# BUILDING NOETHERIAN DOMAINS INSIDE AN IDEAL-ADIC COMPLETION

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Abstract. Suppose $a$ is a nonzero nonunit of a Noetherian integral domain $R$. An interesting construction introduced by Ray Heitmann addresses the question of how ring-theoretically to adjoin a transcendental power series in $a$ to the ring $R$. We apply this construction, and its natural generalization to finitely many elements, to exhibit Noetherian extension domains of $R$ inside the $(a)$-adic completion $R^*$ of $R$. Suppose $\tau_1, \dots, \tau_s \in aR^*$ are algebraically independent over $K$, the field of fractions of $R$. Starting with $U_0 := R[\tau_1, \dots, \tau_s]$, there is a natural sequence of nested polynomial rings $U_n$ between $R$ and $A := K(\tau_1, \dots, \tau_s) \cap R^*$. It is not hard to show that if $U := \cup_{n=0}^{\infty} U_n$ is Noetherian, then $A$ is a localization of $U$ and $R^*[1/a]$ is flat over $U_0$. We prove, conversely, that if $R^*[1/a]$ is flat over $U_0$, then $U$ is Noetherian and $A := K(\tau_1, \dots, \tau_s) \cap R^*$ is a localization of $U$. Thus the flatness of $R^*[1/a]$ over $U_0$ implies the intersection domain $A$ is Noetherian.

**1. Introduction.** Suppose $a$ is a nonzero nonunit of a Noetherian integral domain $R$. The $(a)$-adic completion $R^*$ of $R$ is isomorphic to the ring $R[[x]]/(x-a)$ [N, (17.5), page 55]. Thus elements of the $(a)$-adic completion may be regarded as formal power series in $a$. Of course if $R$ is already complete in its $(a)$-adic topology, then $R = R^*$, but often it is the case that there are elements of $R^*$ that are transcendental over $R$. An interesting construction first introduced by Ray Heitmann in [H, page 126] addresses the question of how ring-theoretically to adjoin a transcendental (over $R$) power series in $a$ to the ring $R$. We have made use of this construction of Heitmann in [HRW3] in a local or semilocal context. Our purpose here is to consider this construction in the more general context of an arbitrary Noetherian integral domain.

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There are numerous articles in the literature that have relevance for the building of Noetherian domains inside an ideal-adic completion, for example [BR1], [BR2], [HRS], [H1], [H2], [H3], [L], [N2], [O1], [O2], [R1], [R2], [R3], and [W].

Let $R$ be a Noetherian integral domain with field of fractions $K$ and let $a$ be a nonzero nonunit of $R$. We are interested in the structure of certain intermediate integral domains between $R$ and $R^* := \widehat{(R,(a))} = R[[x]]/(x-a)$, the $(a)$-adic completion of $R$. We are particularly interested in domains of the form $A := L \cap R^*$, where $L$ is an intermediate field between $K$ and the total ring of fractions of $R^*$. It is often difficult to compute this intersection ring $A$. Thus we seek conditions in order that $A$ be realizable as a localization of a directed union of polynomial ring extensions of $R$.

This intersection construction inside the completion of $R$ with respect to a principal ideal yields interesting Noetherian rings which are directed unions of localized polynomial rings, as we see below. By contrast, taking the analogous construction inside the completion with respect to a maximal ideal, even of an excellent local domain seems less likely to give Noetherian intersection domains. In [HRW1], it is shown for a countable excellent local domain $(R, \mathbf{m})$ of dimension at least two that there exist infinitely many algebraically independent elements $\tau_1, \tau_2, \dots$ in the $\mathbf{m}$-adic completion $\widehat{R}$ of $R$ such that the corresponding intersection domain is a localized polynomial ring in infinitely many variables over $R$; that is, $\widehat{R} \cap K(\tau_1, \tau_2, \dots) = R[\tau_1, \tau_2, \dots]_{(\mathbf{m}, \tau_1, \tau_2, \dots)}$.

