonnegative integers.

A well-known result of Rees [Re1, Theorem 3.2] implies that for each $\mathbf{m}$-primary ideal I of R the integral closure of I is the unique largest ideal containing I and having the same multiplicity. A result of Shah in [Sh, Theorem 1] implies the existence of a unique largest ideal $I_{\{1\}}$ containing I

and having the same coefficients e_0 and e_1 of its Hilbert polynomial. We call $I_{\{1\}}$ the *e_1-ideal associated with I*. If $I = I_{\{1\}}$, we call I a *first coefficient ideal* or an *e_1-ideal*.

There is an interplay between the internal structure of the ideals in R and the external structure of certain birational extensions of R. In this connection, for an $\mathbf{m}$-primary ideal I, the *blowup* of I,

$$\mathcal{B}(I) = \text{Proj}(R[It]) = \{R[I/a]_P : a \in I - 0,\ P \in \text{Spec}(R[I/a])\}\ ,$$

is the projective model over R (in the sense of [ZS, page 120]) consisting of the local domains containing R that are minimal with respect to domination among all the local domains containing R in which the extension of I is principal. There is a nonempty finite subset of $\mathcal{B}(I)$ consisting of local domains in which I generates an ideal primary for the maximal ideal; each of these local domains is one-dimensional and their intersection D is a one-dimensional semilocal domain called the *first coefficient domain* of I. As noted in [HJL, (1.3) and (3.2)], we have $ID \cap R = I_{\{1\}}$; indeed, since all powers I^n of I have the same blowup, we have $I^n D \cap R = (I^n)_{\{1\}}$, for each positive integer n.

Our goal in this paper is a better understanding of e_1-ideals and their first coefficient domains over a two-dimensional regular local domain. The situation where the first coefficient domain is a semilocal PID is well understood in view of the Zariski theory concerning complete ideals and prime divisors on R (see, e.g., [Z], [ZS, Appendix 5] or [Hu]). In particular, if V is a DVR birationally dominating R which is a spot over R (i.e., in Zariski's terminology a *prime divisor of the second kind* on R; in [A2] a *hidden prime divisor* of R), then the ideals of R contracted from V form a descending chain $\mathbf{m} = \mathbf{a}_0 > \mathbf{a}_1 > \mathbf{a}_2 > \dots$ of complete $\mathbf{m}$-primary ideals, the valuation ideals of R with respect to V. The Zariski theory associates to the prime divisor of the second kind V a unique simple (i.e., not factorable into a product of proper ideals) complete ideal $\mathbf{b}$. One way of characterizing $\mathbf{b}$ is that $\mathbf{b}$ is maximal among $\mathbf{m}$-primary ideals $\mathbf{c}$ of R with the property that all powers of $\mathbf{c}$ are contracted from V. We have $\mathbf{b} = \mathbf{a}_n$ for some n. For example, $\mathbf{b} = \mathbf{m}$ if and only if V is the ord-valuation domain $R[y/x]_{\mathbf{m}R[y/x]}$ where $\mathbf{m} = (x, y)R$. If $n > 0$, then certain of the ideals $\mathbf{a}_0, \dots \mathbf{a}_{n-1}$ are also simple complete ideals. If we label the simple complete ideals in this chain as $\mathbf{b}_0 = \mathbf{m}$, $\mathbf{b}_1$, ..., $\mathbf{b}_s = \mathbf{b} = \mathbf{a}_n$, then Zariski proves that each of the valuation ideals $\mathbf{a}_i$, $i \geq 0$, is a product of powers of $\mathbf{b}_0, \ldots, \mathbf{b}_s$ [ZS, page 392]. For example, if $\mathbf{m} = (x, y)R$ and V is the integral closure of $R[x^2/y^3]_{\mathbf{m}R[x^2/y^3]}$, then $\mathbf{b} = (x^2, xy^2, y^3)R$ is the simple complete ideal associated to V, and $\mathbf{m} = \mathbf{a}_0 > \mathbf{a}_1 = \mathbf{b}_1 = (x, y^2)R > \mathbf{a}_2 = \mathbf{m}^2 > \mathbf{b}$ is the beginning of the chain of ideals $\{\mathbf{a}_i\}$ of R contracted from V. The result of Zariski just mentioned implies that each $\mathbf{a}_i$ is a power product of $\mathbf{m}, \mathbf{b}_1$ and $\mathbf{b}$. More detailed information as to which products of the $\mathbf{b}_j$ are actually contracted from V is given by Noh in [No, Theorem 3.1].

(0.2) To describe the same situation from a different starting point, let I be a complete $\mathbf{m}$-primary ideal of R. The first coefficient domain D of I is then a semilocal PID which is the intersection of the Rees valuation domains of I, i.e., the DVR's on $\mathcal{B}(I)$ that dominate R. In this case, D is uniquely determined as the largest one-dimensional semilocal subdomain E of the fraction field of R having the property that all the powers of I are contracted from E (see (3.4) below). If $V_1, \ldots, V_n$ are the Rees valuation domains of I, then the Zariski theory implies that I has the form

$$(*) \qquad \mathbf{b}_1^{r_1} \ldots \mathbf{b}_n^{r_n}$$

where the r_j are positive integers and $\mathbf{b}_j$ is the simple complete ideal of R associated to V_j, $j = 1, \ldots, n$.

