STRONGLY IRREDUCIBLE IDEALS OF A COMMUTATIVE RING

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Abstract

An ideal I of a ring R is said to be **strongly irreducible** if for ideals J and K of R, the inclusion $J \cap K \subseteq I$ implies that either $J \subseteq I$ or $K \subseteq I$. The relationship among the families of irreducible ideals, strongly irreducible ideals, and prime ideals of a commutative ring R is considered, and a characterization is given of the Noetherian rings which contain a non-prime strongly irreducible ideal. [1]

1 INTRODUCTION

Although ideal theory in cancellative abelian monoids is similar in many ways to ideal theory in commutative rings, one important difference is that the set of ideals of such a monoid M is closed under *unions* as well as sums and intersections, and of course the distributive laws hold for unions and intersections. Thus the set of ideals of such a monoid M is always a distributive lattice. However the set of ideals in a ring is usually not closed under unions, and intersection usually does not distribute over addition. Indeed, a ring R is said to be **arithmetical** if for all ideals I, J, and K of R, we have $(I + J) \cap K = (I \cap K) + (J \cap K)$. This property is equivalent to the condition that for all ideals I, J, and K of R, we have $(I \cap J) + K = (I + K) \cap (J + K)$. A commutative ring R is arithmetical if and only if for each maximal ideal M of R the ideals of the localization R_M are totally ordered with respect to inclusion [4, page 321], [7, pages150-151].

An ideal I of a commutative ring R is said to be **irreducible** if I is not the intersection of two ideals of R that properly contain it. Thus if I is an irreducible

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ideal and if J and K are ideals in R such that $(J \cap K) + I = (J + I) \cap (K + I)$, and if $J \cap K \subseteq I$, then either $J \subseteq I$ or $K \subseteq I$. The equality $(J \cap K) + I = (J+I) \cap (K+I)$ holds, for example, if the ideals I, J, K are generated by monomials in an R-sequence $a_1, \ldots, a_n$ and each contains a power of a_i for $1 \leq i \leq n-1$ [10, Theorem 5]. If R is a polynomial ring in the variables $X_1, \ldots, X_d$ over a field or over the ring of integers, and if I, J, K are generated by monomials in $X_1, \ldots, X_d$, then $(J \cap K) + I = (J + I) \cap (K + I)$ [2, page 68].

These considerations motivated us to define an ideal I of a ring R to be **strongly irreducible** if for ideals J and K of R, the inclusion $J \cap K \subseteq I$ implies that either $J \subseteq I$ or $K \subseteq I$. The strongly irreducible ideals are also mentioned in [1, Page 301, exercise 34] where they are called *quasi-prime*. In this paper we consider the relationship among the families of irreducible ideals, strongly irreducible ideals, and prime ideals of a commutative ring R. We observe in Lemma 2.2 that a prime ideal is strongly irreducible and that a strongly irreducible ideal is irreducible.

In Theorem 2.6, we prove that if I is an M-primary strongly irreducible ideal of a quasi-local ring (R, M) and if I is properly contained in $I :_R M$, then (1) $I :_R M$ is a principal ideal, (2) $I = (I :_R M)M$, and (3) for each ideal J of R, either $J \subseteq I$ or $I :_R M \subseteq J$. Using this, we observe in Corollary 3.2 that if $I \neq M$ is a strongly irreducible M-primary ideal in a local ring (R, M), then $I = \cup\{q \mid q$ is an ideal in R and $q \subset I :_R M\}$ and $I :_R M = \cap\{q \mid q$ is an ideal in R and $I \subset q\}$.

Our main result, Theorem 3.6, states that if I is a non-prime ideal with $\mathrm{ht}(I) > 0$ in a Noetherian ring R, then I is strongly irreducible if and only if I is primary, R_P is a DVR, where $P = \mathrm{Rad}(I)$, and $I = P^n$ for some integer $n > 1$. In Proposition 3.4 we prove that an ideal I of a Noetherian ring R is a non-prime strongly irreducible ideal if and only if there exist ideals C and P of R such that $I \subset C \subseteq P$ and: (1) P is prime; (2) I is P-primary; and, (3) for all ideals J in R either $J \subseteq I$ or $CR_P \subseteq JR_P$. Also if this holds, then $CR_P = IR_P :_{R_P} PR_P$. In particular, a Noetherian ring R contains a non-prime strongly irreducible ideal if and only if there exists an ideal I of R satisfying these conditions.

