Mixed Polynomial/Power Series Rings and Relations among their Spectra

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This article is dedicated to Robert Gilmer, an outstanding commutative algebraist, scholar and teacher. It relates to his work in ideal theory.

1 Introduction and Background

In this article we study the nested mixed polynomial/power series rings

$$A := k[x,y] \hookrightarrow B := k[[y]]\,[x] \hookrightarrow C := k[x]\,[[y]] \hookrightarrow E := k[x,1/x]\,[[y]], \quad (1)$$

where k is a field and x and y are indeterminates over k. In Equation 1 the maps are all flat. Also we consider

$$C \hookrightarrow D_1 := k[x]\,[[y/x]] \hookrightarrow \cdots \hookrightarrow D_n := k[x]\,[[y/x^n]] \hookrightarrow \cdots \hookrightarrow E. \quad (2)$$

With regard to Equation 2, for n a positive integer, the map $C \hookrightarrow D_n$ is not flat, but $D_n \hookrightarrow E$ is a localization followed by an adic completion of a Noetherian ring and therefore is flat. We discuss the spectra of these rings and consider the maps induced on the spectra by the inclusion maps on the rings. For example, we determine whether there exist nonzero primes of one of the larger rings that intersect a smaller ring in zero. We were led to consider these rings by questions that came up in two contexts.

The first motivation is from the introduction to the paper [AJL] by Alonzo-Tarrio, Jeremias-Lopez and Lipman: If a map between Noetherian formal schemes can be factored as a closed immersion followed by an open one, can this map also be factored as an open immersion followed by a closed one? This is not true in general. As mentioned in [AJL], Brian Conrad observed that a counterexample can be constructed for every triple (R, x, p), where

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(1) R is an adic domain, that is, R is a Noetherian domain that is separated and complete with respect to the powers of a proper ideal I.
(2) x is a nonzero element of R such that the completion of $R[1/x]$ with respect to the powers of $IR[1/x]$, denoted $S := R_{\{x\}}$, is an integral domain.
(3) p is a nonzero prime ideal of S that intersects R in (0).

The composition $R \to S \to S/p$ determines a map on formal spectra Spf $(S/p) \to$ Spf $(S) \to$ Spf (R) that is a closed immersion followed by an open one. To see this, recall that a surjection such as $S \to S/p$ of adic rings gives rise to a closed immersion Spf $(S/p) \to$ Spf (S) while a localization, such as that of R with respect to the powers of x, followed by the completion of $R[1/x]$ with respect to the powers of $IR[1/x]$ to obtain S gives rise to an open immersion Spf $(S) \to$ Spf (R) [EGA, (10.2.2)].

The map Spf $(S/p) \to$ Spf (R) cannot be factored, however, as an open immersion followed by a closed one. This is because a closed immersion into Spf (R) corresponds to a surjective map of adic rings $R \to R/J$, where J is an ideal of R [EGA, page 441]. Thus if the immersion Spf $(S/p) \to$ Spf (R) factored as an open immersion followed by a closed one, we would have R-algebra homomorphisms from $R \to R/J \to S/p$, where Spf $(S/p) \to$ Spf (R/J) is an open immersion. Since $p \cap R = (0)$, we must have $J = (0)$. This implies Spf $(S/p) \to$ Spf (R) is an open immersion, that is, the composite map Spf $(S/p) \to$ Spf $(S) \to$ Spf (R), is an open immersion. But also Spf $(S) \to$ Spf (R) is an open immersion. It follows that Spf $(S/p) \to$ Spf (S) is both open and closed. Since S is an integral domain this implies Spf $(S/p) \cong$ Spf (S). This is a contradiction since p is nonzero.

An example of such a triple (R, x, p) is described in [AJL]: For w, x, y, z indeterminates over a field k, set $R := k[w, x, z]\,[[y]]$, $S := k[w, x, 1/x, z]\,[[y]]$. Notice that R is complete with respect to yR and S is complete with respect to yS. An indirect proof is given in [AJL] that there exist nonzero primes p of S for which $p \cap R = (0)$. In Proposition 4.5 below we give a direct proof of this fact.

A second motivation is from a question raised by Mel Hochster: "Can one describe or somehow classify the local maps $R \hookrightarrow S$ of complete local domains R and S such that every nonzero prime ideal of S has nonzero intersection with R?" In [HRW2] we study this question and define:

Definition 1.1. For R and S integral domains with R a subring of S we say that S is a *trivial generic fiber* extension of R, or a **TGF** extension of R, if every nonzero prime ideal of S has nonzero intersection with R.

In some correspondence to Lipman regarding closed and open immersions, Conrad asked: "Is there a nonzero prime ideal of $E := k[x, 1/x]\,[[y]]$ that intersects $C = k[x]\,[[y]]$ in zero?" If there were such a prime ideal, then $C := k[x]\,[[y]] \hookrightarrow E := k[x, 1/x]\,[[y]]$ would be a simpler counterexample to the assertion that a closed immersion followed by an open one also has a factorization as an open immersion followed by a closed one. In the terminology of Definition 1.1, one can ask:

Question 1.2. Let x and y be indeterminates over a field k. Is $C := k[x]\,[[y]] \hookrightarrow E := k[x, 1/x]\,[[y]]$ a TGF extension?

We show in Proposition 2.6.2 below that the answer to Question 1.2 is "yes". This is part of our analysis of the prime spectra of A, B, C, D_n and E, and the maps induced on these spectra by the inclusion maps on the rings.

The following example is a local map of the type described in Hochster's question.

Example 1.3. Let x and y be indeterminates over a field k and consider the extension $R := k[[x, y]] \hookrightarrow S := k[[x]]\,[[y/x]]$. To see this extension is TGF, it suffices to show $P \cap R \neq (0)$ for each $P \in \operatorname{Spec} S$ with $\operatorname{ht} P = 1$. This is clear if $x \in P$, while if $x \notin P$, then $k[[x]] \cap P = (0)$, and so $k[[x]] \hookrightarrow R/(P \cap R) \hookrightarrow S/P$ and S/P is finite over $k[[x]]$. Therefore $\dim R/(P \cap R) = 1$, and so $P \cap R \neq (0)$.

Remarks 1.4. (1) The extension $k[[x, y]] \hookrightarrow k[[x, y/x]]$ is, up to isomorphism, the same as the extension $k[[x, xy]] \hookrightarrow k[[x, y]]$.