In [HRW2], [HRW3] and the present paper, we study the following element-wise form of the problem. Let $\tau_1, \dots, \tau_s \in aR^*$ be elements which are algebraically independent over $K$. Starting with $U_0 := R[\tau_1, \dots, \tau_s]$, we define a sequence of nested polynomial rings $U_n$ in $s$ variables over $R$ inside $A := K(\tau_1, \dots, \tau_s) \cap R^*$. In [HRW3] we consider in the case where $R$ is a semilocal Noetherian integral domain and $a$ is an element of the Jacobson radical of $R$ the condition that the embedding $U_0 \to R^*[1/a]$ is flat. Our goal here is to examine flatness of the embedding $U_0 \to R^*[1/a]$ in a more general context, and to prove the following theorem.[1]

**Theorem 1.1.** Suppose $R$ is a Noetherian domain, $a \in R$ is a nonzero nonunit, and $\tau_1, \dots, \tau_s$ are elements of the $(a)$-adic completion $R^*$ of $R$ that are algebraically

[1] This result generalizes [HRW3, Theorem 2.12].

independent over $R$.[2] Then the following conditions are equivalent:

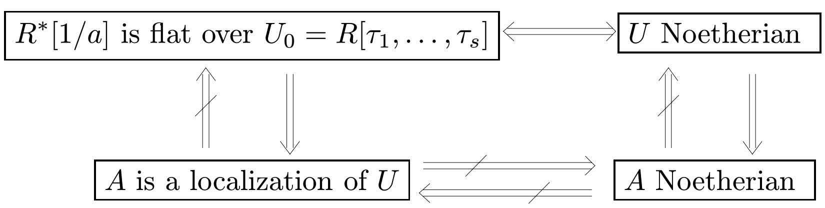
(1) The ring $R^*[1/a]$ is flat over $U_0 = R[\tau_1, \ldots, \tau_s]$.
(2) The directed union $U := \cup_{n=0}^{\infty} U_n$ is Noetherian.[3]

Moreover, if these equivalent conditions hold, then the integral domain $A := K(\tau_1, \ldots, \tau_s) \cap R^*$ is a localization of $U$, and hence $A$ is Noetherian.

**Remark 1.2.** An example given in [HRW3, (4.4)] shows that it is possible for $A$ to be a localization of $U$ and yet for $A$, and therefore also $U$, to fail to be Noetherian. Thus the equivalent conditions of (1.1) are not implied by the property that $A$ is a localization of $U$.

We present in (2.5) an example that is a modification of [HRW2, Example 2.1] to show that $A$ being Noetherian does not imply that $U$ is Noetherian.

The following diagram displays the situation concerning possible implications between $A$ being a localization of $U$ and $A$ or $U$ being Noetherian:




## 2. The general setting.

**(2.1)** Let $R$ be a Noetherian integral domain of dimension $d > 0$ with fraction field $K$. Let $a$ be a nonzero element nonunit of $R$, let $R^* := (\widehat{R, (a)})$ be the $(a)$-adic completion of $R$ and let $R^*_a := R^*[1/a]$. Suppose $\tau_1, \ldots, \tau_s \in aR^*$ are regular elements[4] of $R^*$ that are algebraically independent over $K$. We consider the polynomial ring

$$U_0 := R[\tau_1, \ldots, \tau_s].$$

For every $\gamma \in R^*$ and every $n > 0$, we define the $n^{\text{th}}$-endpiece $\gamma_n$ with respect

[2]We say that elements are algebraically independent over an integral domain if they are algebraically independent over its fraction field.

[3]Heitmann in [H, page 126] considers the case where there is one transcendental element $\tau$ and defines the corresponding extension $U$ to be a *simple PS-extension of $R$ for $a$.* Heitmann proves in this case that a certain monomorphism condition on a sequence of maps is equivalent to $U$ being Noetherian [H, Theorem 1.4].

[4]We say an element of a ring is a *regular element* if it is not a zero divisor.

to $a$ of $\gamma$ to be

$$(2.1.1)\qquad \gamma_n := \sum_{j=n+1}^{\infty} c_j a^{j-n}, \text{ where } \gamma := \sum_{j=1}^{\infty} c_j a^j \text{ with each } c_j \in R.$$

In particular, we represent each of the $\tau_i$ by a power series expansion in $a$; we use these representations to obtain for each positive integer $n$ the $n^{\text{th}}$-endpieces $\tau_{in}$ and the corresponding $n^{\text{th}}$-polynomial ring $U_n$: For $1 \le i \le s$, and $\tau_i := \sum_{j=1}^{\infty} r_{ij} a^j$, where the $r_{ij} \in R$, $\tau_{in} := \sum_{j=n+1}^{\infty} r_{ij} a^{j-n}, U_n := R[\tau_{1n}, \dots, \tau_{sn}]$, for each $n \in \mathbb{N}$. We have a birational inclusion of polynomial rings $U_n \subset U_{n+1}$. We define

$$(2.1.2)\qquad U := \cup_{n=0}^{\infty} U_n = \varinjlim U_n \qquad \text{and} \qquad A := K(\tau_1, \dots, \tau_s) \cap R^*.$$

It is readily seen that $A$ is a birational extension of $U$. We say that the $\tau_i$ have *good limit-intersecting behavior* if $A$ is a localization of $U$.