All rings considered in the paper are assumed to be commutative rings with identity. We use "$\subset$" for strict inclusion. If S is a multiplicatively closed subset of a ring R and A is an ideal of R_S, then we denote by $A \cap R$ the ideal $\varphi^{-1}(A)$, where $\varphi : R \to R_S$ is the canonical map.

2 STRONGLY IRREDUCIBLE IDEALS

Definition 2.1 *An ideal I of a ring R is* **strongly irreducible** *if for ideals J and K of R, the inclusion $J \cap K \subseteq I$ implies that either $J \subseteq I$ or $K \subseteq I$.*

In Lemma 2.2 we list some basic properties concerning strongly irreducible ideals.

Lemma 2.2 *Let I be an ideal in a ring R. Then:*

(1) *If I is strongly irreducible, then I is irreducible. (Therefore, if R is Noetherian, then I is a primary ideal.)*

(2) *If I is a prime ideal, then I is strongly irreducible.*

(3) *If R is an arithmetical ring, I is irreducible if and only if I is strongly irreducible if and only if the set of zero-divisors on R/I is a prime ideal of R.*

(4) *If S is a multiplicatively closed set in R and if IR_S is strongly irreducible, then $IR_S \cap R$ is strongly irreducible.*

(5) *If I is a strongly irreducible primary ideal and S is a multiplicatively closed subset of R such that $\mathrm{Rad}(I) \cap S = \emptyset$, then IR_S is strongly irreducible.*

(6) *If I is P-primary and IR_P is strongly irreducible, then I is strongly irreducible.*

(7) *If T is a faithfully flat extension ring of R and if IT is strongly irreducible, then I is strongly irreducible.*

(8) *If I is strongly irreducible and if H is an ideal contained in I, then I/H is strongly irreducible in R/H.*

(9) *To show that I is strongly irreducible, it suffices to show that if bR and cR are principal ideals in R such that $bR \cap cR \subseteq I$, then either $b \in I$ or $c \in I$.*

(10) *A principal primary ideal of a UFD is strongly irreducible.*

Proof. For (1) assume that I is strongly irreducible and let J and K be ideals in R such that $J \cap K = I$. Then $J \cap K \subseteq I$, so either $J \subseteq I$ or $K \subseteq I$, since I is strongly irreducible, and it then follows that either $J = I$ or $K = I$, so I is irreducible. (If R is Noetherian, then [11, Lemma 2, p. 209] shows that an irreducible ideal is a primary ideal.)

For (2) assume that I is prime and let J and K be ideals in R such that $J \cap K \subseteq I$. Then $JK \subseteq I$, so either $J \subseteq I$ or $K \subseteq I$, since I is prime, so it follows that I is strongly irreducible.

The first assertion in (3) is clear from the discussion in the Introduction. To prove the second assertion in (3), assume the set of zero-divisors on R/I is a prime ideal P of R. Then $I = IR_P \cap R$. Since the ideals of R_P are linearly ordered with respect to inclusion, IR_P is strongly irreducible in R_P. Hence by part (4) below, I is strongly irreducible. For the other direction, if I is strongly irreducible, then I is irreducible. It is then easily seen that the zero-divisors on R/I form an ideal and hence a prime ideal of R.

For (4) assume that IR_S is strongly irreducible and let J and K be ideals in R such that $J \cap K \subseteq IR_S \cap R$. Then $JR_S \cap KR_S \subseteq IR_S$, so either $JR_S \subseteq IR_S$ or $KR_S \subseteq IR_S$, so either $J \subseteq IR_S \cap R$ or $K \subseteq IR_S \cap R$, hence $IR_S \cap R$ is strongly irreducible.

For (5) assume that I is a strongly irreducible primary ideal of R and let J and K be ideals in R_S such that $J \cap K \subseteq IR_S$. Then $(J \cap R) \cap (K \cap R) \subseteq IR_S \cap R = I$ (by [8, Theorem 6.6], since I is primary). So either $J \cap R \subseteq I$ or $K \cap R \subseteq I$, since I is strongly irreducible. Therefore it follows that either $J = (J \cap R)R_S \subseteq IR_S$ or $K = (K \cap R)R_S \subseteq IR_S$, and hence IR_S is strongly irreducible.