(2) We show in [HRW2] that the extension $R := k[[x, y, xz]] \hookrightarrow S := k[[x, y, z]]$ is not TGF.

2 Trivial generic fiber (TGF) extensions and prime spectra

The following two propositions about trivial generic fiber extensions from [HRW2] are useful and straightforward to check.

Proposition 2.1. *Let $R \hookrightarrow S$ be an injective map where R and S are integral domains. Then the following are equivalent:*

(1) *S is a TGF extension of R.*
(2) *Every nonzero element of S has a nonzero multiple in R.*
(3) *For $U = R \setminus \{0\}$, $U^{-1}S$ is a field.*

Proposition 2.2. *Let $R \hookrightarrow S$ and $S \hookrightarrow T$ be injective maps where R, S and T are integral domains.*

(1) *If $R \hookrightarrow S$ and $S \hookrightarrow T$ are TGF extensions, then so is $R \hookrightarrow T$. Equivalently if $R \hookrightarrow T$ is not TGF, then at least one of the extensions $R \hookrightarrow S$ or $S \hookrightarrow T$ is not TGF.*
(2) *If $R \hookrightarrow T$ is TGF, then $S \hookrightarrow T$ is TGF.*
(3) *If the map Spec $T \to$ Spec S is surjective, then $R \hookrightarrow T$ is TGF implies $R \hookrightarrow S$ is TGF.*

More information about TGF extensions is in [HRW1] and [HRW2].

Remarks 2.3. Let R be a commutative ring and let $R[[y]]$ denote the formal power series ring in the variable y over R. Then

(1) Each maximal ideal of $R[[y]]$ is of the form $(\mathbf{m}, y)R[[y]]$ where $\mathbf{m}$ is a maximal ideal of R. Thus y is in every maximal ideal of $R[[y]]$.
(2) If R is Noetherian with $\dim R[[y]] = n$ and $x_1, \dots, x_m$ are independent indeterminates over $R[[y]]$, then y is in every height $n+m$ maximal ideal of the polynomial ring $R[[y]]\,[x_1, \dots, x_m]$.

Proof. Item (1) follows from [N, Theorem 15.1]. For item (2), let $\mathbf{m}$ be a maximal ideal of $R[[y]]\,[x_1, \dots, x_m]$ with $\operatorname{ht}(\mathbf{m}) = n+m$. By [K, Theorem 39], $\operatorname{ht}(\mathbf{m} \cap R[[y]]) = n$; thus $\mathbf{m} \cap R[[y]]$ is maximal in $R[[y]]$, and so, by item (1), $y \in \mathbf{m}$. □

Proposition 2.4. *Let n be a positive integer, let R be an n-dimensional Noetherian domain, let y be an indeterminate over R, and let $\mathbf{q}$ be a prime ideal of height n in the power series ring $R[[y]]$. If $y \notin \mathbf{q}$, then $\mathbf{q}$ is contained in a unique maximal ideal of $R[[y]]$.*

Proof. The assertion is clear if $\mathbf{q}$ is maximal. Otherwise $S := R[[y]]/\mathbf{q}$ has dimension one. Moreover, S is complete with respect to the yS-adic topology [M, Theorem 8.7] and every maximal ideal of S is a minimal prime of the principal ideal yS. Hence S is a complete semilocal ring. Since S is also an integral domain, it must be local by [M, Theorem 8.15]. Therefore $\mathbf{q}$ is contained in a unique maximal ideal of $R[[y]]$. □

In Section 3 we use the following corollary to Proposition 2.4.

Corollary 2.5. *Let R be a one-dimensional Noetherian domain and let $\mathbf{q}$ be a height-one prime ideal of the power series ring $R[[y]]$. If $\mathbf{q} \neq yR[[y]]$, then $\mathbf{q}$ is contained in a unique maximal ideal of $R[[y]]$.*

Proposition 2.6. *Consider the nested mixed polynomial/power series rings*

$$\begin{aligned} A := k[x,y] \hookrightarrow B := k[[y]]\,[x] \hookrightarrow C := k[x]\,[[y]] \\ \hookrightarrow D_1 := k[x]\,[[y/x]] \hookrightarrow D_2 := k[x]\,[[y/x^2]] \hookrightarrow \cdots \\ \hookrightarrow D_n := k[x]\,[[y/x^n]] \hookrightarrow \cdots \hookrightarrow E := k[x, 1/x]\,[[y]], \end{aligned}$$

where k is a field and x and y are indeterminates over k. Then

(1) *If $S \in \{B, C, D_1, D_2, \cdots, D_n, \cdots, E\}$, then $A \hookrightarrow S$ is not TGF.*
(2) *If $\{R, S\} \subset \{B, C, D_1, D_2, \cdots, D_n \cdots, E\}$ are such that $R \subseteq S$, then $R \hookrightarrow S$ is TGF.*
(3) *Each of the proper associated maps on spectra fails to be surjective.*

Proof. For item (1), let $\sigma(y) \in yk[[y]]$ be such that $\sigma(y)$ and y are algebraically independent over k. Then $(x-\sigma(y))S \cap A = (0)$, and so $A \hookrightarrow S$ is not TGF.

For item (2), observe that every maximal ideal of C, D_n or E is of height two with residue field finite algebraic over k. To show $R \hookrightarrow S$ is TGF, it suffices to show $\mathbf{q} \cap R \neq (0)$ for each height-one prime ideal $\mathbf{q}$ of S. This is clear if $y \in \mathbf{q}$. If $y \notin \mathbf{q}$, then $k[[y]] \cap \mathbf{q} = (0)$, and so $k[[y]] \hookrightarrow R/(\mathbf{q} \cap R) \hookrightarrow S/q$ are injections. By Corollary 2.5, $S/\mathbf{q}$ is a one-dimensional local domain. Since the residue field of $S/\mathbf{q}$ is finite algebraic over k, it follows that $S/\mathbf{q}$ is finite over $k[[y]]$. Therefore $S/\mathbf{q}$ is integral over $R/(\mathbf{q} \cap R)$. Hence $\dim\ (R/(\mathbf{q} \cap R) = 1$ and so $\mathbf{q} \cap R \neq (0)$.

