PROJECTIVE LINES OVER ONE-DIMENSIONAL SEMILOCAL DOMAINS AND SPECTRA OF BIRATIONAL EXTENSIONS

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Dedicated to Shreeram S. Abhyankar on his 60-th birthday

1. Introduction. In [Na1], Nashier asked if the condition on a one-dimensional local domain R that each maximal ideal of the Laurent polynomial ring $R[y, y^{-1}]$ contracts to a maximal ideal in $R[y]$ or in $R[y^{-1}]$ implies that R is Henselian. Motivated by this question, we consider the structure of the projective line $\mathrm{Proj}(R[s, t])$ over a one-dimensional semilocal domain R (the projective line regarded as a topological space, or equivalently as a partially ordered set). In particular, we give an affirmative answer to Nashier's question. (Nashier has also independently answered his question [Na3].) Nashier has also studied implications on the prime spectrum of the Henselian property in [Na2] as well as in the papers cited above.

We also investigate the structure of prime spectra of finitely generated birational extensions of $R[y]$ and of blowups of parameter ideals of a two-dimensional Cohen-Macaulay local domain. In each case we note some analogies with $\mathrm{Spec}(R[y])$, which was analyzed in [HW].

Since the Henselian property is so crucial to this work, it seems appropriate to thank Professor Abhyankar here for his inspiration and contributions to an earlier paper [AHW]. In [AHW] an example was constructed of a non-Henselian local two-dimensional domain D such that D/P is Henselian for each height-one prime ideal P of D.

The present paper is in part an extension and generalization of work in [HW]. One of the results of that paper is the following:

Theorem. *Let R be a countable one-dimensional semilocal domain.*

(1) If R is not Henselian and has exactly n maximal ideals, then $\mathrm{Spec}(R[y])$ *is isomorphic (as topological spaces or partially ordered sets) to* $\mathrm{Spec}(L[y])$*, where L is a localization of the integers* $\mathbf{Z}$ *outside n distinct nonzero prime ideals.*

(2) If R is Henselian (which implies R is local), then $\mathrm{Spec}(R[y])$ *is isomorphic to* $\mathrm{Spec}(H[y])$*, where H is a Henselization within the complex numbers of* $\mathbf{Z}$ *localized outside* $2\mathbf{Z}$.

In analogy with the affine case given in the Theorem above, we prove in Theorem 2.3 that if R is a countable one-dimensional Noetherian domain with n maximal ideals, then up to homeomorphism or isomorphism, there are exactly two possibilities for $\mathrm{Proj}(R[s, t])$ if $n = 1$, and only one if $n > 1$. As before, the two cases distinguish between Henselian and non-Henselian rings.

In Section 3 we consider certain birational extensions of the polynomial ring $R[y]$, where R is a one-dimensional semilocal domain. For example, if $(R, \mathbf{m})$ is a countable one-dimensional local domain and $f \in R[y] - \mathbf{m}[y]$, then $\operatorname{Spec}(R[y]) \cong \operatorname{Spec}(R[y, 1/f])$. But if the ideal $fR[y]$ has prime radical and B is a finitely generated R-algebra that is properly between $R[y]$ and $R[y, 1/f]$, we show in Proposition 3.1 that $\operatorname{Spec}(B)$ is not homeomorphic to $\operatorname{Spec}(R[y])$.

Section 4 concerns the blowup of a parameter ideal of a two-dimensional Cohen-Macaulay local domain. We show in Proposition 4.1 that affine pieces of this blowup satisfy many of the axioms satisfied by the spectrum of a polynomial ring in one variable over a one-dimensional local domain. Proposition 4.2 gives similar results for the entire blowup.

All rings we consider are commutative and contain a multiplicative identity. The terms "local" and "semilocal" include "Noetherian." The symbol $<$ between sets means proper inclusion.

It will be convenient to set some notation for partially ordered sets from earlier papers:

1.1 Notation. For U a partially ordered set, $u \in U$, and T a subset of U,

$$\mathrm{G}(u) = \{w \in U \mid w > u\}\ , \quad \mathrm{L}(u) = \{w \in U \mid w < u\}\ ,$$
$$\mathrm{L_e}(T) = \{w \in U \mid \mathrm{G}(w) = T\}.$$

Note that the set called $\mathrm{L}(T)$ in [HW] is denoted $\mathrm{L_e}(T)$, the "exactly-less-than" set, here.

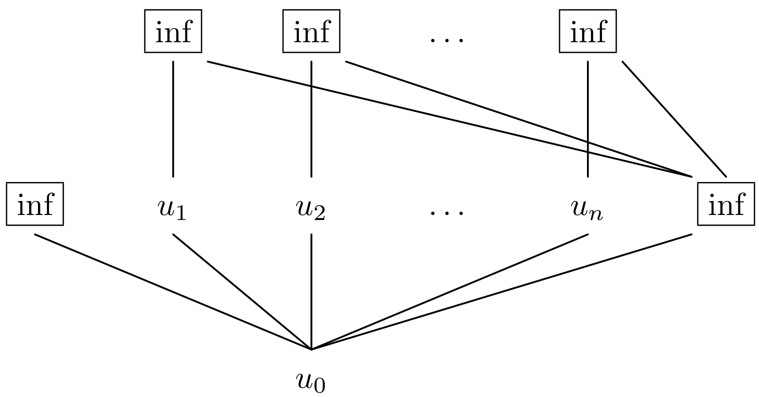
We will be concerned with partially ordered sets of dimension two with a unique minimal element, specifically the spectra of two-dimensional integral domains. In this context, if P is a height-one prime, then $\mathrm{G}(P)$ is the set of maximal ideals containing P, while if T is a set of height-two maximal ideals, then $\mathrm{L_e}(T)$ is the set of height-one primes contained in the intersection of the elements of T and not contained in any other maximal ideal of the ring.