For (6), by (4) $IR_P \cap R$ is strongly irreducible. But since I is P-primary, $IR_P \cap R = I$ by [8, Theorem 6.6].

For (7) assume that T is a faithfully flat extension ring of R and that IT is strongly irreducible. Let J and K be ideals in R such that $J \cap K \subseteq I$, so $JT \cap KT \subseteq IT$, hence either $JT \subseteq IT$ or $KT \subseteq IT$. Therefore either $J = JT \cap R \subseteq IT \cap R = I$, or $K = KT \cap R \subseteq IT \cap R = I$, hence I is strongly irreducible.

For (8) let J and K be ideals in R such that $(J/H) \cap (K/H) \subseteq I/H$. Then $(J+H) \cap (K+H) \subseteq I+H = I$, since $H \subseteq I$. Since I is strongly irreducible it follows that either $J \subseteq I$ or $K \subseteq I$, hence either $J/H \subseteq I/H$ or $K/H \subseteq I/H$, so I/H is strongly irreducible.

For (9), assume that I has the property that whenever $bR \cap cR \subseteq I$ it holds that either $b \in I$ or $c \in I$. To see that I is strongly irreducible let J and K be ideals in R such that $J \cap K \subseteq I$. Assume that $J \not\subseteq I$, so there exists $b \in J$ such that $b \notin I$. Then for all $c \in K$ it holds that $bR \cap cR \subseteq J \cap K \subseteq I$, so $c \in I$. It follows that $K \subseteq I$, hence I is strongly irreducible.

Finally, for (10), let pA be a principal prime ideal in the UFD A, and let n be a positive integer. To show that $p^n A$ is strongly irreducible, it suffices by (6) to show that $p^n A_{pA}$ is strongly irreducible, and this is clear since A_{pA} is a DVR. $\square$

Concerning conditions (4), (5) and (6) of Lemma 2.2, it is well known that in a Noetherian ring, irredicible ideals are primary and primary ideals are not necessarily irreducible. In an arithmetical ring the opposite holds. That is primary ideals are irreducible [5, Theorem 6], and irredicible ideals are not necessarily primary. Recall that an integral domain is arithmetical if and only if it is Prüfer [5, Corollary 3], and that a Prüfer domain R has the property that each ideal of R with prime radical is irreducible if and only if each prime ideal of R is contained in a unique maximal ideal [5, Theorem 8]. A general necessary and sufficient condition for an irreducible ideal I of a commutative ring R to be primary is that each chain of the form $I \subseteq I :_R a \subseteq I :_R a^2 \subseteq I :_R a^3 \ldots$, $a \in R$, must be finite [3].

In Example 2.3, we give several examples of strongly irreducible ideals (the first of which is alluded to in the proof of parts (3) and (10) of Lemma 2.2).

Example 2.3 **(1)** If the ideals of R are linearly ordered, then each ideal in R is strongly irreducible. So, for example, if R is either a DVR or a homomorphic image of a DVR, then each ideal in R is strongly irreducible (and also principal). In particular, if F is a field, X is an indeterminate, and n is a positive integer, then each ideal in $R = F[[X]]/(X^n)$ is strongly irreducible.
(2) If R is any ring such that the zero ideal of R is irreducible, then the zero ideal of R is strongly irreducible.

(3) If R is Gorenstein of altitude zero, then the zero ideal is irreducible, so it is strongly irreducible, by (2). In particular, if $R = F[X_1, \ldots, X_g]/({X_1}^{n_1}, \ldots, {X_g}^{n_g})$, where F is a field, $X_1, \ldots, X_g$ are indeterminates, and $n_1, \ldots, n_g$ are positive integers, then the zero ideal in R is strongly irreducible.
(4) If P is a height-one prime ideal of a Krull domain R, then each P-primary ideal is strongly irreducible (by Lemma 2.2(6)).