We observe the following properties of $(a)$-adic completions and an implication of this concerning good limit-intersecting behavior.

**Proposition 2.2 (cf.** [HRW2],[HRW3, (2.2)]**).** *Assume the notation and setting of (2.1), and let $U^*$ and $A^*$ denote the $(a)$-adic completions of $U$ and $A$. Then*

(1) $a^kU = a^kA \cap U = a^kR^* \cap U$ *for each positive integer $k$.*
(2) $U^* = A^* = R^*$, *so* $R/aR = U/aU = A/aA = R^*/aR^*$.
(3) *If $U$ is Noetherian, then $R^*$ is flat over $U$ and $A$ is the localization of $U$ at the multiplicative system $1 + aU$ of $U$.*

*Proof.* We have $R \subseteq U \subseteq A \subseteq R^*$. Since $R$ is Noetherian, $R^*$ is flat over $R$ [M1, Theorem 8.8, page 60]. Moreover, $a^kR$ is closed in the $(a)$-adic topology on $R$, so we have $a^kR^* \cap R = a^kR$ for each positive integer $k$ [ZS, Theorem 8, page 261]. Furthermore, $A = R^* \cap K(\tau_1, \dots, \tau_s)$ implies $a^kA = a^kR^* \cap A$. It is clear that $a^kU \subseteq a^kR^* \cap U$, thus for (1) and (2) it suffices to show $a^kR^* \cap U \subseteq a^kU$. Moreover, if $aR^* \cap U = aU$, it follows that $a^kR^* \cap U = a^kR^* \cap aU = a(a^{k-1}R^* \cap U)$, and by induction we see that $a^kR^* \cap U = a^kU$. Thus we show $aR^* \cap U \subseteq aU$.

Let $g \in aR^* \cap U$. Then there is a positive integer $n$ with $g \in U_n = R[\tau_{1n}, \dots, \tau_{sn}]$. Write $g = r_0 + g_0$ where $g_0 \in (\tau_{1n}, \dots, \tau_{sn})U_n$ and $r_0 \in R$. From the definition of $\tau_{in}$, we have $\tau_{in} = a\tau_{in+1} + a_{in}a$, where $a_{in} \in R$, for each $i$ with $1 \le i \le s$. Thus $r_0 \in aR^* \cap R = aR$, $\tau_{in}U_n \subseteq aU_{n+1}$ and $g \in aU$. This completes the proof of (1) and (2). If $U$ is Noetherian, then $U^* = R^*$ is flat over $U$. Let $S$ be the multiplicative system $1 + aU$ and let $B = S^{-1}U$. Then $B$ is Noetherian, the $(a)$-adic completion of $B$ is $R^*$ and $R^*$ is faithfully flat over $B$ [M1, Theorem 8.14, page 62]. Therefore $B = K(\tau_1, \dots, \tau_s) \cap R^* = A$. □

With the notation and setting of (2.1), the representation of the $\tau_i$ as power series in $a$ with coefficients in $R$ is, in general, not unique. However, as we observe in (2.3), the rings $U$ and $U_n$ are uniquely determined by the $\tau_i$.

**Proposition 2.3 (cf.** [HRW3, (2.3)]**).** *Assume the notation and setting of (2.1). Then $U$ and the $U_n$ are independent of the representation of the $\tau_i$ as power series in $a$ with coefficients in $R$.*

*Proof.* For $1 \le i \le s$, assume that $\tau_i$ and $\omega_i = \tau_i$ have representations

$$\tau_i = \sum_{j=1}^{\infty} a_{ij} a^j \qquad \text{and} \qquad \omega_i = \sum_{j=1}^{\infty} b_{ij} a^j,$$

where each $a_{ij}, b_{ij} \in R$. We define the $n^{\text{th}}$-endpieces $\tau_{in}$ and $\omega_{in}$ as in (2.1.1):

$$\tau_{in} = \sum_{j=n+1}^{\infty} a_{ij} a^{j-n} \qquad \text{and} \qquad \omega_{in} = \sum_{j=n+1}^{\infty} b_{ij} a^{j-n}.$$