In the present paper we pursue the study of e_1-ideals and first coefficient domains begun in [HJL]. In particular, we consider implications of the Zariski theory for these broader classes of ideals and integral domains. Our objective, only partially realized, is to identify the first coefficient domains over a two-dimensional regular local domain and the ideals of which they are first coefficient domains.

In Section 1 we illustrate with several examples properties that one-dimensional spots birationally dominating a two-dimensional regular local domain may have or fail to have. We also observe in Proposition 1.1 that the condition of being a spot descends from an integral extension.

In Section 2 we consider implications of residual transcendence. As part of Theorem 2.2, we prove that if R is a two-dimensional RLR of characteristic $p > 0$ with algebraically closed residue field and D is a one-dimensional local domain birationally dominating R such that the integral closure of D is a prime divisor on R, then D is the first coefficient domain of an ideal of R.

In Section 3 we examine asymptotic behavior of ideals and implications for first coefficient domains. Suppose $(R, \mathbf{m})$ is a local domain that is the intersection of its localizations at height-one primes and D is a one-dimensional semilocal domain birationally dominating R. In Theorem 3.3 we prove that if J is an $\mathbf{m}$-primary ideal of R such that JD is principal and $J^n D \cap R = J^n$ for each positive integer n, then the first coefficient domain of J is a localization of D. In particular, if D is local, then D is the first coefficient domain of J.

As usual, we abbreviate "regular local domain" by RLR and "rank-one discrete valuation domain" by DVR. The words "local" and "semilocal" include the hypothesis of Noetherian. The symbol $<$ between sets denotes proper inclusion. For an ideal I in a Noetherian domain R the blowup of I and the first coefficient domain of I are defined as in (0.1) above. The Rees valuation domains of I are the localizations of the integral closure of the first coefficient domain of I at its maximal ideals. It is convenient to extend some familiar terminology to the case of rings that are not necessarily Noetherian or that have more than one maximal ideal: A ring D containing a domain R having a unique maximal ideal $\mathbf{m}$ is said to *birationally dominate* R if D is contained in the fraction field of R and for each maximal ideal N of D, $N \cap R = \mathbf{m}$. An extension ring D of a ring R is said to be *affine* over R if D is finitely generated as an algebra over R. We say that a ring D with finitely many maximal ideals is a *semispot* over a subring R if D is a ring of fractions of a ring containing and affine over R. If such a D has only one maximal ideal, then we call it a *spot* over R.

1. One-dimensional birational spots.

We are interested in considering one-dimensional semilocal domains D that birationally dominate a two-dimensional RLR R. A DVR V birationally dominating R is a spot over R if and only if the the residue field of V is not algebraic as an extension of $R/\mathbf{m}$ [A1, Proposition 3, page 336]. An interesting property of such a DVR V (also proved in [A1]) is that the residue field F of V is ruled as an extension field of $R/\mathbf{m}$, i.e., F is obtained as a simple transcendental extension of a field intermediate between $R/\mathbf{m}$ and F. In view of the fact that R is a two-dimensional RLR, it follows that F is a simple transcendental extension of a finite algebraic extension of $R/\mathbf{m}$.

In general, if D is a one-dimensional semilocal domain birationally dominating R, then the integral closure D' of D is a semilocal PID birationally dominating R. If R is complete, then D' is necessarily a semispot over R; but for certain R (such as $R = k[x,y]_{(x,y)k[x,y]}$ where x, y are indeterminates over the field k) there exist DVR's birationally dominating R that are not spots over R (cf., e.g., [HRS]).

We begin by proving a result (Corollary 1.3) that implies that if D is a one-dimensional semilocal domain birationally dominating a two-dimensional RLR R and if the integral closure D' of D is a semispot over R, then D is a semispot over R and D' is a finitely generated D-module.

Proposition 1.1. *Let R be a Noetherian ring, and let V be a semispot over R. Suppose $R \subseteq D \subseteq V$ with D quasilocal and V integral over D. Then D is a spot over R and V is a finitely generated D-module.*

Proof. Since V is a semispot over R, there exist elements $a_1, \dots, a_n \in V$ such that V is a ring of fractions of $R[a_1, \dots, a_n]$. Let $b_1, \dots, b_m$ be the coefficients of monic polynomials over D satisfied by $a_1, \dots, a_n$; set $B = R[b_1, \dots, b_m]$ and $A = B[a_1, \dots, a_n]$. Let Q be the center of D on B, and let A_1 and B_1 be the rings of fractions of A and B at the multiplicative set $B - Q$. Then B_1 is local, with maximal ideal $Q_1 = QB_1$, and A_1 is a finite integral extension of B_1. Hence A_1 has only finitely many maximal ideals. Let $P_1, \dots, P_r$ denote the centers on A_1 of the maximal ideals of V, and let $S = A_1 - (\bigcup_{i=1}^r P_i)$. Since V is a ring of fractions of $R[a_1, \dots, a_n]$, we have $S^{-1}A_1 = V$. Choose $a \in S$ such that a is in each maximal ideal of A_1 distinct from $P_1, \dots, P_r$ (if any — otherwise let $a = 1$). Then $1/a$ is in V and hence is integral over D. Let $c_1, \dots, c_p$ be the coefficients of a monic polynomial over D satisfied by

$1/a$; let (B_2, Q_2) be the localization of $B_1[c_1, \ldots, c_p]$ at the center of D on this ring, and set $A_2 = B_2[1/a, A_1]$.