A strongly irreducible ideal of a Noetherian ring is primary and thus, in particular, has prime radical. Also, as noted in part (3) of Lemma 2.2, if I is a strongly irreducible ideal of an arithmetical ring R, then the set of zero-divisors on R/I is a prime ideal P of R. Since R is arithmetical, the prime ideals of R contained in P are linearly ordered with respect to inclusion. Therefore in an arithmetical ring a strongly irreducible ideal has prime radical. In general, however, a strongly irreducible ideal may fail to have prime radical as we show in Example 2.4.

Example 2.4 Let (R, M) be a Noetherian local ring having more than one minimal prime. Let $E = E(R/M)$ denote the injective envelope of the residue field R/M of R as an R-module. Then the zero ideal of the idealization $A = R + E$ [8, page 2]is irreducible, and hence strongly irreducible by part (2) of Example 2.3. But the radical of zero in A has more than one minimal prime since R has more than one minimal prime.

We believe Lemma 2.5 is known, but do not know an appropriate reference, so we include a proof.

Lemma 2.5 *Let b and c be elements in a ring R. Then $bR \cap cR = b(cR :_R bR) = c(bR :_R cR)$. Moreover, if I is an ideal in R such that $I \subseteq bR$, then $I = b(I :_R bR)$.*

Proof. For the last statement, assume that I is an ideal in R such that $I \subseteq bR$. Then it is clear that $b(I :_R bR) \subseteq I$, and if $i \in I \subseteq bR$, then $i = rb$ for some $r \in R$, so $r \in iR :_R bR \subseteq I :_R bR$, so $i = rb \in b(I :_R bR)$.

Then, since $bR \cap cR \subseteq bR$, it follows that $bR \cap cR = b((bR \cap cR) :_R bR) = b(cR :_R bR)$. By symmetry it follows that $bR \cap cR = c(bR :_R cR)$. $\square$

Theorem 2.6 gives some properties of a strongly irreducible M-primary ideal in a quasi-local ring (R, M). (The hypothesis in Theorem 2.6 that I is properly contained in $I :_R M$ is clearly satisfied if $I = M$. It is also satisfied if R is local (Noetherian) and $I \neq M$, so this result plays an important role in the next section where we restrict attention to the case where R is Noetherian.)

Theorem 2.6 *Let (R, M) be a quasi-local ring and let I be a strongly irreducible M-primary ideal in R. Assume that $I \subset I :_R M$. Then:*
(1) *$I :_R M$ is a principal ideal.*
(2) *$I = (I :_R M)M$.*
(3) *For each ideal J in R either $J \subseteq I$ or $I :_R M \subseteq J$.*

Proof. By hypothesis, $I \subset I :_R M$, so there exist $x \in (I :_R M) - I$. If $(I :_R M) \neq xR$, let $y \in (I :_R M) - xR$. Then $xR \cap yR = y(xR :_R yR)$ (by Lemma 2.5) $\subseteq I$ (since $xR :_R yR \subseteq M$ and $y \in I :_R M$). However, I is strongly irreducible, so $xR \cap yR \subseteq I$ implies that either $xR \subseteq I$ or $yR \subseteq I$, hence $y \in I$. Therefore it follows that $I :_R M = xR \cup I$. But then $I :_R M \subseteq xR$ or $I :_R M \subseteq I$ [6, Theorem 81]. Therefore $I :_R M = xR$, so (1) holds.

For (2), $I \subset I :_R M = xR$ (by (1)), so $I = x(I :_R xR)$ (by Lemma 2.5) $= xM$ (since $x \in (I :_R M) - I$ implies (since R is quasi-local with maximal ideal M) that $I :_R xR = M$), hence (2) holds.

To prove (3) let J be an ideal in R. It may clearly be assumed that $J \not\subseteq I$, so it remains to show that $I :_R M \subseteq J$; that is, that $x \in J$. For this, if $x \notin J$, then let $j \in J$, so $x \notin jR$. Therefore $xR \cap jR = xR(jR :_R xR) \subseteq xM \subseteq I$, hence $jR \subseteq I$ (since I is strongly irreducible and $xR \not\subseteq I$). Since this holds for each $j \in J$, it follows that $J \subseteq I$, and this is a contradiction. Therefore $x \in J$, hence (3) holds. $\square$

Recall that an ideal I of a ring R is said to be **sheltered** if there exists a least element in the set of nonzero submodules of R/I [1, Page 238, exercise 18]. Thus a strongly irreducible M-primary ideal in a local ring (R, M) is sheltered. The converse is false since, for example in a one-dimensional Gorenstein local ring, any regular principal I of R is sheltered (since it is irreducible), but it will be seen in Corollary 3.7, that if I is strongly irreducible, then either (R, M) is a DVR or $I = M$. In particular the zero ideal of R/I can be strongly irreducible while I fails to be strongly irreducible. An example of a sheltered ideal of a non-Noetherian ring is the zero ideal of $A = R + E$ in Example 2.4.

