MAXIMAL PRIME DIVISORS IN ARITHMETICAL RINGS

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ABSTRACT. We investigate for an ideal A of an arithmetical ring R the relationship between the set $\mathrm{Max}(A)$ of maximal prime divisors of A and the set $\mathcal{X}_A$ of maximal members of the set of Krull associated primes of A. We show that the arithmetical rings R such that $\mathcal{X}_A \subseteq \mathrm{Max}(A)$ for every regular ideal A are precisely those satisfying the "strong" separation property. For a Prüfer domain R, we prove that every branched prime ideal of height greater than one is the radical of a finitely generated ideal if and only if $\mathrm{End}(A)_M = \mathrm{End}(A_M)$ for every nonzero ideal A and maximal ideal M of R. We use this to prove that if in addition R is a QR-domain, then every maximal prime divisor of an ideal A of R is a Krull associated prime of A (i.e. $\mathcal{X}_A = \mathrm{Max}(A)$) if and only if each branched prime ideal of R of height greater than one is the radical of a finitely generated ideal.

INTRODUCTION

Let R be a commutative ring with identity. Associated to a proper ideal A of R is the set $S(A) = \{x \in R : xy \in A \text{ for some } y \in R \setminus A\}$ of elements of R that are non-prime to A. The complement $R \setminus S(A)$ of elements prime to A is closed under multiplication. Krull proves [14, page 732] that ideals maximal with respect to not meeting a multiplicatively closed set are prime ideals and defines the *maximal prime divisors* of A as the prime ideals P of R that contain A and are maximal with respect to the property $P \subseteq S(A)$.

Let $\mathrm{Max}(A)$ denote the set of maximal prime divisors of A. For $P \in \mathrm{Max}(A)$, Krull defines the ideal $A_{(P)} = \{x \in R : xy \in A \text{ for some } y \in R \setminus P\}$ to be the *principal component* of A with respect to P and establishes the decomposition of every proper ideal A of R as the intersection of its principal components $A = \bigcap_{P \in \mathrm{Max}(A)} A_{(P)}$ [14, Satz 2]. However, as we discuss in [4], a drawback to this decomposition is that we do not know what kind of ideals the $A_{(P)}$ are. For instance, there sometimes exist elements of P that are prime to the principal component $A_{(P)}$; indeed, the ideal $A_{(P)}$ is not in general a primal ideal, where an ideal B is said to be *primal* if the set $S(B)$ of elements non-prime to B is an ideal. If B is primal, then $S(B)$ is a prime ideal called the *adjoint prime* of B.

To obtain in a commutative ring without finiteness conditions a decomposition of the ideal A that involves components closely tied to A with structure we know, in [4] we define the set $\mathcal{X}_A$ of maximal Krull associated primes of A and obtain using the set $\mathcal{X}_A$ a canonical primal decomposition of A.

If R is a Noetherian ring and $\mathrm{Ass}(A) = \{P_1, P_2, \ldots, P_n\}$ is the set of associated primes of a proper ideal A of R, then $S(A) = \bigcup_{i=1}^n P_i$. It follows that the maximal prime divisors of A are exactly the prime ideals of R that are maximal members of $\mathrm{Ass}(A)$. In this sense, the maximal prime divisors of an ideal of a Noetherian ring are well-understood.

However if R is not Noetherian, the set $\mathrm{Max}(A)$ of maximal prime divisors of A is generally less transparent, and the primes in $\mathrm{Max}(A)$ need not be "associated." There are several inequivalent notions of an associated prime of an ideal of a general commutative ring, but from our point of view, it is the *Krull associated primes* that are most natural. We review this and related notions in Section 1. Motivated by the Noetherian case, we are thus interested in the question of when the set of maximal prime divisors of an ideal A coincides with the set of maximal members of the set of Krull associated primes of A. We shall examine this question for the class of *arithmetical rings*, those rings R such that for every maximal ideal M of R, R_M is a *valuation ring*, that is, a ring for which the set of ideals is linearly ordered by inclusion. Of particular interest is the class of *Prüfer domains*, namely the arithmetical integral domains.