For item (3), observe that xD_n is a prime ideal of D_n and x is a unit of E. Thus Spec $E \to$ Spec D_n is not surjective. Now, considering $C = D_0$ and $n > 0$, we have $xD_n \cap D_{n-1} = (x, y/x^{n-1})D_{n-1}$. Therefore xD_{n-1} is not in the image of the map Spec $D_n \to$ Spec D_{n-1}. The map from Spec $C \to$ Spec B is not onto, because $(1+xy)$ is a prime ideal of B, but $1+xy$ is a unit in C. Similarly Spec $B \to$ Spec A is not onto, because $(1+y)$ is a prime ideal of A, but $1+y$ is a unit in B. This completes the proof. □

Question/Remarks 2.7. Which of the Spec maps of Proposition 2.6 are one-to-one and which are finite-to-one?

(1) For $S \in \{B, C, D_1, D_2, \cdots, D_n, \cdots, E\}$, the generic fiber ring of the map $A \hookrightarrow S$ has infinitely many prime ideals and has dimension one. Every height-two maximal ideal of S contracts in A to a maximal ideal. Every maximal ideal of S containing y has height two. Also $yS \cap A = yA$ and the map Spec $S/yS \to$ Spec A/yA is one-to-one.

(2) Suppose $R \hookrightarrow S$ is as in Proposition 2.6.2. Each height-two prime of S contracts in R to a height-two maximal ideal of R. Each height-one prime of R is the contraction of at most finitely many prime ideals of S and all of these prime ideals have height one. If $R \hookrightarrow S$ is flat, which is true if $S \in \{B, C, E\}$, then "going-down" holds for $R \hookrightarrow S$, and so, for P a height-one prime of S, we have $\text{ht}\ (P \cap R) \leq 1$.

(3) As mentioned in [HW, Remark 1.5], C/P is Henselian for every nonzero prime ideal P of C other than yC.

3 Spectra for two-dimensional mixed polynomial/power series rings

Let k be a field and let x and y be indeterminates over k. We consider the prime spectra, as partially ordered sets, of the mixed polynomial/power series rings A, B, C, $D_1, D_2, \cdots, D_n, \cdots$ and E as given in Equations 1 and 2 of the introduction.

Even for k a countable field there are at least two non-order-isomorphic partially ordered sets that can be the prime spectrum of the polynomial ring

$A := k[x,y]$. Let $\mathbb{Q}$ be the field of rational numbers, let F be a field contained in the algebraic closure of a finite field and let $\mathbb{Z}$ denote the ring of integers. Then, by [rW1] and [rW2], Spec $\mathbb{Q}[x,y] \ncong$ Spec $F[x,y] \cong$ Spec $\mathbb{Z}[y]$.

The prime spectra of the rings B, C, $D_1, D_2, \cdots, D_n, \cdots$, and E of Equations 1 and 2 are simpler since they involve power series in y. Remark 2.3.2 implies that y is in every maximal ideal of height two of each of these rings.

The partially ordered set Spec B = Spec $k[[y]]\,[x]$ is similar to a prime ideal space studied in [HW] and [Shah]. The difference from [HW] is that here $k[[y]]$ is uncountable, even if k is countable. It follows that Spec B is also uncountable. As a partially ordered set, Spec B can be described uniquely up to isomorphism by the axioms of [Shah] (similar to the CHP axioms of [HW]), since $k[[y]]$ is Henselian and has cardinality at least equal to c, the cardinality of the real numbers $\mathbb{R}$.

The following theorem characterizes $U :=$ Spec B as a *Henselian affine* partially ordered set (where the "$\leq$" relation is "set containment"):

Theorem 3.1. [HW, Theorem 2.7] [Shah, Theorem 2.4] *Let $B = k[[y]]\,[x]$ be as in Equation 1, where k is a field, the cardinality of the set of maximal ideals of $k[x]$ is α and the cardinality of $k[[y]]$ is β. Then the partially ordered set $U :=$ Spec B is called Henselian affine of type (β, α) and is characterized as a partially ordered set by the following axioms:*

(1) $|U| = \beta$.
(2) *U has a unique minimal element.*
(3) *dim $(U) = 2$ and $|\{$ height-two elements of $U\}| = \alpha$.*
(4) *There exists a unique special height-one element $u \in U$ such that u is contained in every height-two element of U.*
(5) *Every nonspecial height-one element of U is in at most one height-two element.*
(6) *Every height-two element $t \in U$ contains cardinality β many height-one elements that are only contained in t. If $t_1, t_2 \in U$ are distinct height-two elements, then the special element from (4) is the unique height-one element less than both.*
(7) *There are cardinality β many height-one elements that are maximal.*

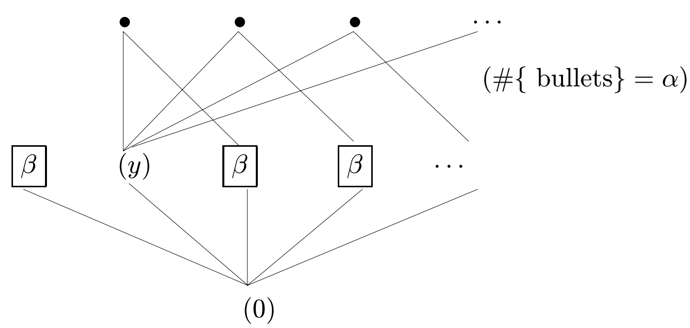
Remarks 3.2. (1) The axioms of Theorem 3.1 are redundant. We feel this redundancy helps in understanding the relationships between the prime ideals.

(2) The theorem applies to the spectrum of B by defining the unique minimal element to be the ideal (0) of B and the special height-one element to be the prime ideal yB. Every height-two maximal ideal $\mathbf{m}$ of B has nonzero intersection with $k[[y]]$. Thus $\mathbf{m}/yB$ is principal and so $\mathbf{m} = (y, f(x))$, for some monic irreducible polynomial $f(x)$ of $k[x]$. Consider $\{f(x) + ay \,|\, a \in k[[y]]\}$. This set has cardinality β and each $f(x)+ay$ is contained in a nonempty finite set of height-one primes contained in $\mathbf{m}$. If $\mathbf{p}$ is a height-one prime contained in $\mathbf{m}$ with $\mathbf{p} \neq yB$, then $\mathbf{p} \cap k[[y]] = (0)$, and so $\mathbf{p}k((y))[x]$ is generated by a monic polynomial in $k((y))[x]$. But for $a, b \in k[[y]]$ with $a \neq b$, we have

$(f(x)+ay, f(x)+by)k((y))[x] = k((y))[x]$. Therefore no height-one prime contained in $\mathbf{m}$ contains both $f(x)+ay$ and $f(x)+by$. Since B is Noetherian and $|B| = \beta$ is an infinite cardinal, we conclude that the cardinality of the set of height-one prime ideals contained in $\mathbf{m}$ is β. Examples of height-one maximals are $(1+xyf(x,y))$, for various $f(x,y) \in k[[y]]\,[x]$. The set of height-one maximal ideals of B also has cardinality β.