3 STRONGLY IRREDUCIBLE IDEALS IN NOETHERIAN RINGS

In this section we first prove a corollary of Theorem 2.6. We then give a characterization for a Noetherian ring to have a strongly irreducible non-prime ideal I. We observe that such an ideal I has several properties similar to an ideal in a homomorphic image of a DVR, and then show that a strongly irreducible ideal I of positive height is either prime or $R_{\mathrm{Rad}(I)}$ is a DVR. We often use the fact (Lemma 2.2(1)) that a strongly irreducible ideal in a Noetherian ring is a primary ideal.

Corollary 3.1 *Let I be a strongly irreducible ideal in a Noetherian ring R, let* $\mathrm{Rad}(I)$ $= P$*, and assume that $I \neq P$. Then:*

(1) *$(I :_R P)R_P$ is a principal ideal (hence* $\mathrm{ht}(I) \leq 1$*).*

(2) *$IR_P = ((I :_R P)P)R_P$.*

(3) *For each ideal J in R either $J \subseteq I$ or $(I :_R P)R_P \subseteq JR_P$.*

Proof. I is a primary ideal, since I is strongly irreducible, hence I is P-primary (where $P = \mathrm{Rad}(I)$). Also, IR_P is strongly irreducible, by Lemma 2.2(1), so (1) - (3) follow immediately from Theorem 2.6. (Since R is Noetherian, it follows from the Principal Ideal Theorem (and the fact that $IR_P \subset (I :_R P)R_P$ and $(I :_R P)R_P$ is a principal ideal) that $\mathrm{ht}(I) = \mathrm{ht}(P) \leq 1$.) □

Corollary 3.2 *Let (R, M) be a local ring and let $I \neq M$ be a strongly irreducible M-primary ideal in R (so* $\mathrm{ht}(M) \leq 1$*, by Corollary 3.1). Then I and $I :_R M$ are* *comparable (under containment) to all ideals in R; in fact, $I = \cup\{q \mid q$ is an ideal in R and $q \subset I :_R M\}$ and $I :_R M = \cap\{q \mid q$ is an ideal in R and $I \subset q\}$.*

Proof. $I :_R M = \cap\{q \mid q$ is an ideal in R and $I \subset q\}$ by Theorem 2.6(3). Also, if q is an ideal in R such that $q \subset I :_R M$, then $I :_R M \not\subseteq q$, so $q \subseteq I$ by Corollary 3.1, hence $I = \cup\{q \mid q$ is an ideal in R and $q \subset I :_R M\}$. $\square$

Remark 3.3 **(1)** It follows from Corollary 3.1(1) and (2) and Corollary 3.2 that if I is a strongly irreducible non-prime ideal in a Noetherian ring R, then IR_P has the following three properties that are similar to the ideals in a DVR (where $P = \mathrm{Rad}(I)$): (a) $IR_P :_{R_P} PR_P$ is principal; (b) $IR_P = PR_P(IR_P :_{R_P} PR_P)$; and, (c) IR_P and $IR_P :_{R_P} PR_P$ are comparable to all ideals in R_P.
(2) If I is a non-prime strongly irreducible ideal in a local ring R that is primary for the maximal ideal of R, then I is comparable to all ideals in R, by (1)(c). However, an ideal in a local ring that is comparable to all ideals in R need not be strongly irreducible. For example, the zero ideal in every ring has this property, but need not even be irreducible.
(3) If I is an irreducible M-primary ideal in a local ring (R, M), then I is strongly irreducible if and only if I is comparable to all ideals in R.