Roger Wiegand has conjectured in [rW] that the spectrum of any two-dimensional domain that is a finitely generated algebra over $\mathbf{Z}$ is homeomorphic to the spectrum of the polynomial ring $\mathbf{Z}[y]$. It is shown in [rW] that if k is a field and A is a two-dimensional domain that is finitely generated as a k-algebra, then $\operatorname{Spec}(A) \cong \operatorname{Spec}(\mathbf{Z}[y])$ if and only if k is contained in the algebraic closure of a finite field. His method was to provide an axiom system characterizing $\operatorname{Spec}(\mathbf{Z}[y])$ up to homeomorphism or isomorphism. Motivated by his result, the following axiom systems were formulated in [HW]:

1.2 Definition. *A partially ordered set U is "$C\mathbf{Z}(n)P$" if it satisfies:*

- (P0) *U is countable.*
- (P1) *U has a unique minimal element u_0.*
- (P2) *U has dimension two.*
- (P3) *There exist infinitely many height-one maximal ideals.*
- (P4) *There exist n height-one nonmaximal "special" elements $u_1, u_2, \dots u_n$ satisfying: (i) $\mathrm{G}(u_1) \cup \dots \cup \mathrm{G}(u_n) = \{$ height-two elements of U $\}$, (ii) $\mathrm{G}(u_i) \cap \mathrm{G}(u_j) = \emptyset$ for $i \neq j$, and (iii) $\mathrm{G}(u_i)$ is infinite for each i, $1 \leq i \leq n$.*
- (P5) *For each height-one nonspecial element u, $G(u)$ is finite.*
- (P6) *For each nonempty finite subset T of $\{$ height-two elements of U $\}$, $\mathrm{L_e}(T)$ is infinite.*

(Pictorially, a $C\mathbf{Z}(n)P$ partially ordered set looks like this:

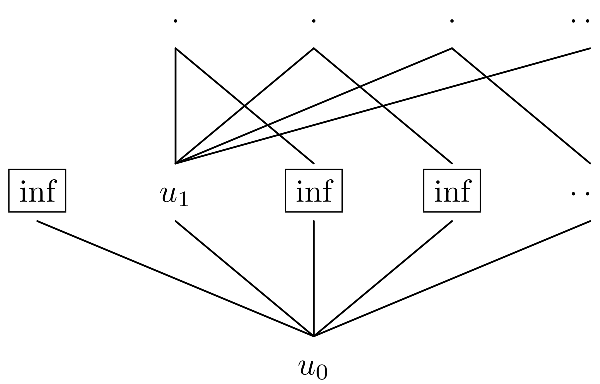


The relationships of the lower right boxed section, determined by (P5) and (P6), are too complicated to display.)

1.3 Definition. *A partially ordered set U is "CHP" if it satisfies:*

(P0)–(P5) *Same as $C\mathbf{Z}(1)P$ above.*

(P6) *For each finite subset T of $\{$ height-two elements of U $\}$ of cardinality greater than one, $\mathrm{L_e}(T) = \emptyset$. For each singleton $t \in \{$ height-two elements of U $\}$, $\mathrm{L_e}(\{t\})$ is infinite.*



It was shown in [HW] that (1) these axiom systems are categorical; (2) if $(R, \mathbf{m}_1, \ldots, \mathbf{m}_n)$ is a countable semilocal one-dimensional domain that is not Henselian, then Spec($R[y]$ is $C\mathbf{Z}(n)P$; and (3) if R is a countable Henselian one-dimensional (local) domain, then Spec($R[y]$) is CHP. We use these facts in the present paper.

R. Wiegand proves in [rW] that if D is an order in an algebraic number field, then $\mathrm{Spec}(D[y]) \cong \mathrm{Spec}(\mathbf{Z}[y])$. A crucial point in this proof is his axiomatic characterization of Spec($\mathbf{Z}[y]$), and the crucial axiom here is (rW5), called (P5) in [rW], which states that if $P_1, \ldots, P_r$ are height-one primes and $M_1, \ldots, M_s$ are maximal ideals, then there exists a height-one prime Q such that $Q \subset M_i$, for each $i = 1, \ldots, s$, and if M is a maximal ideal containing Q and some P_i, then M is one of the M_j. If A is a two-dimensional domain that is finitely generated as a $\mathbf{Z}$-algebra and if P is a height-one prime of A, then it is known that every maximal ideal of A/P is the radical of a principal ideal. It follows that Spec(A) satisfies a restricted version of axiom (rW5) where $r = 1$ and $s = 1$. This motivates the following.

Question. Suppose A is a two-dimensional Noetherian domain having the property that $\mathrm{Spec}(A)$ is countable, every height-one prime of A is contained in infinitely many maximal ideals, and for each height-one prime P and each maximal ideal M containing P, there exists a height-one prime Q such that $P+Q$ is M-primary, does it follow that $\mathrm{Spec}(A)$ satisfies axiom (rW5) mentioned above?

Our work in this paper is part of an on-going study of the general question: What partially ordered sets arise as the prime spectrum of a Noetherian ring? This question is entirely open, even for two-dimensional rings. It is even unknown how to characterize polynomial rings over one-dimensional countable rings (even polynomial rings in two variables over a countable field).

2. The projective line over a one-dimensional semilocal domain.

Let $(R, \mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_n)$ be a one-dimensional semilocal domain and s, t be indeterminates. In this section, we study the projective line X over R. It will be convenient to use two interpretations of the projective line: (1) $X = \mathrm{Proj}(R[s,t])$, the set of relevant homogeneous primes in the polynomial ring in two indeterminates over R, and (2) X is the union of its affine pieces $\mathrm{Spec}(R[y])$ and $\mathrm{Spec}(R[1/y])$, where $y = s/t$. (The only elements in the second affine piece that are not in the first are the height-one prime $(1/y)R[1/y]$ and the height-two maximals $(\mathbf{m}_i, 1/y)R[1/y]$, and the two pieces intersect in $\mathrm{Spec}(R[y, 1/y])$.) We will refer to homogeneous relevant prime ideals of $R[s,t]$ as points of X. Each height-two point of X has the form $(\mathbf{m}, f(s,t))R[s,t]$ where $\mathbf{m}$ is a maximal ideal of R and f is a homogeneous polynomial of which the image $\mod \mathbf{m}$ is irreducible in $(R/\mathbf{m})[s,t]$. In such an f the highest power of at least one of s, t has coefficient not in $\mathbf{m}$; and if only one (say s) has coefficient not in $\mathbf{m}$, then $f(s,t)$ can be taken to be s times an element of $R - \mathbf{m}$. (Warning: If R is not integrally closed, the ideal $f(s,t)R[s,t]$ need not be prime despite the fact that its image in $(R/\mathbf{m})[s,t]$ is a prime ideal.)