Our purpose in the present paper is to investigate for ideals A of an arithmetical ring or Prüfer domain the interrelationship between the sets $\mathrm{Max}(A)$ and $\mathcal{X}_A$. The connection between these two sets of prime ideals is of special interest in the context of arithmetical rings. Indeed, if A is a proper ideal of an arithmetical ring R, then $\mathrm{Max}(A) = \mathcal{X}_A$ if and only if every prime ideal containing A but not prime to A is a Krull associated prime (Corollary 1.3). Also, Krull mentions [15, p. 16] that he does not know whether the principal components $A_{(P)}$ of A, $P \in \mathrm{Max}(A)$, are always P-primal ideals. (A primal ideal B is Q-primal if $Q = S(B)$.) Examples show that the answer is in the negative (see [17],[5] and [4, Example 3.8]); we investigate in Section 3 arithmetical rings for which the answer to this question is in the affirmative for all regular ideals. (An ideal A of R is *regular* if it contains a regular element, i.e., an element that is not a zero-divisor.) This is also of special interest, because an ideal of an arithmetical ring is primal exactly if it is irreducible (Theorem 1.8 of [4]).

In Section 3 we examine conditions on an arithmetical ring R in order that each maximal prime divisor of a regular ideal A of R be a Krull associated prime of A. It follows from Theorem 2.3 that a necessary condition for this to hold is that R satisfy the strong separation property: for regular prime ideals $P \subset Q$ and regular element $r \in P$, there exists $s \in Q$ such that $P \subset (r, s)R \subseteq Q$. We conclude that Prüfer domains such as the ring of entire functions and the ring $\mathrm{Int}(\mathbb{Z})$ of integer-valued polynomials contain ideals A having the property that there exists a maximal prime divisor of A that is not a Krull associated prime of A.

For a regular ideal A of an arithmetical ring, we consider in Theorem 3.7 conditions in order that the set of maximal prime divisors of A is precisely the set of prime ideals that are maximal among the Krull associated primes of A. We deduce a complete characterization of the QR-domains with this property. (By contrast, ideals of Noetherian rings always exhibit equality between these two sets of prime ideals.)

The results that lay the groundwork for this theorem touch on several interesting technical aspects of arithmetical rings and Prüfer domains. In particular, there is a connection between our problem and that of when $\mathrm{End}(X)$ localizes, that is, when $\mathrm{End}(X)_M = \mathrm{End}(X_M)$ for a submodule X of the quotient field of R and maximal ideals M of R.

1. Krull associated primes

In this section we briefly review the notion of a Krull associated prime of a proper ideal of a ring. We became interested in the Krull associated primes of an ideal because of their connection to the primal isolated components of the ideal. For a more complete treatment of these primes and their relation to issues involving the primal decompositions of an ideal, see [4].

Let A be a proper ideal of the ring R. Following [12], we define a prime ideal P of R to be a *Krull associated prime* of the ideal A if for every element $x \in P$, there exists $y \in R$ such that $x \in (A : y) \subseteq P$. A prime ideal P of R is called a *weak-Bourbaki associated prime* of A if it is a minimal prime divisor of $(A : x)$ for some $x \in R \backslash A$. Following [11, page 279], we call P a *Zariski-Samuel associated prime* of A if $P = \sqrt{(A : x)}$ for some $x \in R \setminus A$. If R is a Noetherian ring, then

all three of these notions of an associated prime coincide, but for non-Noetherian rings these notions are in general distinct (see [11], for example).

It is clear that a Zariski-Samuel associated prime is a weak-Bourbaki associated prime. Moreover, it is noted in Lemma 2.1 of [4] that P is a Krull associated prime of A if and only if P is a set-theoretic union of weak-Bourbaki primes of A. It is this characterization of Krull associated primes that we shall use in what follows.

We denote by $\mathrm{Ass}(A)$ the set of Krull associated primes of A. Notice that if A is a proper ideal of R, then $\mathrm{Ass}(A)$ is nonempty. By contrast, Nakano [19] gives an example of a Prüfer domain such that no finitely generated ideal has a Zariski-Samuel associated prime.