We claim that $A_2 = V$. To see this, it suffices to show each s in S is a unit in A_2: Assume by way of contradiction that s in S is in a maximal ideal M of A_2. Since A_2 is integral over B_2, we have $M \cap B_2 = Q_2$, and so $Q_1 = M \cap B_1 = (M \cap A_1) \cap B_1$. Since A_1 is integral over B_1, $M \cap A_1$ is maximal in A_1. Moreover, $M \cap A_1$ survives in A_2 , so our choice of a assures that $M \cap A_1$ is the center on A_1 of one of the maximal ideals of V. But this yields $s \in S \subseteq A_1 - (M \cap A_1)$, a contradiction.

Therefore, V is an affine extension of B_2 and hence a finitely generated D-module. Thus, by Artin-Tate [Ku, Lemma 3.3, page 16], D is an affine extension of B_2 and hence a spot over R. $\square$

To extend this result to the case where D has finitely many maximal ideals, we use:

Proposition 1.2. *Let R be an integral domain. Suppose D is an extension domain of R having only finitely many maximal ideals $N_1, \ldots, N_r$ and having the property that D_{N_i} is a spot over R for each $i = 1, \ldots, r$. Then D is a semispot over R.*

Proof. For each maximal ideal N_i of D there is a finite subset T_i of D_{N_i} such that D_{N_i} is a localization of $R[T_i]$. And there is an element s_i of $D - N_i$ for which $s_i T_i \subseteq D$. Let $A = R[(\bigcup_{i=1}^r s_i T_i) \cup \{s_1, \ldots, s_r\}]$. If P_i denotes the center of D_{N_i} on A, then $A \subseteq D \subseteq D_{N_i} = A_{P_i}$; so D is the ring of fractions of A at the complement of the union of the P_i's. $\square$

As an immediate corollary of Propositions 1.1 and 1.2, we have:

Corollary 1.3. *Let D be a semilocal extension domain of a Noetherian domain R, and let V be a domain integral over D. If V is a semispot over R, then D is also a semispot over R.* $\square$

(1.4) It follows from Corollary 1.3 that a one-dimensional semilocal domain D that birationally dominates a two-dimensional RLR R is a semispot over R if and only the integral closure of D is an intersection of prime divisors of the second kind on R, or equivalently, if and only if each DVR birationally containing D is a prime divisor of the second kind on R.

We are interested in the question of which one-dimensional semilocal domains birationally dominating R are first coefficient domains of ideals of R. The first coefficient domains of complete ideals of R are well understood. They are precisely the one-dimensional semilocal PID's birationally dominating R that are semispots over R. Moreover, if I and J are complete $\mathbf{m}$-primary ideals of R with first coefficient domains D_I and D_J, respectively, then $D_I \cap D_J$ is a PID semispot over R and is the first coefficient domain of IJ. More generally, by the Theorem on Independence of Valuations (e.g., [N, (11.11)] or [ZS, Theorem 18, p. 45]) the intersection of two semilocal PID's birationally dominating a local domain is again a semilocal PID birationally dominating the local domain. But for arbitrary $\mathbf{m}$-primary ideals I and J of R, the relation of D_I and D_J with the first coefficient domain of IJ is more delicate. It is not necessarily $D_I \cap D_J$; indeed, in Example 1.5 we show that $D_I \cap D_J$ need not be a first coefficient domain of R. In this example we make use of the description of the first coefficient domain of an ideal generated by a regular sequence given in [HJL, (3.8)].

Example 1.5. Let k be a field of characteristic 0 and x, y be indeterminates over k; set $R = k[x, y]_{(x,y)}$. Then the first coefficient domains of the ideals $(x^2, y^2)R$ and $(x^2, xy + y^2)R$ are

$$D_1 = k((y/x)^2) + M \qquad \text{and} \qquad D_2 = k((y/x) + (y/x)^2) + M \ ,$$

respectively, where M is the maximal ideal of the ord-valuation domain $V = R[y/x]_{\mathbf{m}R[y/x]} = k(y/x) + M$ over R. (The maximal ideals M_1 and M_2 of D_1 and D_2, respectively, are contained in M, and a module basis for V over either D_1 or D_2 is $1, y/x$. Since $M_i(y/x) \subseteq D_i$ and $M_i V = M$, we have $M_i = M$.) Since k is of characteristic zero, we have $k((y/x)^2) \cap k((y/x) + (y/x)^2) = k$. It follows that the residue field of $D_1 \cap D_2$ at the center of V on $D_1 \cap D_2$ is not residually transcendental over the residue field k of R, so $D_1 \cap D_2$ is not a semispot over R by (1.4) and hence is not the first coefficient domain of an ideal of R.