In analogy with the axiom systems in [rW] and [HW], we introduce the following:

2.1 Definition. *We say that the partially ordered set U is "$\mathbf{P}C\mathbf{Z}(n)P$" if it satisfies:*

- (P0) *U is countable.*
- (P1) *U has a unique minimal element u_0.*
- (P2) *U has dimension two.*
- (P3) *Every maximal element has height two.*
- (P4) *There exist n height-one nonmaximal "special" elements $u_1, u_2, \dots u_n$ satisfying: (i) $\mathrm{G}(u_1) \cup \dots \cup \mathrm{G}(u_n) = \{$ height-two elements of U $\}$, (ii) $\mathrm{G}(u_i) \cap \mathrm{G}(u_j) = \emptyset$ for $i \neq j$, and (iii) $\mathrm{G}(u_i)$ is infinite for each i, $1 \leq i \leq n$.*
- (P5) *For each height-one nonspecial element u, $G(u)$ is finite and $\mathrm{G}(u) \cap \mathrm{G}(u_i) \neq \emptyset$ for each i, $1 \leq i \leq n$.*
- (P6) *For each nonempty finite subset T of $\{$ height-two elements of U $\}$ such that $\{u_1, \dots, u_n\} \subseteq \bigcup \{L(t) \mid t \in T\}$, $\mathrm{L_e}(T)$ is infinite. (Here $\mathrm{L_e}(T)$ is the exactly-less-than set.)*

2.2 Definition. *We say that the partially ordered set U is "$\mathbf{P}CHP$" if it satisfies:*

- (P0)–(P5) *Same as $\mathbf{P}C\mathbf{Z}(1)P$ above.*
- (P6) *For each finite subset T of $\{$ height-two elements of U $\}$ of cardinality greater than one, $\mathrm{L_e}(T) = \emptyset$. For each singleton $t \in \{$ height-two elements of U $\}$, $\mathrm{L_e}(\{t\})$ is infinite. ($\mathrm{L_e}(T)$ as above.)*

2.3 Theorem. *Let R be a countable one-dimensional semilocal Noetherian domain with n maximal ideals. If $n = 1$, then the projective line over R is $\mathbf{P}CHP$ if R is Henselian and $\mathbf{P}C\mathbf{Z}(1)P$ otherwise. If $n > 1$, then the projective line over R is $\mathbf{P}C\mathbf{Z}(n)P$.*

The proof of this result will occupy most of this section. We show first that these axiom systems are categorical:

2.4 Lemma. *Every two partially ordered sets which satisfy the axioms $\mathbf{P}CHP$ are order-isomorphic. The same is true for $\mathbf{P}C\mathbf{Z}(n)P$ for any fixed positive integer n.*

Proof. We show this for $\mathbf{P}C\mathbf{Z}(n)P$; the argument for $\mathbf{P}CHP$ is similar, and both are only slight adaptations of those of [rW] or [HW]: Given two posets U, V satisfying $\mathbf{P}C\mathbf{Z}(n)P$, define the order-isomorphism $f : U \to V$ by sending the minimal element u_0 to the minimal element v_0, the n height-one special elements $u_1, \ldots, u_n$ bijectively to the n height-one special elements $v_1, \ldots, v_n$, and for each i, $1 \leq i \leq n$, the elements of $\mathrm{G}(u_i)$ to the elements of $\mathrm{G}(f(u_i))$, each in any bijective way. Now enumerate the nonspecial height-one elements of U: $u_{n+1}, u_{n+2}, \ldots$, and for $k > n$, enumerate $\mathrm{L_e}(f(\mathrm{G}(u_k)))$ in such a way that if $k' < k$ but $\mathrm{G}(u_{k'}) = \mathrm{G}(u_k)$, then $\mathrm{L_e}(f(\mathrm{G}(u_k)))$ is enumerated in the same order as $\mathrm{L_e}(f(\mathrm{G}(u_{k'})))$. Then inductively define $f(u_k)$ to be the first element of $\mathrm{L_e}(f(\mathrm{G}(u_k)))$ that is not of the form $f(u_{k'})$ for some $k' < k$. $\square$

We now begin to show that for a countable one-dimensional semilocal domain R, $X = \mathrm{Proj}(R[s,t])$ is either $\mathbf{P}C\mathbf{Z}(n)P$ or $\mathbf{P}CHP$. Since we are assuming that R is countable, so is $R[s,t]$. The relevant homogeneous primes in $R[s,t]$ are generated by finite subsets, so X is also countable, and (P0) holds. This is the only use we make of the hypothesis of countability on R.

Of course (0) is the unique minimal element of X, so (P1) holds. Since $R[s,t]$ has Krull dimension 3 and the irrelevant maximal ideals $(\mathbf{m}_i, s, t)$ are not elements of X, we see that $\dim(X) = 2$, i.e., (P2) holds.

Axiom (P3) follows from the second assertion in (P5). For (P4), as in the affine case, the "special" elements are the extensions $\mathbf{m}_i[s,t]$ to $R[s,t]$ of the maximal ideals $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n$ of R. Since any two of these extensions generate the unit ideal $R[s,t]$, it is clear that no point of X contains two of them; so (P4)*(ii)* holds. Since $\mathrm{Proj}((R/\mathbf{m}_i)[s,t])$ is the (infinite) projective line over the field $R/\mathbf{m}_i$, we also have (P4)*(iii)*.

To see that X satisfies (P4)*(i)* and (P5), we picture X as the union of its affine pieces $\mathrm{Spec}(R[y])$ and $\mathrm{Spec}(R[1/y])$. Since these affine spectra are either $C\mathbf{Z}(n)P$ or CHP [HW, p. 583], we see that each height-two point in X contains one of the special elements, i.e., that (P4)*(i)* holds; and that each nonspecial height-one element is contained in only finitely many height-two points, i.e., that the first part of (P5) holds.

To see that the second part of (P5) holds, assume by way of contradiction that the height-one nonspecial prime P in $R[y]$ is comaximal with the special prime $\mathbf{m}[y]$ in X. We may safely localize all the rings in question at the complement of $\mathbf{m}$ in R, so we assume that R is local and P is a height-one maximal in $R[y]$. Since $yR[y]$ is not maximal, $y \notin P$, so P survives in the localization $R[y, 1/y]$ of $R[y]$ at the powers of y, i.e., $PR[y, 1/y] \in \mathrm{Spec}(R[y, 1/y])$. There are polynomials $f(y)$ in P and $g(y)$ in $\mathbf{m}[y]$ for which $1 = f(y) + g(y)$, so the coefficients of $f(y)$, except the constant term, are in $\mathbf{m}$, but the constant term is a unit. Hence the extension of P to $R[y, 1/y]$ contains $f(0)^{-1}f(y)/y^{\deg(f)}$, a monic polynomial in $1/y$. Let $Q = R[1/y] \cap PR[y, 1/y]$ (i.e., the element in $\mathrm{Spec}(R[1/y])$ corresponding to P in $\mathrm{Spec}(R[y])$). Then since Q contains a monic polynomial and meets R in (0), $R[1/y]/Q$ is integral over R, so it has a maximal ideal lying over $\mathbf{m}$, and hence Q is contained in a maximal ideal of $R[1/y]$ that also contains $\mathbf{m}[1/y]$. It follows that the element of X represented by P or Q is not comaximal with the special element represented by the extension of $\mathbf{m}$, the desired contradiction.