The set $\mathrm{Ass}(A)$ behaves well with respect to localization:

Lemma 1.1. (Lemma 2.4(ii) of [4]) *Let A be an ideal of a ring R and P be a prime ideal of R containing A. Then $P \in \mathrm{Ass}(A)$ if and only if $P_M \in \mathrm{Ass}(A_M)$ for some (or equivalently every) maximal ideal M of R containing P.*

It is not hard to see that if A is a proper ideal of a ring R, then every member of $\mathrm{Ass}(A)$ is contained in a maximal member of $\mathrm{Ass}(A)$. We define $\mathcal{X}_A$ to be the maximal members of $\mathrm{Ass}(A)$, that is, $\mathcal{X}_A$ consists of the maximal Krull associated primes of A. There exist examples of rings with ideals A such that $\mathcal{X}_A$ is infinite. (See Example 2.6 of [4] for example, or use Lemma 1.5 below.) These examples are necessarily non-Noetherian.

In Lemma 2.3 of [4], we observe that $S(A) = \bigcup_{P \in \mathcal{X}_A} P$. In particular, $\bigcup_{P \in \mathcal{X}_A} P = \bigcup_{Q \in \mathrm{Max}(A)} Q$. Despite this close connection between $\mathcal{X}_A$ and $\mathrm{Max}(A)$, one cannot conclude in general that $\mathcal{X}_A \subseteq \mathrm{Max}(A)$ or $\mathrm{Max}(A) \subseteq \mathcal{X}_A$. If $\mathrm{Max}(A)$ has only one member, then it is easy to see $\mathrm{Max}(A) = \mathcal{X}_A$. However, in Example 3.8 of [4] we construct a ring R with Noetherian prime spectrum and an ideal A of R such that $\mathrm{Max}(A)$ has only two elements, but neither maximal prime divisor of A is in $\mathcal{X}_A$. We also construct in Example 2.9 of [4] a 2-dimensional Prüfer domain R for which $\mathrm{Spec}(R)$ is Noetherian and for which there exist an ideal A and a finitely generated maximal ideal M such that M is a maximal prime divisor of A and yet $M \notin \mathrm{Ass}(A)$.

When R is arithmetical, the set $\mathrm{Ass}(A)$ has some striking properties. In the next sections we will need the following lemmas.

Lemma 1.2. (Proposition 2.7 of [4]) *Let R be an arithmetical ring and let A be a proper ideal of R. If $P \in \mathrm{Ass}(A)$, then $Q \in \mathrm{Ass}(A)$ for every prime ideal Q of R such that $A \subseteq Q \subseteq P$.*

Corollary 1.3. *Let A be a proper ideal of an arithmetical ring R. Then $\mathcal{X}_A = \mathrm{Max}(A)$ if and only if every prime ideal of R that contains A and is not prime to A is a Krull associated prime of A.*

Proof. Assume that $\mathcal{X}_A = \mathrm{Max}(A)$. If P contains A and is not prime to A, then P is contained in some $Q \in \mathcal{X}_A$. Hence by Lemma 1.2, $Q \in \mathrm{Ass}(A)$. The converse is clear. $\square$

Let A be a nonzero proper ideal of a Prüfer domain R and let Q be a prime ideal of R with $A \subseteq Q$. Since R_Q is a valuation domain, $\mathrm{End}(AR_Q) = R_P$ for some $P \in \mathrm{Spec}\, R$ with $P \subseteq Q$. Define

$$\mathcal{E}_A = \{P \in \mathrm{Spec}\, R : R_P = \mathrm{End}(AR_Q) \text{ for some prime ideal } Q \text{ containing } A\}.$$

Lemma 1.4. (Proposition 2.8 of [4]) *Let A be a nonzero proper ideal of a Prüfer domain R. Then $\mathrm{Ass}(A) = \mathcal{E}_A$. In particular, if $P \in \mathcal{X}_A$, then there exists a maximal ideal M of R such that $\mathrm{End}(A_M) = R_P$.*