(1.6) Suppose I and J are $\mathbf{m}$-primary ideals of R, where $(R, \mathbf{m})$ is a two-dimensional RLR, or more generally, a quasi-unmixed analytically unramified local domain. We want to relate the first coefficient domain D of IJ to the first coefficient domains D_I and D_J of I and J. A first remark is that since the set of Rees valuation domains of IJ is the union of the sets of Rees valuation domains of I and J, the integral closure of D is the intersection of the integral closures of D_I and D_J. With each DVR V that is a localization of the integral closure of D_I (of which there are only finitely many) we associate a one-dimensional semilocal domain $D_V = (D_I)_P[B]$, where P is the center of V on D_I and B is the unique local domain on the blowup of J that is dominated by V. In an analogous way we construct D_W for each DVR W that is a localization of the integral closure of D_J. The first coefficient domain D of IJ is the the intersection of the one-dimensional semilocal domains D_V and D_W as V and W vary over the sets of the Rees valuation domains of I and J respectively.

(1.7) The proofs of several results below rely on Theorem 3.12 of [HJL]; and on rereading the proof of that result, we feel one point deserves a fuller discussion. The relevant hypotheses in that result are as follows: R is a normal, analytically unramified, quasi-unmixed, local domain with infinite residue field, I is an ideal primary for the maximal ideal of R, D is the first coefficient domain of I, E is a domain birational and integral over D, and a is an element of I for which $ID = aD$. In the proof, we set $S = R[1/a] \cap D$ and $T = R[1/a] \cap E$, and we assert that D, E are rings of fractions of S, T respectively. This is true under the hypothesis of Theorem 3.12 of [HJL], but in Example 1.8 below we show that for $a \in \mathbf{m}$ with $aD \neq ID$ it can happen that D is not a ring of fractions of $S = R[1/a] \cap D$. So we felt these assertions should be given a more explicit justification: The hypothesis that D is the first coefficient domain of I means that there exists an element b of I such that D is an intersection of a finite number of one-dimensional localizations of $R[I/b]$ and hence is itself a ring of fractions of $R[I/b]$. Moreover, $bD = ID = aD$. Thus, b/a is an element of $R[I/a]$ that is not in any of the prime ideals of D, so the ring of fractions of $R[I/a]$ with respect to the complement in $R[I/a]$ of the union of the primes in D contains $R[I/b]$ and hence is all of D. Since $S = R[1/a] \cap D \supseteq R[I/a]$, we see that D is also a ring of fractions of S. Now we turn to $T = R[1/a] \cap E$, which is almost integral over S since there is a nonzero conductor from E into D (because R is analytically unramified [Re2, Theorem 1.2]). Since

$$S = \cap\{R[I/a]_P \ : \ P \text{ is a height-one prime } \}$$

and since $R[I/a]$ is universally catenary, S is contained in the integral closure of $R[I/a]$. Moreover, the fact that R is analytically unramified implies that the integral closure of $R[I/a]$ is a finitely generated $R[I/a]$-module. Therefore S is Noetherian and hence T is integral over S. Since D is a ring of fractions of S, the maximal ideals of D are centered on height-one primes of S. It follows that the maximal ideals of E are centered on height-one primes of T. Since the essential valuation domains of $R[1/a]$ are all localizations of S and of T, it follows that E is a ring of fractions of T.

Example 1.8. Let $R = k[x,y]_{(x,y)k[x,y]}$, where k is a field and x, y are indeterminates over k. Let $V = k(y/x)[x]_{(x)}$ be the ord-valuation domain of R. Then $V = k(y/x) + M$, where M is the maximal ideal of V. Let $D = k((y^2+x^2)/xy) + M$. Then D is the first coefficient domain of the ideal $(xy, y^2 + x^2)R$, a one-dimensional local domain that birationally dominates R, and V is the integral closure of D. Let $T = R[1/x] \cap V$ and $S = R[1/x] \cap D$. Then $T = R[y/x]$, so $S = R[y/x] \cap D$. Using that $k[y/x] \cap k((y^2+x^2)/xy) = k$ and considering the unique expression of each element of a subdomain of V as the sum of an element of $k(y/x)$ and an element of M, we see that $S = k + (M \cap R[y/x])$. Hence D is centered on a maximal ideal of S and is not a localization of S. We also have in this example that S is not Noetherian and T is almost integral but not integral over S. The localization of S at each of its height-one primes contains $R[1/x]$.

2. Residually transcendental elements.

Let $(R, \mathbf{m})$ be a two-dimensional RLR with residue field $k = R/\mathbf{m}$. A first coefficient domain of an $\mathbf{m}$-primary ideal of R is a one-dimensional semispot birationally dominating R. As a partial converse, we observe in

Proposition 2.1 that a domain satisfying these hypotheses is at least a ring of fractions of a first coefficient domain of R.