(3) These axioms *characterize* Spec B in the sense that every two partially ordered sets satisfying these axioms are order-isomorphic.

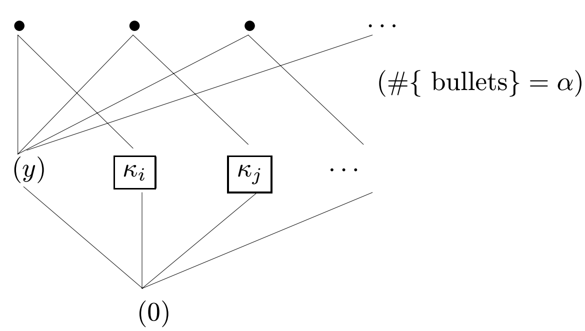
The picture of Spec B is shown below:



Spec $k[[y]]\,[x]$

In the diagram, β is the cardinality of $k[[y]]$, and α is the cardinality of the set of maximal ideals of $k[x]$ (and also the cardinality of the set of maximal ideals of $k[[y]]\,[x]$); the boxed β means there are cardinality β height-one primes in that position with respect to the partial ordering.

Next we consider Spec $R[[y]]$, for R a Noetherian one-dimensional domain. Then Spec $R[[y]]$ has the following picture by Theorem 3.4 below:



Spec($R[[y]]$)

Here α is the cardinality of the set of maximal ideals of R (and also the cardinality of the set of maximal ideals of $R[[y]]$ by Remarks 2.3.1); the boxed κ_i (one for each maximal ideal of R) means that there are cardinality κ_i prime ideals in that position, where each κ_i is uncountable. If R satisfies certain cardinality conditions described in Remark 3.3.3, for example, if $R = k[x]$, for k a countable field, then each κ_i equals the cardinality of $R[[y]]$. By Remark 3.3.2 below, each κ_i is at least $\gamma^{\aleph_0}$, where γ is the cardinality of $R/\mathbf{m}$ and $\mathbf{m}$ is the maximal ideal of R such that $(\mathbf{m}, y)$ is the maximal ideal of $R[[y]]$ above the κ_i height-one primes in the picture above.

Remarks 3.3. Let $\aleph_0$ denote the cardinality of the set of natural numbers. Suppose that T is a commutative ring of cardinality δ, that $\mathbf{m}$ is a maximal ideal of T and that γ is the cardinality of $T/\mathbf{m}$. Then

(1) The cardinality of $T[[y]]$ is $\delta^{\aleph_0}$, because the elements of $T[[y]]$ are in one-to-one correspondence with $\aleph_0$-tuples having entries in T. If T is Noetherian, then $T[[y]]$ is Noetherian, and so every prime ideal of $T[[y]]$ is finitely generated. Since the cardinality of the finite subsets of $T[[y]]$ is $\delta^{\aleph_0}$, it follows that $T[[y]]$ has at most $\delta^{\aleph_0}$ prime ideals.

(2) If T is Noetherian, there are at least $\gamma^{\aleph_0}$ distinct height-one prime ideals (other than $(y)T[[y]]$) of $T[[y]]$ contained in $(\mathbf{m}, y)T[[y]]$. To see this, choose a set $C = \{c_i \mid i \in I\}$ of elements of T so that $\{c_i + \mathbf{m} \mid i \in I\}$ gives the distinct coset representatives for $T/\mathbf{m}$. Thus there are γ elements of C, and for $c_i, c_j \in C$ with $c_i \neq c_j$, we have $c_i - c_j \notin \mathbf{m}$. Now also let $a \in \mathbf{m}, a \neq 0$. Consider the set

$$G = \{a + \sum_{n \in \mathbb{N}} d_n y^n \mid d_n \in C \, \forall n \in \mathbb{N}\}.$$

Each of the elements of G is in $(\mathbf{m}, y)T[[y]] \setminus yT[[y]]$ and hence is contained in a height-one prime contained in $(\mathbf{m}, y)T[[y]]$ distinct from $yT[[y]]$.

Moreover, $|G| = |C|^{\aleph_0} = \gamma^{\aleph_0}$. Let P be a height-one prime ideal of $T[[y]]$ contained in $(\mathbf{m}, y)T[[y]]$ but such that $y \notin P$. If two distinct elements of G, say $f = a + \sum_{n \in \mathbb{N}} d_n y^n$ and $g = a + \sum_{n \in \mathbb{N}} e_n y^n$, with the $d_n, e_n \in C$, are both in P, then so is their difference; that is

$$f - g = \sum_{n \in \mathbb{N}} d_n y^n - \sum_{n \in \mathbb{N}} e_n y^n = \sum_{n \in \mathbb{N}} (d_n - e_n) y^n \in P.$$

Now let t be the smallest power of y so that $d_t \neq e_t$. Then $(f-g)/y^t \in P$, since P is prime and $y \notin P$, but the constant term, $d_t - e_t \notin \mathbf{m}$, which contradicts the fact that $P \subseteq (\mathbf{m}, y)T[[y]]$. Thus there must be at least $|C|^{\aleph_0} = \gamma^{\aleph_0}$ distinct height-one primes contained in $(\mathbf{m}, y)T[[y]]$.

(3) Using (1) and (2), if T is Noetherian and if $\gamma^{\aleph_0} = \delta^{\aleph_0}$, then there are exactly $\gamma^{\aleph_0} = \delta^{\aleph_0}$ distinct height-one prime ideals (other than $yT[[y]]$) of $T[[y]]$ contained in $(\mathbf{m}, y)T[[y]]$. This is the case, for example, if T is countable, say $T = k[x]$ where k is a countable field, for then $|T| = \aleph_0$ and for every γ with $1 < \gamma \leq \aleph_0$, $\gamma^{\aleph_0} = \aleph_0^{\aleph_0} = c$, the cardinality of the real numbers.