Proof. For (3), it follows from (1)(c) that it suffices to show that an irreducible ideal that is comparable to all ideals in R is strongly irreducible. For this, it follows from Proposition 3.4 below that it suffices to show that $I :_R M$ is comparable to all ideals in R. For this, if J is an ideal in R that is not contained in I, then $I \subset J$, by hypothesis. Since I is irreducible, it follows that $I :_R M \subseteq J$. $\square$

Part (3) of Corollary 3.1 characterizes a non-prime strongly irreducible ideal in a Noetherian ring, as we observe in Proposition 3.4.

Proposition 3.4 *Let R be a Noetherian ring. An ideal I of R is a non-prime strongly irreducible ideal if and only if there exist ideals C and P of R such that $I \subset C \subseteq P$ and: (1) P is prime; (2) I is P-primary; and, (3) for all ideals J in R either $J \subseteq$*

I or $CR_P \subseteq JR_P$. Also if this holds, then $CR_P = IR_P :_{R_P} PR_P$. In particular, a Noetherian ring R contains a non-prime strongly irreducible ideal if and only if there exists an ideal I of R satisfying these conditions.

Proof. We have already noted in Remark 3.3(1) that a non-prime strongly irreducible ideal in a Noetherian ring satisfies the stated conditions. For the converse, assume that I is P-primary. By Lemma 2.2(5), it suffices to show that IR_P is strongly irreducible, so it may be assumed that R is local with maximal ideal P.

Let J and K be ideals in R such that $J \cap K \subseteq I$. If $J \not\subseteq I$ and $K \not\subseteq I$, then $I \subset C \subseteq J \cap K$, and this is a contradiction. Therefore either $J \subseteq I$ or $K \subseteq I$, hence I is strongly irreducible.

Finally, the ideal C is clearly uniquely determined by the properties (a) $I \subset C \subseteq P$, and (b) for all ideals J in R either $J \subseteq I$ or $C \subseteq J$. Since $I :_R P$ also has these properties by Corollary 3.1(3), $C = I :_R P$. $\square$

Proposition 3.5 *Let I be a strongly irreducible ideal in a Noetherian ring R, let $\mathrm{Rad}(I) = P$, and assume that $I \neq P$ and that $\mathrm{ht}(P) > 0$. Then IR_P is a regular ideal.*

Proof. By Lemma 2.2(5) it may be assumed that R is local with maximal ideal P, so it must be shown that $(0) :_R I = (0)$.

For this, let $J = \cup\{(0) :_R I^n \mid n \geq 0\}$. Then either $J \subseteq I$ or $I :_R P \subseteq J$, by Corollary 3.1(3). If $I :_R P \subseteq J$, then $I :_R P \subseteq (0) :_R I^n$ for all large integers n, so $I^{n+1} \subseteq I^n(I :_R P) \subseteq I^n((0) :_R I^n) = (0)$, and this contradicts the hypothesis that $\mathrm{ht}(I) > 0$.

Therefore it may be assumed that $J \subseteq I$. Also, $I :_R P = xR$ for some element x in P, by Corollary 3.1(1). Therefore $(0) \subseteq J \subseteq I \subseteq I :_R P = xR$. However, it is readily checked that J is the isolated component of zero determined by the height-zero prime ideals in R, so $J :_R xR = J$ (since $\mathrm{ht}(xR) = 1$). By Lemma 2.5 we have $J = x(J :_R xR) = xJ$, and since R is a local ring, $J = (0)$ by Nakayama's Lemma. Hence I is a regular ideal. $\square$

It follows from Example 2.3(1) that if R is a discrete valuation ring, then each ideal in R is strongly irreducible. We show in Theorem 3.6 that in the Noetherian ring case this is the only case of a non-prime strongly irreducible ideal of positive height; that is, if I is a strongly irreducible ideal in a Noetherian ring R and if $\mathrm{ht}(I) > 0$ and R_P is not a DVR, (where $P = \mathrm{Rad}(I)$), then $I = P$.

Theorem 3.6 *Let I be a non-prime ideal with $\mathrm{ht}(I) > 0$ in a Noetherian ring R. Then I is strongly irreducible if and only if I is primary, R_P is a DVR, where $P = \mathrm{Rad}(I)$, and $I = P^n$ for some integer $n > 1$.*

Proof. ($\Leftarrow$) Since R_P is a DVR, IR_P is strongly irreducible, and since I is P-primary, this implies I is strongly irreducible by Lemma 2.2(6).