Proposition 2.1. *Let $(R, \mathbf{m})$ be a two-dimensional RLR and E be a one-dimensional semispot birationally dominating R. Then there exists a first coefficient domain D of R such that E is a ring of fractions of D.*

Proof. Let $a_1, \ldots, a_n, b$ be elements of R such that E is a ring of fractions of $R[a_1/b, \ldots, a_n/b]$. We may assume that $a_1, \ldots, a_n, b$ have no common factor in R, so that the ideal $I = (a_1, \ldots, a_n, b)R$ is $\mathbf{m}$-primary. Let D_0 denote the first coefficient domain of I. Since E is a semispot over R, the dimension formula [M, (14.D)] shows that for each maximal ideal N of E the image of at least one of the quotients a_i/b in E/N is transcendental over $R/\mathbf{m}$. Thus, the center of N on $R[a_1/b, \ldots, a_n/b]$ is one-dimensional, so that $D_0 \subseteq E_N$. Since this holds for each maximal ideal N of E, $D_0 \subseteq E$. But there may be prime divisors dominating R that contain D_0 but not E. The intersection D of all these prime divisors and E is an integral extension of D_0 and hence a first coefficient domain (of an ideal integral over a power of I) by [HJL, Theorem 3.12]. We have $D \subseteq E$ are one-dimensional semilocal domains with E birational over D and D integrally closed in E. Forming the ring of fractions of D with respect to the elements of D that are units of E and applying [N, (33.1)], we see that E is a ring of fractions of D. $\square$

A variant of the process used in this proof is as follows: With R, E, etc. as in Proposition 2.1 and its proof, let $(c, d)R$ be a reduction of $I = (a_1, \ldots, a_n, b)R$ (or of a power of I if the residue field of R is finite and I fails to have a 2-generated reduction). For each maximal ideal N of E, the image of c/d in E/N is transcendental over $R/\mathbf{m}$, so $N \cap R[c/d] = \mathbf{m}R[c/d]$. It follows that E is a localization of the integral closure of $R[c/d]_{\mathbf{m}R[c/d]}$ in E. To realize E itself as a first coefficient domain in this manner amounts to answering in the affirmative the following question: Does there exist a single element a/b of E such that $J = (a, b)R$ is a reduction of a complete ideal of the form $(*)$ in (0.2) above, where the r_j are positive integers and the $\mathbf{b}_j$ are the simple complete ideals corresponding to the DVR localizations of the integral closure of E? If so, then E and $R[a/b]_{\mathbf{m}R[a/b]}$ have the same integral closure. Thus, E is integral over $R[a/b]_{\mathbf{m}R[a/b]}$ and hence a first coefficient domain in its own right. The proof of Theorem 2.2 below is essentially the construction of such an element a/b in a special case.

In the proof of Theorem 2.2 is a reference to $R(t)$, where t is an indeterminate over R. In general, for a ring A, the symbol $A(t)$ denotes the ring of fractions of the polynomial ring $A[t]$ with respect to the multiplicative system of polynomials whose coefficients generate the unit ideal in A (cf. [N, page 18]). In the present local case, this means only that not all of the coefficients of the polynomial are in $\mathbf{m}$. There is a natural epimorphism from $R(t)$ onto the simple transcendental field extension $k(t)$ of k, with kernel generated by $\mathbf{m}$; images under this epimorphism (as well as under other extensions of the epimorphism $R \to k$) are denoted by overbars (vincula).

Theorem 2.2. *Let D be a one-dimensional local domain birationally dominating a two-dimensional RLR R. Assume that $k = R/\mathbf{m}$ is algebraically closed, that the integral closure D' of D is a prime divisor on R, and that either (1) R has nonzero characteristic or (2) D contains the maximal ideal of D'. Then there is an $\mathbf{m}$-primary ideal of which D is the first coefficient domain.*

Proof. By (1.1), D is a spot over R and D' is a finitely generated D-module. In view of the last sentence of General Example 3.8 and Theorem 3.12 of [HJL], it is enough to find a 2-generated $\mathbf{m}$-primary ideal $(a, b)R$ of R for which $a/b \in D$ and the integral closure of $R[a/b]_{\mathbf{m}R[a/b]}$ is D'. Also, since D' is a prime divisor of the second kind of R, there is a simple complete $\mathbf{m}$-primary ideal $\mathbf{b}$ with which D' is associated, in the sense of the Zariski theory. It will suffice to find elements a, b of R so that $a/b \in D$ and the ideal $(a, b)R$ is a reduction of a power of $\mathbf{b}$.

Let $(c, d)R$ be a minimal reduction of $\mathbf{b}$ (or of a power of $\mathbf{b}$). Then the residue field of D' is of transcendence degree 1 over k, generated by the image $\overline{c/d}$ of c/d (because k is algebraically closed [HuS, Remark 3.5]), but algebraic over the residue field of D, and for any other prime divisor of the second kind of R, either c/d is not in that prime divisor or its image in the residue field is not transcendental over the image of k (i.e., c/d is not "residually transcendental" for any other prime divisor of the second kind). Thus, for an element z of D of which the image $\overline{z}$ in the residue field of D (or D') is transcendental over k, there is an element $\varphi(t)$ of $R(t)$ such that if $\overline{\varphi}(t) \in k(t)$ is the image of $\varphi(t)$ in $R(t)/\mathbf{m}R(t)$, then $\overline{z} = \overline{\varphi}(\overline{c/d})$. We may assume that the numerator and denominator of $\overline{\varphi}(t)$ are relatively prime polynomials over k. Now $z - \varphi(c/d)$ is in the maximal ideal of D', so under assumption (2) of the statement, we immediately have that $\varphi(c/d) \in D$. To reach a similar (though not identical) conclusion under assumption (1), we note that since D' is local and is a finitely generated D-module, the maximal ideal of D contains a power of the maximal ideal of D'; so we can raise $z - \varphi(c/d)$ to a sufficiently high power q, a power of the characteristic of R, to conclude that $\varphi(c/d)^q \in D$. Multiplying the numerator and denominator of φ or φ^q by the same power of d, we convert them into forms $a = a(c, d)$ and $b = b(c, d)$ in c, d of the same degree n such that their images in the degree-n piece of the fiber ring $F((c, d)) = R[(c, d)t] \otimes_R R/\mathbf{m}$, a polynomial ring in two variables over k, are relatively prime.