Then we have

$$\tau_i = \Sigma_{j=1}^{\infty} a_{ij} a^j = \Sigma_{j=1}^{n} a_{ij} a^j + a^n \tau_{in} = \Sigma_{j=1}^{\infty} b_{ij} a^j = \Sigma_{j=1}^{n} b_{ij} a^j + a^n \omega_{in} = \omega_i.$$

Therefore, for $1 \le i \le s$ and each positive integer $n$,

$$a^n \tau_{in} - a^n \omega_{in} = \Sigma_{j=1}^{n} b_{ij} a^j - \Sigma_{j=1}^{n} a_{ij} a^j, \quad \text{and so} \quad \tau_{in} - \omega_{in} = \frac{\Sigma_{j=1}^{n}(b_{ij} - a_{ij}) a^j}{a^n}.$$

Since $\Sigma_{j=1}^{n}(b_{ij} - a_{ij}) a^j \in R$ is divisible by $a^n$ in $R^*$ and since $a^n R = R \cap a^n R^*$ because $a^n R$ is closed in the $(a)$-adic topology, it follows that $a^n$ divides the sum $\Sigma_{j=1}^{n}(b_{ij} - a_{ij}) a^j$ in $R$. Therefore $\tau_{in} - \omega_{in} \in R$. It follows that $U_n$ and $U = \cup_{n=1}^{\infty} U_n$ are independent of the representation of the $\tau_i$. □

**Remark 2.4.** With notation as in (2.1), if the embedding $U_0 = R[\tau_1, \dots, \tau_s] \to R^*[1/a]$ is flat, then every nonzero element of $U_0$ is a regular element of $R^*$.

**Example 2.5.** (cf. [HRW2, Example 2.1]) In $\mathbb{Q}[[x,y]]$, the power series ring in the two variables $x$ and $y$ over the rational numbers, let $\gamma := e^x - 1$ and $\tau := e^y - 1$; take $\gamma_n$ to be the $n^{\text{th}}$-endpiece of $\gamma$ with respect to $x$ and take $\tau_n$ to be the $n^{\text{th}}$-endpiece of $\tau$ with respect to $y$, as described in (2.1). Set $R := \cup_{n\in\mathbb{N}}\mathbb{Q}[x,y,\gamma_n]_{(x,y,\gamma_n)}$. Then $R = \mathbb{Q}[y]_{(y)}[[x]] \cap \mathbb{Q}(x,y,\gamma)$ is an excellent two-dimensional regular local domain. Now define $U$ in the $(y)$-adic completion of $R$ using the endpieces $\tau_n$ as above. Then $U \supseteq V := \cup_{n\in\mathbb{N}}\mathbb{Q}[x,y,\gamma_n,\tau_n]$. The ring $A := \mathbb{Q}[[x,y]] \cap \mathbb{Q}(x,y,\gamma,\tau)$ is Noetherian but is different from $B := \cup\mathbb{Q}[x,y,\gamma_n,\tau_n]_{(x,y,\gamma_n,\tau_n)}$. The ring $B$ is the localization of $U$ at the multiplicative system $1+yU$, and the rings $B$ and $U$ are not Noetherian. It follows that $A$ is not a localization of $U$.

*Proof.* Consider the element $\theta = \frac{\gamma-\tau}{x-y} \in A$. If $\theta$ is an element of $B$, then

$$\gamma - \tau \in (x-y)B \cap V = (x-y)V.$$

Now

$$V = \cup_{n\in\mathbb{N}}\mathbb{Q}[x,y,\gamma_n,\tau_n] \subseteq \mathbb{Q}[x,y,\gamma,\tau][1/x,1/y] \subseteq \mathbb{Q}[x,y,\gamma,\tau]_{(x-y)},$$

and so

$$\gamma - \tau \in (x-y)\mathbb{Q}[x,y,\gamma,\tau]_{(x-y)} \cap \mathbb{Q}[x,y,\gamma,\tau] = (x-y)\mathbb{Q}[x,y,\gamma,\tau],$$

but this contradicts the fact that $x,y,\gamma,\tau$ are algebraically independent over $\mathbb{Q}$.