We now begin the proof that $X = \mathrm{Proj}(R[s, t])$ satisfies P(6) of $\mathbf{P}C\mathbf{Z}(n)P$ or $\mathbf{P}CHP$. We deal first with the non-Henselian case. Note first that by adjoining to the field of fractions K of R the roots and a $\deg(f)$-th root of the leading coefficient of a dehomogenized version of $f(s, t)$ (i.e., $f(s/t, 1)$ or $f(1, t/s)$) to obtain a field L, and letting S be the integral closure of R in L, we have that each of the points of $\mathrm{Proj}(S[s, t])$ lying over a height-two point of X is of the form $(\mathbf{n}, as + bt)S[s, t]$ where $\mathbf{n}$ is a maximal ideal of S and $a, b \in S$, not both in $\mathbf{n}$.

We use the following lemma to deduce the existence of a generic point (in the sense of [K, Def. 4.7, p. 25]) for a certain subset of $\mathrm{Proj}(R[s, t])$ from the fact that an appropriate set in $\mathrm{Proj}(S[s, t])$ has a generic point.

2.5 Lemma. *Let $B = \bigoplus_{n=0}^{\infty} B_n$ be a graded ring and $A = \bigoplus_{n=0}^{\infty} A_n$ be a graded subring (in the sense that $A \cap B_n = A_n$ for each n) such that $A \subseteq B$ satisfies the going-up property. (In particular, this holds if B is integral over A.) Let $\mathcal{Q}$ be a set of homogeneous prime ideals in B. If there exists a homogeneous prime ideal $\mathbf{q}$ of B such that $\mathcal{Q} = \{Q : Q$ is a homogeneous prime ideal in B containing $\mathbf{q}\}$, then $\mathbf{p} = \mathbf{q} \cap A$ is a homogeneous (prime) ideal in A, and*

$$\{Q \cap A : Q \in \mathcal{Q}\} = \{P : P \textit{ is a homogeneous prime ideal in } A \textit{ containing } \mathbf{p}\} .$$

Proof. The homogeneous components of an element of $\mathbf{q} \cap A$ are in both $\mathbf{q}$ and A (the latter because of the uniqueness of the expression of an element of $B = \bigoplus_{n=0}^{\infty} B_n$ as a sum of its homogeneous components); so $\mathbf{p}$ is homogeneous. Any $Q \cap A$, for Q in $\mathcal{Q}$, clearly contains $\mathbf{p}$, so let P be a homogeneous prime of A containing $\mathbf{p}$. By going-up, there is a prime ideal Q_1 of B containing $\mathbf{q}$ and such that $Q_1 \cap A = P$. The homogeneous ideal $\mathbf{q} + PB$ of B is contained in Q_1, so Q_1 contains a minimal prime Q of $\mathbf{q} + PB$. By [K, Proposition 5.11, p. 34], Q is homogeneous, and $\mathbf{q} \subseteq Q$, so $Q \in \mathcal{Q}$. Also, since $PB \subseteq Q \subseteq Q_1$, $Q \cap A = P$. □

We can now verify that, if R is not Henselian, then for a set T satisfying the hypothesis of (P6) of $\mathbf{P}C\mathbf{Z}(n)P$, $\mathrm{L_e}(T)$ is at least nonempty. Note that by the second assertion of (P5), if the set T does not satisfy the hypothesis of (P6), then $\mathrm{L_e}(T)$ is empty.

2.6 Theorem. *Suppose R is not Henselian. Then for each finite set $M_1, \dots, M_r$ of height-two points of $X = \mathrm{Proj}(R[s, t])$ such that each maximal ideal of R is contained in at least one M_i, there is a height-one element P of X that is contained in $M_1, \dots, M_r$ but not in any other height-two point of X.*

Proof. In view of Lemma 2.5, we may replace R by its integral closure in a finite algebraic extension of its field of fractions K, and the collection $\{M_1, \dots, M_r\}$ by the (possibly larger) set of points in the projective line over that integral closure that lie over these M_i. Therefore, we may assume that each M_i has the form $(\mathbf{m}, as + bt)R[s, t]$ for some $\mathbf{m}$ maximal in R and some $a, b \in R$, not both in $\mathbf{m}$.

Now we use the fact that R is not Henselian. Since each of the maximal ideals of R is ∞-split [HW, Theorem 1.1], there exists a finite algebraic extension L of K for which the integral closure S of R in L has the property that, for each $\mathbf{m}$ maximal in R, the number of maximal ideals $\mathbf{n}$ of S lying over $\mathbf{m}$ is greater than or equal to the number of M_i containing $\mathbf{m}$. For each $\mathbf{n}$, we pick an $M_i = (\mathbf{n} \cap R, as+bt)R[s,t]$ in such a way that every M_i is picked at least once, and set $N_{\mathbf{n}} = (\mathbf{n}, as+bt)S[s,t]$. Then $N_{\mathbf{n}} \cap R[s,t] = M_i$.

Since the a, b now vary with $\mathbf{n}$, we write them as $a_{\mathbf{n}}, b_{\mathbf{n}}$. By the Chinese Remainder Theorem, there are elements a, b of S for which $a \equiv a_{\mathbf{n}} \mod \mathbf{n}$ and $b \equiv b_{\mathbf{n}} \mod \mathbf{n}$ for every maximal ideal $\mathbf{n}$ of S. Let $Q = (as+bt)S[s,t]$. Since not both $a_{\mathbf{n}}, b_{\mathbf{n}}$ are in $\mathbf{n}$ for each $\mathbf{n}$, a, b generate the unit ideal in S; so Q is a prime ideal, that is, $Q \in Y = \mathrm{Proj}(S[s,t])$. Observe that for each maximal ideal $\mathbf{n}$ of S, the polynomial $as+bt$ is in exactly one height-two point of Y containing $\mathbf{n}$ (because the image of $as+bt$ in the polynomial ring $(S/\mathbf{n})[s,t]$ over the field $S/\mathbf{n}$ is a nonzero linear form). Therefore, the set $\{N_{\mathbf{n}} : \mathbf{n} \in \mathrm{Mspec}(S)\}$ is precisely the set of height-two points of Y that contain Q. Since $\{N_{\mathbf{n}} \cap R[s,t] : \mathbf{n} \in \mathrm{Mspec}(S)\} = \{M_1, \ldots, M_r\}$, it follows from Lemma 2.5 that $P = Q \cap R[s,t]$ is contained in $M_1, \ldots, M_r$ but not in any other height-two point of X. $\square$