Theorem 3.4. *Suppose that R is a one-dimensional Noetherian domain with cardinality $\delta := |R| =$ and that the cardinality of the set of maximal ideals of R is α (α can be finite). Let $U =$ Spec $R[[y]]$, where y is an indeterminate over R. Then*
(a) *U as a partially ordered set (where the "$\leq$" relation is "set containment") satisfies the following axioms:*

(1) $|U| \leq \delta^{\aleph_0}$.
(2) *U has a unique minimal element, namely* (0).
(3) *dim* $(U) = 2$ *and* $|\{$ *height-two elements of* $U \}| = \alpha$.
(4) *There exists a unique special height-one element $u \in U$ (namely $u = (y)$) such that u is contained in every height-two element of U.*
(5) *Every nonspecial height-one element of U is in exactly one height-two element.*
(6) *Every height-two element $t \in U$ contains uncountably many height-one elements that are contained only in t. (The number of height-one elements contained only in t is at least $\gamma^{\aleph_0}$, where γ is the cardinality of the residue field of the corresponding maximal ideal of R.) If $t_1, t_2 \in U$ are distinct height-two elements, then the special element from (4) is the unique height-one element less than both.*
(7) *There are no height-one maximal elements in U. Every maximal element has height two.*

(b) *If R is countable, or more generally if $\delta = |R|$ satisfies the condition of Remarks 3.3.3, that is, $\gamma^{\aleph_0} = \delta^{\aleph_0}$, for every γ that occurs as $|R/\mathbf{m}|$, where $\mathbf{m}$ is a maximal ideal of R, then $U =$ Spec $R[[y]]$ satisfies (1)-(7), with the stronger axioms (1′) and (6′):*

(1′) $|U| = \delta^{\aleph_0}$. *(For R countable, this is c, the cardinality of the real numbers.)*
(6′) *Every height-two element $t \in U$ contains $\delta^{\aleph_0}$ (uncountably many) height-one elements that are contained only in t.*

(c) *With the additional hypotheses of (b), U is characterized as a partially ordered set by the axioms given in (a) and (b). Every partially ordered set satisfying the axioms (1)-(7) in (a) and (b) is order-isomorphic to every other such partially ordered set.*

Proof. In part(a), item (1) is from Remark 3.3.1. Item (2) and the first part of (3) are clear. The second part of (3) follows immediately from Remark 2.3.1.

For items (4) and (5), suppose that P is a height-one prime of $R[[y]]$. If $P = yR[[y]]$, then P is contained in each maximal ideal of $R[[y]]$ by Remark 2.3.1, and so $yR[[y]]$ is the special element. If $y \notin P$, then, by Corollary 2.5, P is contained in a unique maximal ideal of $R[[y]]$.

For item (6) and items (1′) and (6′) of part (b) use Remarks 3.3.2 and 3.3.3.

For item (c), all partially ordered sets satisfying the axioms of Theorem 3.1 are order-isomorphic, and the partially ordered set U of the present theorem satisfies the same axioms as in Theorem 3.1 except axiom (7) that involves height-one maximals. Since U has no height-one maximals, an order-isomorphism between two partially ordered sets as in item (c) can be deduced by adding on height-one maximals and then deleting them. □

Corollary 3.5. *In the terminology of Equations 1 and 2 of the introduction, we have Spec $C \cong$ Spec $D_n \cong$ Spec E, but Spec $B \not\cong$ Spec C.*

Proof. The rings C, D_n, and E are all formal power series rings in one variable over a one-dimensional Noetherian domain R, where R is either $k[x]$ or $k[x, 1/x]$. Thus the domain R satisfies the hypotheses of Theorem 3.4 with the cardinality conditions of parts (b) and (c). If k is finite, then $|R| = |k[x]| = \aleph_0$ and α, the number of maximal ideals of R, is also $\aleph_0$; in this case $|R/\mathbf{m}| = \gamma$ is finite for each maximal ideal $\mathbf{m}$ of R and $\delta = |R| = \gamma \cdot \aleph_0 = \alpha$, and so $\gamma^{\aleph_0} = \delta^{\aleph_0}$. On the other hand, if k is infinite, then $|k| = |k[x]| = |R| = \alpha$, and $|k| = |R/\mathbf{m}| = \gamma$ is the same for every maximal ideal $\mathbf{m}$ of R. Hence also in this case $\delta = |R| = \gamma \cdot \aleph_0 = \alpha$, and so $\gamma^{\aleph_0} = \delta^{\aleph_0}$.

Also the number of maximal ideals is the same for C, D_n, and E, because in each case, it is the same as the number of maximal ideals of R which is $|k[x]| = |k| \cdot \aleph_0$.

Thus in the picture of $R[[y]]$ shown above, for $R[[y]] = C, D_n$ or E, the κ_i are all equal to $|k|^{\aleph_0}$ and $\alpha = |k| \cdot \aleph_0$, and so the spectra are isomorphic. The spectrum of B is not isomorphic to that of C, however, because B contains height-one maximal ideals, such as that generated by $1 + xy$, whereas C has no height-one maximal ideals. □

Remarks 3.6. As mentioned at the beginning of this section, it is shown in [rW1] and [rW2] that Spec $\mathbb{Q}[x, y] \not\cong$ Spec $F[x, y] \cong$ Spec $\mathbb{Z}[y]$, where F is a field contained in the algebraic closure of a finite field. Corollary 3.7 shows that the spectra of power series extensions in y behave differently in that Spec $\mathbb{Z}[[y]] \cong$ Spec $\mathbb{Q}[x]\,[[y]] \cong$ Spec $F[x]\,[[y]]$.