If $U$ were Noetherian, then $B$ would be Noetherian. But the maximal ideal of $B$ is $(x,y)B$, so if $B$ were Noetherian, then it would be a regular local domain with completion $\mathbb{Q}[[x,y]]$. Since the completion of a local Noetherian ring is a faithfully flat extension of it, and since the fraction field of $B$ is $\mathbb{Q}(x,y,\gamma,\tau)$, then $B$ would equal $A$.

That $A$ is Noetherian follows from [V, Proposition 3]. If $A$ were a localization of $U$, then $A$ would be a localization of $B$. But each of $A$ and $B$ has a unique maximal ideal and the maximal ideal of $A$ contains the maximal ideal of $B$. Therefore $B \subsetneq A$ implies that $A$ is not a localization of $B$. □

## 3. The proof of the main theorem.

*Proof of Theorem 1.1.* Assume that $U$ is Noetherian. By (2.2), the $(a)$-adic completion $U^*$ of $U$ is equal to $R^*$. Since $U$ is Noetherian, $U^* = R^*$ is flat over

$U$ [M1, Theorem 8.8]. Therefore the localization $R^*[1/a]$ is flat over $U$. Since $U[1/a] = U_0[1/a]$, the localization $R^*[1/a]$ is also flat over $U_0$.

To prove the converse we use results of Heitmann in [H1, Theorem 1.4].

First we show in (3.1) that the flatness condition for $R^*[1/a]$ over $U_0$ behaves well under certain residue class formations.

**Proposition 3.1.** *Let $R$ be a Noetherian domain, let $a$ be a nonzero nonunit of $R$, let $R^*$ be the $y$-adic completion of $R$ and let $\tau_1, \ldots, \tau_s \in aR^*$ be algebraically independent over $R$. Suppose that $R^*_a := R^*[1/a]$ is flat over $U_0$, using the notation of (2.1) and that $Q$ is a prime ideal of $R$ with $a \notin Q$. Assume that $Q$ is the contraction of a prime ideal of $R^*$. Let $^-$ denote image in $R^*_a/QR^*_a$ and let $(R/Q)^*$ denote the $(\bar{a})$-adic completion of $R/Q$. Then $(R/Q)^*_{\bar{a}} := (R/Q)^*[\bar{1}/\bar{a}]$ is flat over $(R/Q)[\bar{\tau}_1, \ldots, \bar{\tau}_s]$.*

*Proof.* The $(\bar{a})$-adic completion $(R/Q)^*$ of $R/Q$ is canonically isomorphic to $R^*/QR^*$. Therefore $\bar{\tau}_1, \ldots, \bar{\tau}_s$ are regular elements of $(R/Q)^*$. We show $\bar{\tau}_1, \ldots, \bar{\tau}_s$ are algebraically independent over $R/Q$. Since $R[\tau_1, \ldots, \tau_s] \longrightarrow R^*_a$ is flat, $a \notin Q$, and $Q$ is the contraction of a prime ideal of $R^*$, we have $QR[\tau_1, \ldots, \tau_s] = QR^*_a \cap R[\tau_1, \ldots, \tau_s]$. Thus

$$R[\tau_1, \ldots, \tau_s]/(QR^*_a \cap R[\tau_1, \ldots, \tau_s]) \cong (R/Q)[\bar{\tau}_1, \ldots, \bar{\tau}_s]$$

is a polynomial ring in $s$ variables $\bar{\tau}_1, \ldots, \bar{\tau}_s$ over $R/Q$. Therefore $\bar{\tau}_1, \ldots, \bar{\tau}_s$ are algebraically independent over $R/Q$.

We show flatness of the map:

$$\bar{\phi} : (R/Q)[\bar{\tau}_1, \ldots, \bar{\tau}_s] \longrightarrow R^*_a/QR^*_a = (R/Q)^*_{\bar{a}}.$$