Lemma 1.5. *Let A be an ideal of an arithmetical ring R. If A_M is a nonzero finitely generated ideal of R_M for each maximal ideal M of R, then $\mathrm{Ass}(A)$ is precisely the set of prime ideals of R containing A.*

Proof. First assume that R is a valuation ring and A is a nonzero finitely generated proper ideal of R. Then $A = yR$ is a principal ideal of R. Let P be the maximal ideal of R. If $A = P$, then clearly $P \in \mathrm{Ass}(A)$, so suppose there exists $x \in P \setminus A$. Since R is a valuation ring, $A \subset Rx$ and $A = (A : x)xR$. If $(A : x) = A$, then $A = xA$ and $y = xya$ for some $a \in A$. But this means $y(1 - xa) = 0$. Since $1 - xa$ is a unit of R, this implies $y = 0$, a contradiction to our assumption that $A \neq 0$. Therefore each $x \in P$ is non-prime to A, so A is a primal ideal with $S(A) = P$ and $P \in \mathrm{Ass}(A)$. In the general case where R is an arithmetical ring, we conclude from Lemma 2.1 and the preceding argument that every maximal ideal P of R that contains A is in $\mathrm{Ass}(A)$. Hence by Lemma 1.2, every prime ideal of R containing A is in $\mathrm{Ass}(A)$. $\square$

2. The case $\mathcal{X}_A \subseteq \mathrm{Max}(A)$

In this section we characterize when $\mathcal{X}_A \subseteq \mathrm{Max}(A)$ for every regular ideal A of an arithmetical ring. To do this, we first recall the notion of separated prime ideals, but we reformulate this definition to include rings with zero-divisors in such a way that our definition agrees with that of the separation property for domains (see pp. 91-92 of [3]). We define a ring R to have the *separation property* if for each

pair of distinct comparable regular prime ideals $P \subset Q$ of R, there exists a finitely generated ideal A such that $P \subset A \subseteq Q$. The ring R has the *strong separation property* if for each pair of comparable regular prime ideals $P \subset Q$ of R and regular element $r \in P$, there exists $s \in Q$ such that $P \subset (r, s)R \subseteq Q$.

Clearly every one-dimensional domain has the strong separation property. By the Altitude Theorem of Krull [18, page 26] or [13, page 110], a Noetherian domain of dimension greater than two has the separation property but not the strong separation property. By contrast, Prüfer domains need not possess either separation property. The Prüfer domains that have the separation property have been well-studied (see, for example, pp. 91-92 of [3] and Lemma 2.7 of [20]). Note that a Bézout domain R has the separation property if and only if R has the strong separation property. More generally, a Prüfer domain with the QR-property (so in particular a Prüfer domain with torsion Picard group) has the separation property if and only if it has the strong separation property.

In the following lemma, we collect some technical characterizations of the separation property for Prüfer domains that will be needed in the next section.

Lemma 2.1. *The following statements are equivalent for a Prüfer domain R.*

(i) *R has the separation property.*

(ii) *For each nonzero prime ideal P of R, P is a maximal prime ideal of $\mathrm{End}(P)$.*

(iii) *For each nonzero prime ideal P of R, $\mathrm{End}(P_M) = \mathrm{End}(P)_M$ for every maximal ideal M of R.*

(iv) *For each nonzero nonmaximal prime ideal P of R, if $\{M_i\}$ is the collection of maximal ideals of R not containing P, then $R_P \subseteq (\bigcap_i R_{M_i})R_M$ for each maximal ideal M of R containing P.*

(v) *For each nonzero ideal A of R, if M is a maximal ideal with $A \subseteq M$ and P is a prime ideal such that $\mathrm{End}(A)_M = R_P$, then no element of P is prime to A.*

Proof. (i) $\Leftrightarrow$ (ii) Lemma 4.2.38 of [3].

(i) $\Leftrightarrow$ (iii) Lemma 2.7 of [20].