Corollary 3.7. *If $\mathbb{Z}$ is the ring of integers, $\mathbb{Q}$ is the rational numbers, F is a field contained in the algebraic closure of a finite field, and $\mathbb{R}$ is the real numbers, then*

$$\textit{Spec}\ \mathbb{Z}[[y]] \cong \textit{Spec}\ \mathbb{Q}[x]\,[[y]] \cong \textit{Spec}\ F[x]\,[[y]] \not\cong \textit{Spec}\ \mathbb{R}[x]\,[[y]].$$

Proof. The rings $\mathbb{Z}, \mathbb{Q}[x]$ and $F[x]$ are all countable with countably infinitely many maximal ideals. Thus if $R = \mathbb{Z}, \mathbb{Q}[x]$ or $F[x]$, then R satisfies the hypotheses of Theorem 3.4 with the cardinality conditions of parts (b) and (c). On the other hand, $\mathbb{R}[x]$ has uncountably many maximal ideals; thus $\mathbb{R}[x]\,[[y]]$ also has uncountably many maximal ideals. □

4 Higher dimensional mixed power series/polynomial rings

In analogy to Equation (1), we display several embeddings involving three variables.

$$(4.0)\qquad k[x,y,z] \stackrel{\alpha}{\hookrightarrow} k[[z]]\,[x,y] \stackrel{\beta}{\hookrightarrow} k[x]\,[[z]]\,[y] \stackrel{\gamma}{\hookrightarrow} k[x,y]\,[[z]] \stackrel{\delta}{\hookrightarrow} k[x]\,[[y,z]],$$
$$k[[z]]\,[x,y] \stackrel{\epsilon}{\hookrightarrow} k[[y,z]]\,[x] \stackrel{\zeta}{\hookrightarrow} k[x]\,[[y,z]] \stackrel{\eta}{\hookrightarrow} k[[x,y,z]],$$

where k is a field and x, y and z are indeterminates over k.

Remarks 4.1. (1) By Proposition 2.6.2 every nonzero prime ideal of $C = k[x]\,[[y]]$ has nonzero intersection with $B = k[[y]]\,[x]$. In three or more variables, however, the analogous statements fail. We show below that the maps $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta$ in Equation (4.0) fail to be TGF. Thus, by Proposition 2.2.2, no proper inclusion in (4.0) is TGF. The dimensions of the generic fiber rings of the maps in the diagram are either one or two.

(2) For those rings in (4.0) of form $R = S[[z]]$ (ending in a power series variable) where S is a ring, such as $R = k[x,y][[z]]$, we have some information concerning the prime spectra. By Proposition 2.4 every height-two prime ideal not containing z is contained in a unique maximal ideal. By [N, Theorem 15.1] the maximal ideals of $S[[z]]$ are of the form $(\mathbf{m}, z)S[[z]]$, where $\mathbf{m}$ is a maximal ideal of S, and thus the maximal ideals of $S[[z]]$ are in one-to-one correspondence with the maximal ideals of S. As in section 3, using Remarks 2.3, we see that maximal ideals of Spec $k[[z]]\,[x,y]$ can have height two or three, that (z) is contained in every height-three prime ideal, and that every height-two prime ideal not containing (z) is contained in a unique maximal ideal.

(3) It follows by arguments analogous to that in Proposition 2.6.1, that α, δ, ϵ are not TGF. For α, let $\sigma(z) \in zk[[z]]$ be transcendental over $k(z)$; then $(x-\sigma)k[[z]]\,[x,y] \cap k[x,y,z] = (0)$. For δ and ϵ: let $\sigma(y) \in k[[y]]$ be transcendental over $k(y)$; then $(x-\sigma)k[x]\,[[z,y]] \cap k[x]\,[[z]]\,[y] = (0)$, and $(x-\sigma)k[[y,z]]\,[x] \cap k[[z]]\,[x,y] = (0)$.

(4) By [HRW1, Theorem 1.1], η is not TGF and the dimension of the generic fiber ring of η is one.

In order to show in Proposition 4.3 below that the map β is not TGF, we first observe:

Proposition 4.2. *The element $\sigma = \sum_{n=1}^{\infty}(xz)^{n!} \in k[x]\,[[z]]$ is transcendental over $k[[z]]\,[x]$.*

Proof. Consider an expression

$$Z := a_\ell \sigma^\ell + a_{\ell-1}\sigma^{\ell-1} + \cdots + a_1\sigma + a_0,$$

where the $a_i \in k[[z]]\,[x]$ and $a_\ell \neq 0$. Let m be an integer greater than $\ell + 1$ and greater than $\deg_x a_i$ for each i such that $0 \le i \le \ell$ and $a_i \neq 0$. Regard each $a_i\sigma^i$ as a power series in x with coefficients in $k[[z]]$.

For each i with $0 \le i \le \ell$, we have $i(m!) < (m+1)!$. It follows that the coefficient of $x^{i(m!)}$ in σ^i is nonzero, and the coefficient of x^j in σ^i is zero for every j with $i(m!) < j < (m+1)!$. Thus if $a_i \neq 0$ and $j = i(m!) + \deg_x a_i$, then the coefficient of x^j in $a_i\sigma^i$ is nonzero, while for j such that $i(m!) + \deg_x a_i < j < (m+1)!$, the coefficient of x^j in $a_i\sigma^i$ is zero. By our choice of m, for each i such that $0 \le i < \ell$ and $a_i \neq 0$, we have

$$(m+1)! > \ell(m!) + \deg_x a_\ell \ge i(m!) + m! > i(m!) + \deg_x a_i.$$

Thus in Z, regarded as a power series in x with coefficients in $k[[z]]$, the coefficient of x^j is nonzero for $j = \ell(m!) + \deg_x a_\ell$. Therefore $Z \neq 0$. We conclude that σ is transcendental over $k[[z]]\,[x]$. $\square$

Proposition 4.3. *$k[[z]]\,[x, y] \overset{\beta}{\hookrightarrow} k[x]\,[[z]]\,[y]$ is not TGF.*

Proof. Fix an element $\sigma \in k[x]\,[[z]]$ that is transcendental over $k[[z]]\,[x]$. We define $\pi : k[x]\,[[z]]\,[y] \to k[x]\,[[z]]$ to be the identity map on $k[x]\,[[z]]$ and $\pi(y) = \sigma z$. Let $\mathbf{q} = \ker \pi$. Then $y - \sigma z \in \mathbf{q}$. If $h \in \mathbf{q} \cap (k[[z]]\,[x, y])$, then

$$h = \sum_{j=0}^{s}\sum_{i=0}^{t}(\sum_{\ell\in\mathbb{N}} a_{ij\ell}z^\ell)x^iy^j, \text{ for some } \; s, t \in \mathbb{N} \text{ and } a_{ij\ell} \in k, \text{ and so}$$

$$0 = \pi(h) = \sum_{j=0}^{s}\sum_{i=0}^{t}(\sum_{\ell\in\mathbb{N}} a_{ij\ell}z^\ell)x^i(\sigma z)^j = \sum_{j=0}^{s}\sum_{i=0}^{t}(\sum_{\ell\in\mathbb{N}} a_{ij\ell}z^{\ell+j})x^i\sigma^j.$$