Let $\bar{P}$ be a prime ideal of $R^*/QR^*$ with $\bar{a} \notin \bar{P}$. The ideal $\bar{P}$ lifts to a prime ideal $P$ of $R^*$ with $a \notin P$ and $QR^* \subseteq P$. By assumption the map

$$\phi_P : R[\tau_1, \ldots, \tau_s] \longrightarrow R^*_P$$

is flat. The map on the residue class rings:

$$\bar{\phi}_{\bar{P}} : (R/Q)[\bar{\tau}_1, \ldots, \bar{\tau}_s] \longrightarrow (R^*/QR^*)_{\bar{P}}$$

is obtained from $\phi_P$ by tensoring with $(R/Q)[\tau_1, \ldots, \tau_s]$ over the ring $R[\tau_1, \ldots, \tau_s]$. Hence $\bar{\phi}$ is flat. □

**Theorem 3.2.** *Assume the notation and setting of (2.1). Also assume that* $s = 1$, $\tau := \tau_1$ *and that the localization* $R^*[1/a]$ *is flat over* $U_0 = R[\tau]$. *Then* $U$ *is Noetherian and* $A = R^* \cap K(\tau)$ *is a localization of* $U$.

We use the same proof as in [H1, Theorem 1.4] and prove first the following lemma.

**Lemma 3.3.** *With notation as in Theorem 3.2, if* $P$ *is a nonzero prime ideal of* $U$ *such that* $P \cap R = (0)$, *then there exists* $f \in P$, $r \in R$ *and a positive integer* $N$ *such that* $P = (fU :_U ra^N)$.

*Proof.* The localization $D := (R - \{0\})^{-1}U$ of $U$ at the nonzero elements of $R$ is also a localization of the polynomial ring $U_0 := R[\tau]$. Hence $PD$ is a a principal maximal ideal of $D$ and there exists a polynomial $f \in R[\tau]$ such that $PD = fD$.

We use the fact that $U$ is the directed union of the polynomial rings $U_n := R[\tau_n]$, $U = \cup_{n=0}^{\infty} U_n$. Let $P_n = P \cap U_n$. Since $D_{PD} = (U_0)_{P_0}$ and $U_0$ is Noetherian, there exists $r \in R$ such that $P_0 = (fU_0 :_{U_0} r)$. Also for $g \in U$ there exists a positive integer $b(g)$, depending on $g$, such that $a^{b(g)}g \in U_0$. Hence for $g \in P$ we have $ra^{b(g)}g \in fU_0$.

The Artin-Rees Lemma [N1, (3.7)] applied to the ideals $aR^*$ and $fR^*$ of the Noetherian ring $R^*$ implies the existence of a positive integer $N$ such that for $m \geq N$ we have

$$fR^* \cap (aR^*)^m = (aR^*)^{m-N}((fR^* \cap (aR^*)^N) = (a^{m-N})R^*(fR^* \cap a^N R^*).$$

We may assume that $b(g) \geq N$.

Suppose $g \in P$. Then $ra^{b(g)}g \in fU_0 \subseteq fU$, so

$$ra^{b(g)}g \in fR^* \cap a^{b(g)}R^* = a^{b(g)-N}R^*(fR^* \cap a^N R^*).$$

Since $a$ is not a zero-divisor in $R^*$, it follows that $ra^N g \in fR^* \cap a^N R^*$. Thus $ra^N g = ft$, where $t \in R^*$. Since we also have $ra^{b(g)}g \in fU$, it follows that $a^{b(g)-N}ft \in fU$, and therefore $a^{b(g)-N}t \in U$, as $f$ is not a zero-divisor in $R^*$. Therefore $a^{b(g)-N}t \in a^{b(g)-N}R^* \cap U = a^{b(g)-N}U$ by (2.2.1) and so $t \in U$. Hence for every $g \in P$ we have $g \in (fU :_U ra^N)$. It follows that $P = (fU :_U ra^N)$. □

As in [H1, Lemma 1.5], we have:

**Lemma 3.4.** *With notation as in Theorem 3.2, if each prime ideal $P$ of $U$ such that $P \cap R \neq (0)$ is finitely generated, then $U$ is Noetherian.*