We show that (a, b) is a reduction of $(c, d)^n$, which will complete the proof. It suffices to show that $(a, b)(c, d)^n = (c, d)^{2n}$, and by Nakayama's Lemma it suffices to show that the k-vector spaces $[(a, b)(c, d)^n + \mathbf{m}(c, d)^{2n}]/\mathbf{m}(c, d)^{2n}$ and $(c, d)^{2n}/\mathbf{m}(c, d)^{2n}$ have the same dimension. The latter is the degree-$2n$ piece of the fiber ring $F((c, d))$; its dimension is $2n + 1$. The images of the products $ac^i d^{n-i}$, $i = 0, \dots, n$, span a subspace of the former of dimension $n+1$, and similarly with b in place of a; and since the images of a, b are relatively prime, the intersection of these two subspaces is spanned by the image of ab, so it is one-dimensional. Thus, $[(a, b)(c, d)^n + \mathbf{m}(c, d)^{2n}]/\mathbf{m}(c, d)^{2n}$ has dimension $2(n + 1) - 1 = 2n + 1$ as required. $\square$

3. Principal extensions and contracted powers.

(3.1) Suppose D is a one-dimensional semispot birationally dominating a quasi-unmixed, analytically unramified, normal local domain $(R, \mathbf{m})$. In this section we seek conditions for D to be the first coefficient domain of an ideal I of R. If D is the first coefficient domain of I, then ID is principal, and replacing I by the associated e_1-ideal of a high power of I, we obtain an $\mathbf{m}$-primary ideal J such that JD is principal and $J^nD \cap R = J^n$ for each positive integer n [HJLS, Theorem 3.17]. Thus a necessary condition for D to be a first coefficient domain is the existence of an $\mathbf{m}$-primary ideal J of R with the two properties: (1) JD is principal, and (2) $J^nD \cap R = J^n$ for each positive integer n. If D is local, we prove in Theorem 3.3 that this necessary condition is also sufficient, and that D is in fact the first coefficient domain of each ideal J with these two properties.

The case in which V is a prime divisor birationally dominating a two-dimensional RLR $(R, \mathbf{m})$ is illustrative. Suppose a is a nonzero element of $\mathbf{m}$ and consider the descending chain $J_n = a^nV \cap R$, $n = 1, 2, \ldots$, of ideals of R. As noted in the introduction, each J_n is a complete ideal of R, and from the Zariski theory it follows that J_n is a product of powers of the simple complete ideals associated with the finitely many prime divisors that "come out" on the sequence of quadratic transformations of R along V. Let $\mathbf{b}$ be the simple complete ideal of R associated to V, and suppose the V-values of a and $\mathbf{b}$ are p and q respectively. Then $J_q = \mathbf{b}^p$. Since all powers of $\mathbf{b}$ are contracted from V, for each positive integer r we have $J_q^r = J_{qr}$, or equivalently the powers of J_q are contracted from V. Moreover, J_q has V as its first coefficient domain.

(3.2) It was noted in [HJL, (3.7)] that the first coefficient domain of an ideal I of R can be described using the minimal primes of $IR[It]$ of the Rees algebra $R[It]$ or the minimal primes of $t^{-1}R[t^{-1}, It]$ of the extended Rees algebra $R[t^{-1}, It]$ of I (where t is an indeterminate over R). These primes are in one-to-one correspondence with the maximal ideals of the first coefficient domain D of I: If P is one of these minimal primes, then P does not contain the degree-1 piece of the Rees algebra (or extended Rees algebra), say $bt \notin P$ where $b \in I$. Then the localization of the (extended) Rees algebra at P is also a localization of $R[I/b][bt, (bt)^{-1}]$ and has the form $D_N(bt)$ (cf. the paragraph before Theorem 2.2) for the maximal ideal N of D corresponding to P. [Note: The $V(t)$ in the equations on the last line of [HJL, (3.7)] should be $V(bt)$, for b as above.]