Next, we argue that, if R is Henselian, then (P6) in $\mathbf{P}CHP$ holds. Suppose R is Henselian (and hence local, with maximal ideal $\mathbf{m}$). Then no two distinct height-two points of X contain the same nonspecial height-one element of X. For, if $y = s/t$ and P is a height-one prime of the polynomial ring $R[y]$ such that $P \cap R = (0)$, then P is contained in a unique maximal ideal of $R[y]$ [HW, Proposition 1.4]; if P is not itself maximal, it suffices to observe that P contains a monic polynomial in y and therefore is not contained in the height-two point at infinity for $\mathrm{Spec}(R[y])$ in X (i.e., the prime in $R[1/y]$ corresponding to P in X is not contained in the maximal ideal $(\mathbf{m}, 1/y)R[1/y]$). To see that P contains a monic polynomial in y, consider the domain $R[y]/P = D$, an algebraic extension of R. The integral closure S of R in the field of fractions L of D is a local domain since R is Henselian and a finite intersection of DVR's since R is a one-dimensional local domain. Therefore S is the unique DVR of L containing R. Since D is not a field, it follows that $D \subseteq S$, and hence P contains a monic polynomial in y. Thus we have shown that, for t a height-two element of X, $\mathrm{L_e}(\{t\})$ is at least nonempty, since any nonspecial height-one element u contained in t is such that $\mathrm{G}(u) = \{t\}$. (In fact, since a height-two prime in the Noetherian ring $R[y]$ contains infinitely many height-one primes, we get the full strength of the second sentence in (P6) of $\mathbf{P}CHP$ immediately. But the next paragraph treats both Henselian and non-Henselian cases at once.)

Finally, we complete the proof of (P6) in both the Henselian and non-Henselian cases, by showing that if $\mathrm{L_e}(T)$ is nonempty, then it is infinite: For a height-one nonspecial element P of $\mathrm{Proj}(R[s,t])$, recall $\mathrm{G}(P) = \{M \in \mathrm{Proj}(R[s,t]) : \mathrm{ht}(M) = 2 \text{ and } P \subset M\}$. We contend that, given a finite set $\mathcal{M}$ of height-two points of $\mathrm{Proj}(R[s,t])$ such that $\mathcal{M} = \mathrm{G}(P)$ for some height-one nonspecial element P of $\mathrm{Proj}(R[s,t])$, there are infinitely many height-one nonspecial elements P of $\mathrm{Proj}(R[s,t])$ for which $\mathrm{G}(P) = \mathcal{M}$. To see this, let S be a domain that is a finitely generated integral extension of R such that, in $\mathrm{Proj}(S[s,t])$, there is a finite set of maximal ideals $\mathcal{N}$ such that (1) each maximal ideal of S is contained in exactly one element of $\mathcal{N}$ (i.e., the map $\mathcal{N} \to \mathrm{Mspec}(S) : N \mapsto S \cap N$ is a bijection), (2) there is at least one N in $\mathcal{N}$ lying over each M in $\mathcal{M}$, and (3) each N in $\mathcal{N}$ has the form $N = (S \cap N, a_N s + b_N t)S[s,t]$ with a_N, b_N in S, not both in $S \cap N$. (In the non-Henselian case, we saw in the proof of Theorem 2.6 that such an S exists. In the Henselian case, there is only one M; it contains the unique maximal ideal $\mathbf{m}$ of R, and S can be any extension such that the generator of the image of M in $(R/\mathbf{m})[s,t]$ has a linear factor over the residue field of S.) Then choose a, b in S such that $a \equiv a_N \mod (S \cap N)$ and $b \equiv b_N \mod (S \cap N)$ for each N in $\mathcal{N}$ and note that, if $P = (as + bt)L[s,t] \cap R[s,t]$, where L is the field of fractions of S, then $P \subset M$ iff $M \in \mathcal{M}$. Note that $P = f(s,t)K[s,t] \cap R[s,t]$, where K is the field of fractions of R and f is an irreducible element in $K[s,t]$, unique up to constant multiple, of which $as + bt$ is a factor in $L[s,t]$. Now, the choice of a, b above was determined only up to the (infinite) Jacobson radical J of S; we could add any element of J to either of a, b without changing the resulting $\mathrm{G}(P)$. But since a nonzero element f of $K[s,t]$ has only finitely many nonassociate linear factors over an algebraic closure of K, if we fix a nonzero a and add to b nonzero elements of the Jacobson radical of S, then the prime ideals in $L[s,t]$ generated by the elements $as + bt$ are distinct, and only finitely many of these different primes can give the same P. Thus, there are infinitely many P that give the same $\mathrm{G}(P)$.

The proof of Theorem 2.3 is now complete. We close this section by providing our affirmative answer to Nashier's question.

2.7 Proposition. *Let $(R, \mathbf{m})$ be a one-dimensional local domain and y an indeterminate. If for every maximal ideal P in $R[y, 1/y]$, either $P \cap R[y]$ is maximal in $R[y]$ or $P \cap R[1/y]$ is maximal in $R[1/y]$, then R is Henselian.*

Proof. Assume R is not Henselian and let $X = \mathrm{Spec}(R[y]) \cup \mathrm{Spec}(R[1/y])$ be the projective line over R. By the proof of Theorem 2.3, X satisfies (P1)–(P6) of $\mathbf{P}C\mathbf{Z}(1)P$. If P is any height-one element of X that is in $\mathrm{L_e}((\mathbf{m}, y)R[y], (\mathbf{m}, 1/y)R[1/y])$, then $PR[y, 1/y]$ is maximal, while both $PR[y]$ and $PR[1/y]$ are nonmaximal.

An alternative proof, not using Theorem 2.3, is the following: Assuming R is not Henselian, by [N, (43.12)], R has a finite integral extension A that is not local, and the integral closure A' of A is also not local, though it is a semilocal PID. Let $N_1, \dots, N_n$ be all the maximal ideals of A', and pick an element c of the field of fractions K of A such that $c \in N_1 A'_{N_1}$ and $c \notin A'_{N_i}$ for $2 \le i \le n$. Then since none of the maximal ideals of A' survive in $A'[c, 1/c]$, $A'[c, 1/c]$ is a field. Since it is an integral extension of $R[c, 1/c]$, $R[c, 1/c]$ is also a field. Hence the kernel of the R-homomorphism $R[y, 1/y] \to \mathrm{K}$: $y \mapsto c$ is a maximal ideal P. But since $R[c] \subseteq A'_{N_1}$ and $R[1/c] \subseteq A'_{N_2}$, $R[c]$ and $R[1/c]$ are not fields, so neither $P \cap R[y]$ nor $P \cap R[1/y]$ is maximal. □

3. Spectra of birational extensions of the affine line.

In this section we establish the following result:

3.1 Proposition. *Let $(R, \mathbf{m}_1, \dots, \mathbf{m}_n)$ be a one-dimensional semilocal domain, K its field of fractions, y an indeterminate, $A = R[y]$, $f \in A - \bigcup_{i=1}^{n} \mathbf{m}_i[y]$, and B a finitely generated A-algebra strictly between A and $A[1/f]$. Then $\mathrm{Spec}(B)$ satisfies the following axioms from $C\mathbf{Z}(n)P$ or CHP (Definitions 1.2 and 1.3):*

- (a) *(P0) holds if R is countable.*
- (b) *(P1)–(P3) hold without additional hypotheses.*
- (c) *The number m of "special" elements (height-one elements $u_1, \dots, u_m$ for which (P4)(iii) holds, i.e., $\mathrm{G}(u_i)$ is infinite), may be greater than the number n of*

maximal ideals of R, but it is still finite, and (P4)(i) and (P5) hold (the latter trivially). Any "special" element meets R in a maximal ideal.