($\Rightarrow$) Since I is strongly irreducible, it follows from Lemma 2.2(5) that IR_P is strongly irreducible, so it suffices to prove this implication in the case where R is a local ring with maximal ideal P and I is P-primary. Also, since $\mathrm{ht}(I) > 0$, Corollary 3.1(1) shows that $\mathrm{ht}(I) = 1$ and Proposition 3.5 shows that I $(= IR_P)$ is regular.

Assume that P is not a principal ideal. We show that this implies the contradiction that $I = P$. For if $I \neq P$, then $I :_R P = xR$ is a principal ideal and $I = xP$, by Corollary 3.1(1) and (2). Let k be the positive integer such that $x \in P^k - P^{k+1}$, so $I = xP \subseteq P^{k+1}$. If P^k is not principal, then there exists $y \in P^k - P^{k+1}$ such that $x \notin yR$ and $y \notin xR$ (and $y \notin I$, since $I \subseteq P^{k+1}$). Now, $xR \cap yR = x(yR :_R xR) \subseteq I$ (since $x \in I :_R P$ and $yR :_R xR \subseteq P$) and $x \notin I$ and $y \notin I$. This contradicts the hypothesis that I is strongly irreducible. Therefore P^k must be principal. However, by [9, Proposition 1], if some power of P is principal, then either P is principal, or P consists of zero divisors. But P is not principal (by hypothesis) and P is a regular ideal (since I is a regular ideal by Proposition 3.5). Therefore P is principal and $I = P^n$ for some integer $n > 1$. □

Corollary 3.7 *A Noetherian integral domain R has a non-prime strongly irreducible ideal if and only if it contains a height-one prime ideal P such that R_P is a DVR.*

Proof. This is immediate from Theorem 3.6. □

Corollary 3.8 *Let I be a strongly irreducible regular ideal in a one-dimensional Noetherian ring R. Then $(J \cap K) + I = (J + I) \cap (K + I)$ for all ideals J and K of R with $J \cap K \subseteq I$.*

Proof. It suffices to check the equation locally at each prime P, and either IR_P is strongly irreducible or $IR_P = R_P$ by Lemma 2.2(5). Thus assume (R, P) is local. Since the equation clearly holds if $I = R$, we may assume $I \subseteq P$. Then $P = \mathrm{Rad}(I)$ and either $J \subseteq P$ or $K \subseteq P$. Assume $J \subseteq P$. Then by Corollary 3.2, either $J \subseteq I$ or $I \subseteq J$. If $J \subseteq I$ then $(J \cap K) + I = I = I \cap (K + I) = (J + I) \cap (K + I)$. If $I \subseteq J$ then, by the modular law, $(J \cap K) + I = J \cap (K + I) = (J + I) \cap (K + I)$. $\square$

References

[1] N. Bourbaki, *Commutative Algebra*, Addison-Wesley, Reading, Mass., 1972.

[2] J. Eagon and M. Hochster, *R-sequences and indeterminates*, Quart. J. Math. Oxford, 25 (1974), 61-71.

[3] L. Fuchs, *A condition under which an irreducible ideal is primary*, Quart. J. Math., Oxford, 19 (1948), 235-237.

[4] R. Gilmer, *Multiplicative Ideal Theory*, Queens Papers in Pure and Applied Mathematics - No. 90, Queens University, Kingston Ontario, 1992.

[5] C. Jensen, *Arithmetical rings*, Acta Mathematica Scientiarum Hungaricae, 17 (1966), 115-123.

[6] I. Kaplansky, *Commutative Rings*, University of Chicago Press, Chicago, 1974.

[7] M. Larsen and P. McCarthy, *Multiplicative Theory of Ideals*, Academic Press, New York, 1971.

[8] M. Nagata, *Local Rings*, Interscience Tracts in Pure and Applied Math., No. 13, Interscience, New York, NY, 1962.

[9] J. Sally, *On the number of generators of powers of an ideal*, Proc. Amer. Math. Soc., 53 (1975), 24-26.

[10] D. Taylor, *Ideals generated by monomials in an R-sequence*, Ph.D. Dissertation, University of Chicago, 1966.

[11] O. Zariski and P. Samuel, *Commutative Algebra, Vol. I*, D. Van Nostrand Co., Inc., Princeton, NJ, 1958.

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