Theorem 3.3. *Let $(R, \mathbf{m})$ be a local domain that is the intersection of its* *localizations at height-one primes, and let D be a one-dimensional semilocal domain that birationally dominates R. Suppose J is an $\mathbf{m}$-primary ideal of R such that JD is principal and $J^nD \cap R = J^n$ for each positive integer n. Then the first coefficient domain of J is a localization of D. In particular, if D is local, then D is the first coefficient domain of J.*

Proof. Replacing J, if necessary, by a power of J, we may assume that $JD = aD$ where $a \in J$. Let $A = R[t^{-1}, Jt]$ be the extended Rees algebra of the ideal J of R; let $D(at)$ denote the localization of the polynomial ring $D[at]$ at the complement of the union of the extension to $D[at]$ of the maximal ideals of D; and let K be the fraction field of R. Since $D[at, (at)^{-1}]$ is Cohen-Macaulay, it is the intersection of its localizations at height-one primes. It follows that $D[at, (at)^{-1}] = K[at, (at)^{-1}] \cap D(at)$, and hence that

$$\begin{aligned} R[t, t^{-1}] \cap D[at, (at)^{-1}] &= R[t, t^{-1}] \cap K[at, (at)^{-1}] \cap D(at) \\ &= R[t, t^{-1}] \cap D(at) = A \ . \end{aligned}$$

Let P be a minimal prime of $t^{-1}A$ and let $S = A - P$. Then $A_P = S^{-1}(R[t, t^{-1}] \cap D(at)) = S^{-1}(R[t, t^{-1}]) \cap S^{-1}D(at)$. Since $R[t, t^{-1}]$ is the locally finite intersection of its localizations at height-one primes, to show $S^{-1}(R[t, t^{-1}]) = K(t)$, it suffices to show S meets each height-one prime Q of $R[t, t^{-1}]$: If $Q \cap S = \emptyset$, then $Q \cap A \subseteq P$. Since $Q \cap A \neq 0$, we must have $Q \cap A = P$. But $P \cap R = \mathbf{m}$ and $Q \cap R < \mathbf{m}$, a contradiction. Thus S meets each height-one prime of $R[t, t^{-1}]$, so $A_P = S^{-1}D(at)$.

Let E be the first coefficient domain of J. The maximal ideals N of E are in one-to-one correspondence with the minimal primes P of $t^{-1}A$, where $A_P = E_N(at)$. Since each A_P is a localization of $D(at)$, the intersection $E(at)$ of the A_P's is a ring of fractions of $D(at)$. Intersecting with K shows that E is a ring of fractions of D. □

The following corollary implies the uniqueness property of the intersection of the Rees valuation domains of an ideal mentioned in (0.2).

Corollary 3.4. *Let $(R, \mathbf{m})$ be a quasi-unmixed, analytically unramified, normal local domain, and let I be an $\mathbf{m}$-primary ideal of R. The first coefficient* *domain E of I is the unique largest one-dimensional semilocal domain D birationally dominating R and having the properties that ID is principal and $I^nD \cap R$ is contained in the e_1-ideal of I^n for each positive integer n.*

Proof. By [HJLS, Theorem 3.17] for all sufficiently large positive integers r, the ideal $J = I^rE \cap R$ has the property that E is the first coefficient domain of J and for each positive integer n we have $J^nE \cap R = J^n = I^{rn}E \cap R$ is the e_1-ideal associated to I^{rn}. Therefore J^nD is principal and $J^nD \cap R = J^n$ for each n. By Theorem 3.3, E is a localization of D. □

Corollary 3.5. *Let D be a one-dimensional spot birationally dominating a two-dimensional RLR $(R, \mathbf{m})$. If J is an $\mathbf{m}$-primary ideal in R such that JD is principal and all the powers of J are contracted from D, then D is the first coefficient domain of J, and the integral closure of J is a product of powers of the simple complete ideals associated to the localizations of the integral closure of D.*

Proposition 3.6. *Let $(R, \mathbf{m})$ be a quasi-unmixed analytically unramified local domain of dimension $d \geq 2$, and let J be an $\mathbf{m}$-primary ideal of R. Let D be a one-dimensional semilocal domain birationally dominating R, and let V be a finitely generated birational integral extension of D. If all the powers of J are contracted from D, then for each positive integer n, $J^nV \cap R$ is integral over J^n. In particular, if $I = JD \cap R$ is a normal ideal (i.e., the powers of I are integrally closed), then all the powers of I are contracted from V.*

Remark. The hypothesis in Proposition 3.6 (and in Proposition 3.7 below) that V is a finitely generated D-module is necessary (cf., e.g., [HRS, (1.27)]). But if D (or V, by Corollary 1.3) is a (birational) semispot over R, the hypothesis on R assures that V is a finitely generated D-module [Re2, Theorem 1.2].

Proof. For the first assertion, it suffices to show that I is integral over J. Since J is contained in each nonzero prime ideal of D, there exists a positive integer c such that J^c is contained in the conductor of V into D. Thus, for all positive integers n, we have

$$I^{n+c} \subseteq I^{n+c}V \cap R = J^{n+c}V \cap R \subseteq J^n D \cap R = J^n \subseteq I^n.$$

It follows that the length of R/J^n is between those of R/I^n and R/I^{n+c}. Now, for n sufficiently large, the length of I^n/I^{n+c} is a polynomial in n of degree $d-1$, while the lengths of R/I^n and R/J^n are polynomials in n of degree d. Therefore the Hilbert polynomials of I and J have the same highest degree coefficient, i.e., I and J have the same multiplicity. By [Re1, Theorem 3.2], I is integral over J.