(d) *If fA has prime radical, then $m > n$ and (P4)(ii) may fail, i.e., the "special" elements need not be comaximal.*

3.2 Remark. (1) $\mathrm{Spec}(A[1/f]) \cong \mathrm{Spec}(A)$, since $\mathrm{Spec}(A[1/f])$ and $\mathrm{Spec}(A)$ both satisfy the axioms for either $C\mathbf{Z}(n)P$ or CHP. The reason for this is that, in localizing A at f, only finitely many height-one primes of A are lost, none of them special, and consequently only finitely many maximal ideals (those containing those height-one nonspecials) are lost.

(2) If B were a non-Noetherian ring strictly between A and $A[1/f]$, then (P6) of both $C\mathbf{Z}(n)P$ and CHP could fail, and the partially ordered set $\mathrm{Spec}(B)$ could fail to represent $\mathrm{Spec}(C)$ for any Noetherian ring C. For example, if $R = k[x]_{(x)}$, $f = y$, and $B = R[y, x/y, x/y^2, x/y^3, \dots]$, then B has a height-two maximal ideal $M = yB$, that contains only one height-one prime $P = \bigcap_{n=1}^{\infty} y^n B$; cf. [Ka, page 7, Exercise 5]. But this phenomenon is impossible in a Noetherian ring: By Krull's Principal Ideal Theorem, every height-two prime ideal M in a Noetherian ring must contain infinitely many height-one primes. (For, if M contained only r height-one primes $P_1, \dots, P_r$, then for any a in $M - \bigcup_{i=1}^{r} P_i$, the height-two prime ideal M would be minimal over a, a contradiction.)

(3) The stronger hypothesis that B is finitely generated as an algebra over A is used below to insure that the dimension formula holds.

We now begin the proof of Proposition 3.1. If R is countable, then so is B, and since B is also Noetherian, $\mathrm{Spec}(B)$ is countable.

Of course, $\mathrm{Spec}(B)$ always has unique minimal element (0).

We claim that B has dimension two. Indeed, a bit more generally, if $f \in A - \mathrm{Jac}(R)A$ and $B \subseteq A[1/f]$, then $\dim(A[1/f]) = 2$ and since $A[1/f] = B[1/f]$, $\dim(B) \geq 2$. Since B is also a birational extension of the two-dimensional Noetherian domain A, we have $\dim(B) \leq 2$ so $\dim(B) = 2$.

At most finitely many of the height-one maximals in A (those containing f) extend to the unit ideal in B. Let Q be a prime of B lying over a height-one maximal P in A not containing f. Then $B_Q = A_P$ and $Q = PA_P \cap B$ (since $P \cap R = 0$, so A_P is a localization of $K[y]$ and hence a DVR), and $A/P \subseteq B/Q \subseteq A_P/PA_P = A/P$ (the last equality because P is maximal), and hence Q is a height-one maximal in B. Therefore $\mathrm{Spec}(B)$ has infinitely many height-one maximals.

We want to see that the number of height-one primes Q in $\mathrm{Spec}(B)$ such that $\mathrm{G}(Q)$ is an infinite set is finite: Let Q be one of them. If it meets A in a nonspecial height-one prime P, then, because none of the height-two maximals of B containing Q meet A in P (for, if N is a prime in B such that $N \cap A = P$, then B_N is between the one-dimensional Noetherian domain A_P and its field of fractions and hence has dimension at most one), we get an infinite-to-finite map on the maximal spectra $\mathrm{Mspec}(B/Q) \to \mathrm{Mspec}(A/P)$, so that at least one of the extensions of maximals in A/P to the Noetherian ring B/Q would have infinitely many minimal primes, a contradiction. Thus Q meets A in either a special height-one prime or a height-two maximal, and in either case it meets R in a maximal ideal $\mathbf{m}$, and hence Q is a minimal prime of $\mathbf{m}B$. But since R is semilocal, so is $\bigcup\{\{\text{ minimal primes of } \mathbf{m}B \ \} : \mathbf{m} \in \mathrm{Mspec}(R)\}$.

Since $B < A[1/f]$, $fB \neq B$, so fB has at least one minimal prime Q, and since B is Noetherian, $\mathrm{ht}(Q) = 1$. Since R is Cohen-Macaulay, $A = R[X]$ is Cohen-

Macaulay, so every associated prime of fA is of height one. If $P_1, \dots P_m$ are the associated primes of fA, then

$$A = A[1/f] \cap A_{P_1} \cap \dots \cap A_{P_m} = B \cap A_{P_1} \cap \dots \cap A_{P_m} \ .$$

Suppose that $m = 1$, i.e., that fA has prime radical P (e.g., $f = y$). In this case, since $f \notin \bigcup_{i=1}^{n} \mathbf{m}_i A$, $P \cap R = 0$, so P is contracted from $K[y]$, and hence A_P is a DVR. Assume that the center on A of a prime Q of B is exactly P; then $A_P \subseteq B_Q < K(y)$, and hence (since A_P is a DVR) $A_P = B_Q$. So:

$$B \subseteq A[1/f] \cap B_Q = A[1/f] \cap A_P = A \ ,$$

a contradiction. Therefore, for each minimal prime Q of fB, $Q \cap A$ properly contains P and hence is a height-two maximal in A. By the dimension formula, e.g., [M, pages 84–86] (since A is Cohen-Macaulay, it is universally catenary),

$$1 = \text{ht}(Q) = \text{ht}(Q \cap A) + \text{tr.deg.}(B/A) - \text{tr.deg.}(B/Q)/(A/(Q \cap A)) \ .$$

Since $\text{ht}(Q \cap A) = 2$, and $\text{tr.deg.}(B/A) = 0$, we see that $\text{tr.deg.}(B/Q)/(A/(Q \cap A)) = 1$, and since B/Q is finitely generated over the field $A/(Q \cap A)$, B/Q has infinitely many maximal ideals. So Q is contained in infinitely many maximal ideals of B, i.e., the closure $\text{G}(Q)$ is infinite. But for each maximal ideal $\mathbf{m}$ of R, $\mathbf{m}A[1/f] \cap B$ is also a height-one prime with infinite closure.