Since σ is transcendental over $k[[z]]\,[x]$, we have that x and σ are algebraically independent over $k((z))$. Thus each of the $a_{ij\ell} = 0$. Therefore $\mathbf{q} \cap (k[[z]][x, y]) = (0)$, and so the embedding β is not TGF. $\square$

Proposition 4.4. *$k[[y, z]]\,[x] \overset{\zeta}{\hookrightarrow} k[x]\,[[y, z]]$ and $k[x]\,[[z]]\,[y] \overset{\gamma}{\hookrightarrow} k[x, y]\,[[z]]$ are not TGF.*

Proof. For ζ, let $t = xy$ and let $\sigma \in k[[t]]$ be algebraically independent over $k(t)$. Define $\pi : k[x]\,[[y, z]] \to k[x]\,[[y]]$ as follows. For

$$f := \sum_{\ell=0}^{\infty}\sum_{m+n=\ell} f_{mn}(x)y^mz^n \in k[x]\,[[y, z]],$$

where $f_{mn}(x) \in k[x]$, define

$$\pi(f) := \sum_{\ell=0}^{\infty}\sum_{m+n=\ell} f_{mn}(x)y^m(\sigma y)^n \in k[x]\,[[y]].$$

In particular, $\pi(z) = \sigma y$. Let $\mathbf{p} := \ker \pi$. Then $z - \sigma y \in \mathbf{p}$, and so $\mathbf{p} \neq (0)$. Let $h \in \mathbf{p} \cap k[[y,z]]\,[x]$. We show $h = 0$. Now h is a polynomial with coefficients in $k[[y,z]]$, and we define $g \in k[[y,z]]\,[t]$, by, if $a_i(y,z) \in k[[y,z]]$ and

$$h := \sum_{i=0}^{r} a_i(y,z)x^i, \text{ then set } g := y^r h = \sum_{i=0}^{r}(\sum_{\ell=0}^{\infty}\sum_{m+n=\ell} b_{imn}y^m z^n)t^i.$$

The coefficients of g are in $k[[y,z]]$, since $y^r x^i = y^{r-i}t^i$. Thus

$$0 = \pi(g) = \sum_{i=0}^{r}(\sum_{\ell=0}^{\infty}\sum_{m+n=\ell} b_{imn}y^m(\sigma y)^n)t^i = \sum_{i=0}^{r}(\sum_{\ell=0}^{\infty}\sum_{m+n=\ell} b_{imn}\sigma^n y^\ell)t^i$$
$$= \sum_{\ell=0}^{\infty}(\sum_{m+n=\ell}(\sum_{i=0}^{r} b_{imn}t^i)\sigma^n)y^\ell.$$

Now t and y are analytically independent over k, and so the coefficient of each y^ℓ (in $k[[t]]$) is 0; since σ and t are algebraically independent over k, the coefficient of each σ^n is 0. It follows that each $b_{imn} = 0$, that $g = 0$ and hence that $h = 0$. Thus the extension ζ is not TGF.

To see that γ is not TGF, we switch variables in the proof for ζ, so that $t = yz$. Again choose $\sigma \in k[[t]]$ to be algebraically independent over $k(t)$. Define $\psi : k[x,y]\,[[z]] \to k[y]\,[[z]]$ by $\psi(x) = \sigma z$ and ψ is the identity on $k[y]\,[[z]]$. Then ψ can be extended to $\pi : k[y]\,[[x,z]] \to k[y]\,[[z]]$, which is similar to the π in the proof above. As above, set $\mathbf{p} := \ker \pi$; then $\mathbf{p} \cap k[[x,z]]\,[y] = (0)$. Thus $\mathbf{p} \cap k[x]\,[[z]]\,[y] = (0)$ and γ is not TGF. □

Proposition 4.5. *Let k be a field and let x and t be indeterminates over k. Then $\sigma = \sum_{n=1}^{\infty} t^{n!}$ is algebraically independent over $k[[x, xt]]$.*

Proof. Let ℓ be a positive integer and consider an expression

$$\gamma := \gamma_\ell \sigma^\ell + \cdots + \gamma_i \sigma^i + \cdots + \gamma_1 \sigma, \text{ where } \gamma_i := \sum_{j=0}^{\infty} f_{ij}(x)(xt)^j \in k[[x, xt]],$$

that is, each $f_{ij}(x) \in k[[x]]$ and $1 \le i \le \ell$. Assume that $\gamma_\ell \neq 0$. Let a_ℓ be the smallest j such that $f_{\ell j}(x) \neq 0$, and let m_ℓ be the order of $f_{\ell a_\ell}(x)$, that is, $f_{\ell a_\ell}(x) = x^{m_\ell} g_\ell(x)$, where $g_\ell(0) \neq 0$. Let n be a positive integer such that

$$n \geq 2 + \max\{\ell, m_\ell, a_\ell\}.$$

Since $\ell < n$, for each i with $1 \le i \le \ell$, we have

$$\sigma^i = \sigma_{i1}(t) + c_i t^{i(n!)} + t^{(n+1)!}\tau_i(t), \tag{3}$$

where c_i is a nonzero element of k, $\sigma_{i1}(t)$ is a polynomial in $k[t]$ of degree at most $(i-1)n! + (n-1)!$ and $\tau_i(t) \in k[[t]]$.

Claim 4.6. The coefficient of $t^{\ell(n!)+a_\ell}$ in $\sigma^\ell\gamma_\ell = \sigma^\ell(\sum_{j=a_\ell}^{\infty} f_{\ell j}(x)(xt)^j)$ as a power series in $k[[x]]$ has order $m_\ell + a_\ell$, and hence, in particular, is nonzero.