*Proof.* By a Theorem of Cohen [N1, (3.4)], it suffices to show each $P \in \mathrm{Spec}(U)$ such that $P \cap R = (0)$ is finitely generated. Let $P$ be a nonzero prime ideal of $U$ such that $P \cap R = (0)$. Since the localization of $U$ at the nonzero elements of $R$ is also a localization of the polynomial ring $U_0 := R[\tau]$, every prime ideal of $U$ properly containing $P$ has a nonzero intersection with $R$. Therefore the hypothesis implies that $U/P$ is Noetherian. By (3.3), there exist $r \in R$ and $f \in P$ such that $P = (fU :_U ra^N)$. Since $ra^N$ is a nonzero element of $R$, every prime ideal of $U$ containing $ra^N$ is finitely generated, so $U/ra^NU$ is Noetherian. Therefore $U/(P \cap ra^NU)$ is Noetherian [N1, (3.16)]. Since $ra^N \notin P$ and $P$ is prime, we have $ra^NU \cap P = ra^NP$. Therefore $U/ra^NP$ is Noetherian. We have $ra^NP \subseteq fU \subseteq P$. Hence $U/fU$, as a homomorphic image of $U/ra^NP$, is Noetherian, and $P/fU$ is finitely generated. It follows that $P$ is finitely generated. □

*Proof of Theorem 3.2.* Suppose $U$ is not Noetherian and let $Q \in \mathrm{Spec}(R)$ be maximal with respect to being the contraction to $R$ of a non-finitely generated prime ideal of $U$. Since $R/aR = U/aU = R^*/aR^*$ by (2.2), we have $a \notin Q$. Since $U = \cup_{n=0}^{\infty} U_n$ and $QU_n$ is prime, we have $QU$ is prime in $U$. We claim that $Q$ is the contraction of a prime ideal of $R^*$, for otherwise we have $(Q, a)R = R$. But this means that the image of $a$ in $U/QU$ is a unit which implies that $U/QU = U_0/QU_0$ is Noetherian, and this implies that $P$ is finitely generated. Therefore $Q$ is the contraction of a prime of $R^*$, and (3.1) implies that, passing to the image $\bar{\tau}$ of $\tau$ in $U/QU$, the localization $(R/Q)^*_{\bar{a}}$ is flat over $(R/Q)[\bar{\tau}]$. But Lemma 3.4 then implies that $U/QU$ is Noetherian. This contradicts the existence of a non-finitely generated prime ideal of $U$ lying over $Q$ in $R$. We conclude that $U$ is Noetherian. Therefore $U^* = R^*$ is flat over $U$ and if $S$ is the multiplicative system $1 + aU$, then $S^{-1}U = R^* \cap K(\tau)$. □

**Remark 3.5.** The proof of Theorem 3.2 is essentially due to Ray Heitmann. In his paper [H1] Heitmann defines *simple PS-extensions*. For a regular element $x$ in a ring $R$ and a formal power series in $x$ transcendental over $R$, a simple PS-extension of $R$ for $x$ is an infinite direct union of simple transcendental extensions of $R$. If $R$ is Noetherian and $T$ is a simple PS-extension of $R$, Heitmann proves in

[H1, Theorem 1.4] that a certain monomorphism condition is equivalent to $T$ being Noetherian. Heitmann's monomorphism condition insures that the element $f$ in the proof of Lemma 3.3 is a regular element in $R^*$. In our situation our flatness condition on the embedding $U_0 \longrightarrow R^*_a$, and hence on $U \longrightarrow R^*_a$, implies the regularity of $f$ in $R^*$. Thus Proposition 3.1 yields that if $s = 1$ and the embedding $U_0 \longrightarrow R^*_a$ is flat, then the ring $U = \varinjlim R[\tau_n]$ is a simple PS-extension satisfying the monomorphism condition of Heitmann. In view of Theorem 1.1, Heitmann's monomorphism condition on the PS-extension determined by $\tau$ is equivalent to $\tau$ yielding a flat extension. The flat extension concept however extends to more than one element $\tau$.

*Completion of Proof of Theorem 1.1.* If $U$ is Noetherian, we have already shown that $R^*[1/a]$ is flat over $U_0$. Assume, conversely, that $R^*[1/a]$ is flat over $U_0 = R[\tau_1, \dots, \tau_s]$. It follows that $R^*[1/a]$ is flat over $R[\tau_1]$. By Theorem 3.2, $U(1)$, the directed union ring constructed with respect to $\tau_1$ in (2.1) is Noetherian and $R^* \cap K(\tau_1)$ is a localization of $U(1)$. It also follows that $U(1)^*[1/a] = R^*[1/a]$ is flat over $U(1)[\tau_2, \dots, \tau_s]$ (cf. [HRW2, Proposition 5.10]). Hence a simple induction argument implies that $U$ is Noetherian. Hence $U^* = R^*$ is flat over $U$ and $A$ is a localization of $U$. □

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