Finally, to see that the height-one primes Q of $\text{Spec}(B)$ with infinite closure need not be comaximal, we provide an example: Let R be a discrete rank-one valuation domain with maximal ideal $\mathbf{m} = aR$, let $f = y$, and let $B = A[a/y]$. Now yB amd $(a/y)B$ are height-one primes with infinite closures (since $B/yB \cong B/(a/y)B \cong (R/\mathbf{m})[t]$); but they are not comaximal, because $(y, a/y)B$ is a proper ideal of B. This completes the proof of Proposition 3.1.

The example in the last paragraph is somewhat special. We remark that even under the following hypotheses, it is possible that B has exactly two height-one primes with infinite closure, and these two primes are comaximal: Let R be a DVR with $\mathbf{m} = aR$, $A = R[y]$, fA a height-one prime ideal of A such that $fA \cap R = (0)$, $g \in A - fA$ (so that $A < A[g/f]$) such that $(f, g)A < A$ (so that $A[g/f] < A[1/f]$), and $B = A[g/f]$. One such example is obtained by setting $f = y^2 + a^3$ and $g = y$. The two height-one primes of B with infinite closure are $\mathbf{m}B[1/f] \cap B$ and $(a, y)B$; the former contains $a^3/(y^2 + a^3)$, and the latter y, so they are comaximal.

We close this section with two questions suggested by the axiom systems $C\mathbf{Z}(n)P$ and CHP.

Questions. 1. If R is not Henselian and $\mathcal{M}$ is a finite set of height-two maximals of B, is there a height-one prime P of B for which $\mathcal{M} = \text{G}(P)$ (i.e., $\mathcal{M}$ is precisely the set of maximal ideals of B that contain P)? We remark that if R is Henselian and P is a height-one prime of B distinct from the finitely many minimal primes of $\mathbf{m}B$, then P is contained in a unique maximal ideal of B. Therefore, if R is Henselian, then there exist such sets $\mathcal{M}$ for which there is no corresponding P.

2. Given a set $\mathcal{M}$ such that $\mathcal{M} = \text{G}(P)$ for one height-one prime P in B, are there infinitely many P for which $\mathcal{M} = \text{G}(P)$?

4. Spectra of parameter blowups of two-dimensional local domains.

Let $(R, \mathbf{m})$ be a two-dimensional Cohen-Macaulay local domain and let x, y be a system of parameters for R, i.e., the ideal $(x, y)R$ is primary for the maximal ideal $\mathbf{m}$ of R. In this section we examine the "blowup" of the ideal $(x, y)R$, to see how many of the axioms above it satisfies.

We consider first an affine piece $A = R[y/x]$ of the blowup, and we refer to the axiom systems $C\mathbf{Z}(1)P$ and CHP (Definitions 1.2 and 1.3 above). Since x, y form a regular sequence, the kernel of the R-algebra homomorphism of the polynomial ring $R[t] \to A$ defined by $t \mapsto y/x$ is the principal ideal $(xt - y)R[t]$, which is contained in $\mathbf{m}R[t]$, a height-two prime ideal of $R[t]$; so $\mathbf{m}A$ is a height-one prime ideal of A. Moreover, $A/\mathbf{m}A \cong (R/\mathbf{m})[t]$, a polynomial ring in one indeterminate over the residue field of R. Thus, the maximal ideals of A containing $\mathbf{m}A$ are in one-to-one correspondence with the maximal ideals of this polynomial ring; in particular, there are infinitely many height-two maximal ideals of A containing $\mathbf{m}A$. On the other hand, for any height-one prime Q of A distinct from $\mathbf{m}A$, $Q \cap R = P$ is a height-one prime in R; since the ideal $(xt - y)R[t]$ is not contained in $PR[t]$, the image of y/x in A/Q is algebraic over R/P, and since this image generates A/Q over R/P, A/Q is a semilocal Noetherian domain of dimension at most one. Therefore, $\operatorname{Spec}(A)$ satisfies axiom (P5) of either CHP or $C\mathbf{Z}(1)P$ in [HW]. Also, axioms (P1) and (P2) clearly hold for $\operatorname{Spec}(A)$, as does (P0) if R is assumed to be countable. Let us observe that there are infinitely many height-one maximal ideals in A: No two of the elements $x - y^{n+1}$, as n varies over the natural numbers, are in the same height-one prime of R; if P is a minimal prime of such an element, then since $x \notin P$, $A \subseteq R_P$ and $PR_P \cap A = Q$ is maximal in A (since in A/Q the image of y/x is the inverse of the image of y^n, an element in the maximal ideal of R/P). Thus, (P3) also holds. To see (P4), all that remains to show is that every height-two maximal N of A meets R in $\mathbf{m}$; so assume that for some N, $N \cap R = P$ has height one. Then the ring of fractions of A with respect to the complement of P in R lies between the one-dimensional Noetherian domain R_P and its field of fractions, so its dimension is at most one; but N survives in this ring of fractions, a contradiction.

Let Q be a height-one prime of A other than $\mathbf{m}A$, and set $P = Q \cap R$. If R/P is Henselian, then A/Q is algebraic over a one-dimensional Henselian local domain and hence is local (cf. [HW, pp. 577–8]). Thus, Q is contained in a unique maximal ideal of A. Suppose that R/P is Henselian for each height-one prime P of R; then each height-one prime of A other than $\mathbf{m}A$ is contained in a unique maximal ideal. If N is a height-two maximal of A, then N is the union of the height-one primes contained in it. Since each of these height-one primes other than $\mathbf{m}A$ is contained in no maximal ideal except N, we see that $\operatorname{Spec}(A)$ satisfies axiom (P6) of CHP.

Thus we have shown:

4.1 Proposition. *Let R be a two-dimensional Cohen–Macaulay local domain, x, y be a system of parameters of R, and $A = R[y/x]$. Then* $\operatorname{Spec}(A)$ *satisfies axioms (P1)–(P5) of* [HW]*. If R is countable and, for each height-one prime P of R, R/P is Henselian, then* $\operatorname{Spec}(A)$ *is CHP.*

It is shown in [AHW] that the hypotheses in Proposition 4.1, including the assumption that R/P is Henselian for each height-one prime P, do not imply that R is Henselian.