Proof. By the choice of n, $(n+1)! > \ell(n!)+a_\ell$. Hence by the expression for σ^ℓ given in Equation 3, we see that all of the terms in $\sigma^\ell\gamma_\ell$ of the form $bt^{\ell(n!)+a_\ell}$, for some $b \in k[[x]]$, appear in the product

$$(\sigma_{\ell 1}(t) + c_\ell t^{\ell(n!)})(\sum_{j=a_\ell}^{\ell(n!)+a_\ell} f_{\ell j}(x)(xt)^j).$$

One of the terms of the form $bt^{\ell(n!)+a_\ell}$ in this product is

$$c_\ell t^{\ell(n!)} f_{\ell a_\ell}(x)(xt)^{a_\ell} = (c_\ell x^{m_\ell+a_\ell} g_\ell(x))t^{\ell(n!)+a_\ell} = (c_\ell x^{m_\ell+a_\ell} g_\ell(0)+\ldots)t^{\ell(n!)+a_\ell}.$$

Since $c_\ell g_\ell(0)$ is a nonzero element of k, $c_\ell x^{m_\ell+a_\ell} g_\ell(x) \in k[[x]]$ has order $m_\ell + a_\ell$. The other terms in the product $\sigma^\ell\gamma_\ell$ that have the form $bt^{\ell(n!)+a_\ell}$, for some $b \in k[[x]]$, are in the product

$$(\sigma_{\ell 1}(t))(\sum_{j=a_\ell}^{\ell(n!)+a_\ell} f_{\ell j}(x)(xt)^j) = \sum_{j=a_\ell}^{\ell(n!)+a_\ell} f_{\ell j}(x)(xt)^j \sigma_{\ell 1}(t).$$

Since $\deg_t \sigma_{\ell 1} \le (\ell-1)n! + (n-1)!$ and since, for each j with $f_{\ell j}(x) \ne 0$, we have $\deg_t f_{\ell j}(x)(xt)^j = j$, we see that each term in $f_{\ell j}(xt)^j \sigma_{\ell 1}(t)$ has degree in t less than or equal to $j + (\ell-1)n! + (n-1)!$. Thus each nonzero term in this product of the form $bt^{\ell(n!)+a_\ell}$ has

$$j \ge \ell(n!) + a_\ell - (\ell-1)(n!) - (n-1)! = a_\ell + (n-1)!(n-1) > m_\ell + a_\ell,$$

by choice of n. Moreover, for j such that $f_{\ell j}(x) \ne 0$, the order in x of $f_{\ell j}(x)(xt)^j$ is bigger than or equal to j. This completes the proof of Claim 4.6.

Claim 4.7. For $i < \ell$, the coefficient of $t^{\ell(n!)+a_\ell}$ in $\sigma^i\gamma_i$ as a power series in $k[[x]]$ is either zero or has order greater than $m_\ell + a_\ell$.

Proof. As in the proof of Claim 4.6, all of the terms in $\sigma^i\gamma_i$ of the form $bt^{\ell(n!)+a_\ell}$, for some $b \in k[[x]]$, appear in the product

$$(\sigma_{i1} + c_i t^{i(n!)})(\sum_{j=0}^{\ell(n!)+a_\ell} f_{ij}(x)(xt)^j) = \sum_{j=0}^{\ell(n!)+a_\ell} f_{ij}(x)(xt)^j(\sigma_{i1} + c_i t^{i(n!)}).$$

Since $\deg_t(\sigma_{i1} + c_i t^{i(n!)}) = i(n!)$, each term in $f_{ij}(x)(xt)^j(\sigma_{i1} + c_i t^{i(n!)})$ has degree in t at most $j + i(n!)$. Thus each term in this product of the form $bt^{\ell(n!)+a_\ell}$, for some nonzero $b \in k[[x]]$, has

$$j \ge \ell(n!) + a_\ell - i(n!) \ge n! + a_\ell > m_\ell + a_\ell.$$

Thus $\operatorname{ord}_x b \ge j > m_\ell + a_\ell$. This completes the proof of Claim 4.7. Hence $\gamma \notin k[[x, xz]]$ and so Proposition 4.5 is proved.

Question/Remarks 4.8. (1) As we show in Proposition 2.6, the embeddings from Equation 1 involving two-dimensional mixed power series/polynomial rings over a field k with inverted elements are TGF. So far we have not determined whether the same is true in the three-dimensional case. For example, is θ below TGF?

$$k[x,y]\,[[z]] \overset{\theta}{\hookrightarrow} k[x,y,1/x]\,[[z]]$$

(2) For the four dimensional case, as observed in the introduction, it follows from [HR, p. 364, Theorem 1.12] that the extension $k[x,y,u]\,[z] \hookrightarrow k[x,y,u,1/x,]\,[[z]]$ is not TGF. We provide in Proposition 4.9 a direct proof of this fact.

Proposition 4.9. *For k a field and x, y, u and z indeterminates over k, the extension $k[x,y,u]\,[[z]] \hookrightarrow k[x,y,u,1/x,]\,[[z]]$ is not TGF.*

Proof. Let $t = z/x$ and let $\sigma \in k[[t]]$ be algebraically independent over $k[[x,z]]$. (By Proposition 4.5, we may take $\sigma = \sum_{r=1}^{\infty} t^{r!}$.)

Consider

$$\pi : k[[x,y,u]]\,[1/x]\,[[z]] \to k[[x,u]]\,[1/x]\,[[z]]$$

defined by mapping

$$\sum_{i=0}^{\infty} a_i(x,y,u,1/x)z^i \mapsto \sum_{i=0}^{\infty} a_i(x,\sigma u,u,1/x)z^i,$$

where $a_i(x,y,u,1/x) \in k[[x,y,u]][1/x]$. Let $\mathbf{p} = \ker \pi$. Then $y - \sigma u \in \mathbf{p}$. We show that $\mathbf{p} \cap k[[x,y,u,z]] = (0)$, and so also $\mathbf{p} \cap k[x,y,u]\,[[z]] = (0)$. Let

$$f := \sum_{\ell=0}^{\infty} (\sum_{i+j=\ell} d_{ij} u^i y^j) \in k[[x,y,u,z]],$$

where $d_{ij} \in k[[x,z]]$. If $f \in \mathbf{p}$, then

$$0 = \pi(f) = \sum_{\ell=0}^{\infty} (\sum_{i+j=\ell} d_{ij} u^i \sigma^j u^j) = \sum_{\ell=0}^{\infty} (\sum_{i+j=\ell} d_{ij} \sigma^j) u^\ell.$$

This is a power series in u, and so, for each ℓ, $\sum_{i+j=\ell} d_{ij}\sigma^j = 0$. Since σ is algebraically independent over $k[[x,z]]$, each $d_{ij} = 0$. Thus $f = 0$. This completes the proof of Proposition 4.9.

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