For the second assertion, note that $J^n \subseteq I^n \subseteq I^nV \cap R = J^nV \cap R$; the last ideal is integral over J^n, so if I^n is integrally closed, it is equal to $I^nV \cap R$. □

Proposition 3.7. *Let $(R, \mathbf{m})$ be a normal, quasi-unmixed, analytically unramified local domain of dimension $d \geq 2$, and J be an $\mathbf{m}$-primary ideal of R. Let D be a one-dimensional semilocal domain birationally dominating R such that the integral closure V of D is a finitely generated D-module. Suppose that $J^nD \cap R = J^n$ for each positive integer n, and let $I_n = J^nV \cap R$ for each n.*

(1) For sufficiently large r, all the powers of I_r are contracted from V.

(2) Therefore V is contained in each of the Rees valuation domains of J, and so D is contained in the integral closure of the first coefficient domain of J.

Proof. (1) By [Re2, Theorem 1.4], for sufficiently large r, I_r is a normal ideal, so by Proposition 3.6 all the powers of I_r are contracted from V. (2) Since the intersection of the Rees valuation domains is the unique largest one-dimensional semilocal subdomain E of the fraction field of R with the property that the integral closure of J^n is $J^nE \cap R$ for each n, V is contained in each of the Rees valuation domains of J. □

(3.8) Let $(R, \mathbf{m})$ be a two-dimensional RLR, let D be a one-dimensional semispot birationally dominating R, and let V be the integral closure of D. Let $\mathbf{b}_1, \ldots, \mathbf{b}_n$ be the simple complete ideals of R associated to the DVR's which are localizations of V. Then the associated e_1-ideal of an $\mathbf{m}$-primary ideal I of R has the form $(*)$ as in (0.2) above, where the r_j are positive integers, if and only if V is the first coefficient domain of I. By [HJL, Theorem 3.12], D is a first coefficient domain if and only if there exists an ideal J of R such that JD is principal and such that the integral closure of J is of the form $(*)$. Thus, for example, if V is the ord-valuation domain of R, then D is a first coefficient domain if and only if there exists an ideal J such that JD is principal and such that the integral closure of J is a power of $\mathbf{m}$.

(3.9) With R, D as in Corollary 3.5, there always exist $\mathbf{m}$-primary ideals J with the property that all their powers are contracted from D (for, if J is the product of the simple complete ideals associated to the DVR localizations of the integral closure of D, then all the powers of J are contracted from the integral closure of D and hence also from D). Thus, in this case the issue is whether there exists such a J with JD principal. However, if one passes to a more general situation where R is a two-dimensional excellent normal local domain, then there may exist birationally dominating DVR spots V over R for which there does not exist an ideal J of R such that all the powers of J are contracted from V. By definition, an excellent two-dimensional normal local domain $(R, \mathbf{m})$ with the property that each prime divisor of the second kind on R is the first coefficient domain of an $\mathbf{m}$-primary ideal is said to satisfy Muhly's condition (N) (cf. [HL, page 291]). If R is a two-dimensional complete normal local domain, Cutkosky proves in [C, Theorem 4] that R satisfies condition (N) if and only if R has torsion divisor class group. Thus, for example, $R = \mathbb{C}[[x, y, z]]$, where $x^3 + y^3 + z^3 = 0$, has prime divisors of the second kind which are not first coefficient domains of an ideal of R.

(3.10) Let $(R, \mathbf{m})$ be a two-dimensional RLR and let D be the first coefficient domain of an ideal I of R. If D is a prime divisor of R and $a \in \mathbf{m}$ is a nonzero element, then there exists a positive integer n such that D is the first coefficient domain of $a^n D \cap R$ (cf. (3.1)). The case of a general first coefficient domain, however, is different: In Example 1.8, there is no positive integer m for which D is the first coefficient domain of $x^m D \cap R$. This phenomenon is the reef on which founders the following naive approach to realizing a one-dimensional semispot E birationally dominating R as a first coefficient domain. Let $\mathbf{b}_1, \ldots, \mathbf{b}_s$ be the distinct simple complete ideals of R associated with the prime divisors obtained as localizations of the integral closure E' of E, and let $a \in R$ be such that $aE' \cap R = \mathbf{b}_1 \ldots \mathbf{b}_s$. Let $A = R[t^{-1}, t] \cap E(at)$. Then $A = R[t^{-1}, I_1 t, I_2 t^2, \ldots]$, where $I_n = a^n E \cap R$. The integral closure of A is $A' = R[t^{-1}, (I_1)' t, (I_2)' t^2, \ldots]$, while the domain $A'' = R[t^{-1}, t] \cap E'(at)$ is almost integral over A since there is a nonzero conductor from E' to E. The following conditions are equivalent: (1) A is Noetherian. (2) A is affine over R. (3) $A' = A''$. When these conditions hold, $(I_1)' = \mathbf{b}_1 \ldots \mathbf{b}_s$ and E is the first coefficient domain of an ideal integral over a power of I_1. In Example 1.8, however, for $E = D$ and $a = x$, we have $A' < A''$. When we have $A' < A''$, there is no positive integer m for which the powers of I_m are contracted from E, nor for which E is the first coefficient domain of I_m.

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