So we turn our attention to the case where some R/P is not Henselian, and try to prove (P6) of $C\mathbf{Z}(1)P$. An example relevant to our situation here is the following. Let k be a field and let x, y be indeterminates over k. Let R be the ring $k[y(y-1), y^2(y-1)][[x]]$ localized at the maximal ideal generated by $y(y-1), y^2(y-1)$, and x. Let $f = (x - y^2(y-1))/(y(y-1))$, let $A = R[f]$, and let P be the height-one prime of R generated by x. Then $A \subseteq R_P$. Let $Q = PR_P \cap A$. Since the image of f in A/Q is the same as that of y and since adjoining this element to $R/P \cong k[y(y-1), y^2(y-1)]_{(y(y-1), y^2(y-1))}$ gives a ring with two maximal ideals, we see that Q is contained in precisely two maximal ideals of A. Note that if P' is a height-one prime of R that is distinct from P, then R/P' is complete and therefore Henselian. Therefore, if Q' is a height-one prime of A distinct from both Q and $\mathbf{m}A$, then as we observed above Q' is contained in a unique maximal ideal of A. Therefore in this example Spec(A) is neither CHP nor $C\mathbf{Z}(1)P$. So it is natural to ask:

Question. If for each height-one prime P of R the ring R/P is not Henselian, does it follow that Spec(A) satisfies $C\mathbf{Z}(1)P$?

We can provide a first step toward a proof of (P6) of $C\mathbf{Z}(1)P$: For each maximal ideal N of height two of A we show that there exists a height-one prime Q contained in N and not contained in any other maximal ideal of A: If N is a height-two maximal ideal in A, then as we saw above, $\mathbf{m} = N \cap R$. Further above we noted that $A/\mathbf{m}A$ may be identified with the polynomial ring $(R/\mathbf{m})[t]$, where t is the image of y/x. Hence $N = (\mathbf{m}, f)A$, where the image $\overline{f}$ of f in $(R/\mathbf{m})[t]$ is a monic irreducible polynomial. If $\overline{f} = \overline{r_0} + \overline{r_1}t + \ldots + t^n$ for $r_i \in R$, and we set $f = r_0 + r_1(y/x) + \ldots + (y/x)^n$, then N is the unique height-two prime of A that contains f. It follows that there exists a height-one prime Q of A contained in N having the property that N is the unique maximal ideal of A containing Q: Take Q to be a minimal prime of the principal ideal fA.

It seems plausible that, given a height-two maximal ideal N in A, we can find infinitely many height-one primes Q contained in N but not in any other maximal ideal of A. But we wonder whether for every finite set of height-two maximal ideals of A there exists a height-one prime Q of A that is contained in precisely this set of maximal ideals. In certain examples this is the case. For instance, let x, y be indeterminates over a field k, and set $R = k[x, y]_{(x,y)}$ and $A = R[y/x]$. Then using the fact that A is a ring of fractions of $k[x]_{(x)}[y/x]$, we see by Section 2 that Spec(A) satisfies $C\mathbf{Z}(1)P$.

Now let us consider the entire blowup of the ideal $I = (x, y)R$, i.e., $X = \text{Proj}(T)$, where $T = \bigoplus_{n=0}^{\infty} I^n$ is the Rees algebra of I; and refer to the axiom systems $\mathbf{P}C\mathbf{Z}(1)P$ and $\mathbf{P}CHP$. Since X is also the union of its affine pieces Spec$(R[y/x])$ and Spec$(R[x/y])$, Proposition 4.1 provides some of the answers immediately: If R is countable, then so is X. The poset X has a unique minimal element and dimension two. Every height-two point of X contains the extension of the maximal ideal $\mathbf{m}$ of R, and there are infinitely many height-two points. For a height-one element P of X distinct from the extension of the maximal ideal of R, G(P) is finite.

To show that X satisfies (P3), it suffices to show that if P is a height-one prime of R, then at least one of the rings $R[y/x], R[x/y]$ is contained in R_P, and the center of R_P on at least one of these rings is not a maximal ideal. If $x \in P$, then $y \notin P$, so $R[x/y] \subseteq R_P$, and the center of R_P on $R[x/y]$ is properly contained in $(\mathbf{m}, x/y)R[x/y]$. So we may assume that $x, y \notin P$, and hence both $R[y/x]$ and $R[x/y]$ are contained in R_P. Assume by way of contradiction that the center of R_P

on each ring is maximal, and let z denote the image of y/x in R_P/PR_P. Then the images $(R/P)[z]$ and $(R/P)[1/z]$ of $R[y/x]$ and $R[x/y]$ are both the field R_P/PR_P, so their intersection is again R_P/PR_P. But either z or $1/z$ is in every valuation ring between R/P and its field of fractions R_P/PR_P, so $(R/P)[z] \cap (R/P)[1/z]$ is integral over the one-dimensional domain R/P, the desired contradiction.

Suppose that for each height-one prime P of R, R/P is Henselian. Then as we saw above, a height-one element of $\mathrm{Spec}(R[y/x])$ distinct from the extension of $\mathbf{m}$ is contained in a unique maximal ideal. So the first sentence of (P6) of $\mathbf{P}CHP$ can fail for X only if there is a height-one prime P of R such that both $R[y/x]$ and $R[x/y]$ are contained in R_P and the center of R_P on $R[y/x]$ is properly contained in a maximal ideal that is lost in $R[x/y]$ and vice versa. Let P be a height-one of R such that both $R[y/x]$ and $R[x/y]$ are contained in R_P and the center of R_P on each is nonmaximal. Again let z denote the image of y/x in R_P/PR_P. Then $(R/P)[z]$ and $(R/P)[1/z]$ are both properly contained in the field of fractions R_P/PR_P of R/P. Since R/P is one-dimensional and Henselian, both z and $1/z$ are integral over R/P, so $(R/P)[z] = (R/P)[1/z]$ (cf. for example [N, (10.5)]). Therefore the height-two point in $\mathrm{Spec}(R[y/x])$ containing P is the same point of X as the one in $\mathrm{Spec}(R[x/y])$. The second sentence of (P6) of $\mathbf{P}CHP$ also follows, because a height-two maximal of $R[y/x]$ is the union of the height-one primes in it.

Thus we have shown:

4.2 Proposition. *Let R be a two-dimensional Cohen–Macaulay local domain, x, y be a system of parameters of R, $I = (x,y)R$, and $T = \bigoplus_{n=0}^{\infty} I^n$. Then the blowup* $\mathrm{Proj}(T)$ *of I satisfies axioms (P1)–(P5) of $\mathbf{P}C\mathbf{Z}(1)P$ or $\mathbf{P}CHP$. If R is countable and, for each height-one prime P of R, R/P is Henselian, then* $\mathrm{Proj}(T)$ *is* $\mathbf{P}CHP$.

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