(iii) $\Rightarrow$ (iv) By Theorem 3.2.6 of [3] $\mathrm{End}(P) = R_P \cap (\bigcap_i R_{M_i})$ and $\mathrm{End}(P_M) = R_P$. Thus by (iii) $R_P = \mathrm{End}(P_M) = \mathrm{End}(P)_M = R_P \cap (\bigcap_i M_i)R_M$, and (iv) follows.

(iv) $\Rightarrow$ (v) Let A be a nonzero ideal of R and M a maximal ideal with $A \subseteq M$. Let P be a prime ideal such that $\mathrm{End}(A)_M = R_P$. It follows from Lemma 1.4 that

$$\mathrm{End}(A) = (\bigcap_{Q \in \mathcal{X}_A} R_Q) \cap (\bigcap_N R_N),$$

where N ranges over the maximal ideals of R that do not contain A. Let Q' be the unique member of $\mathcal{X}_A$ that is contained in M. By Lemma 1.4, $\text{End}(A_M) = R_{Q'}$. If Q' is a maximal ideal of R, then $Q' = M$ and $\text{End}(A_M) = R_M$, so $R_P \subseteq R_M$ implies that $M = P = Q' \in \text{Ass}(A)$, and the claim is clear. It remains to consider the case where Q' is a non-maximal prime ideal of R. Now

$$R_P = \text{End}(A)_M = (\bigcap_{Q \in \mathcal{X}_A} R_Q)R_M \cap (\bigcap_N R_N)R_M.$$

We claim that $\bigcap_{Q \in \mathcal{X}_A} R_Q \subseteq R_P$. If this is not the case then since R_M is a valuation domain it must be that $R_P \subsetneq (\bigcap_{Q \in \mathcal{X}_A} R_Q)R_M$. Hence from the above representation of $\text{End}(A)_M$ we deduce that since R_P is a valuation domain, $R_P = (\bigcap_N R_N)R_M$. Thus $(\bigcap_N R_N)R_M \subsetneq (\bigcap_{Q \in \mathcal{X}_A} R_Q)R_M$. By (iv), $R_{Q'} \subseteq (\bigcap_N R_N)R_M$ since no N contains Q'. However $Q' \in \mathcal{X}_A$, so this implies $R_{Q'} \subsetneq R_{Q'}R_M$, but since M contains Q', $R_{Q'}R_M = R_{Q'}$. This contradiction implies that $\bigcap_{Q \in \mathcal{X}_A} R_Q \subseteq R_P$, so every element $r \in P$ is contained in some $Q \in \mathcal{X}_A$. Consequently, no element of P is prime to A.

(v) $\Rightarrow$ (iii) Suppose P is a non-maximal prime ideal of R, and let M be a maximal ideal of R containing P. Let Q be a prime ideal of R such that $\text{End}(P)_M = R_Q$. (Since $\text{End}(P)_M$ is a valuation domain such a prime Q must exist). Then by (v), the elements of Q are not prime to P. Consequently, $P = Q$, and (iii) follows. $\square$

Using the strong separation property, we can characterize when $\mathcal{X}_A \subseteq \text{Max}(A)$ for every regular ideal A of an arithmetical ring. In fact, the strong separation property is always sufficient, regardless of whether R is arithmetical, to guarantee that $\mathcal{X}_A \subseteq \text{Max}(A)$:

Lemma 2.2. *If A is a regular ideal of a ring R having the strong separation property, then every $P \in \mathcal{X}_A$ is a maximal prime divisor of A, i.e., $\mathcal{X}_A \subseteq \text{Max}(A)$.*

Proof. Let $P \in \mathcal{X}_A$. There exists $Q \in \text{Max}(A)$ such that $P \subseteq Q$. We show $P = Q$. If $\mathcal{X}_A$ consists only of P, then A is P-primal and $P = Q$. Suppose $\mathcal{X}_A \setminus \{P\}$ is nonempty, and let $\{P_i\} = \mathcal{X}_A \setminus \{P\}$. Since $Q \in \text{Max}(A)$, $Q \subseteq P \cup (\bigcup_i P_i)$. Assume P is properly contained in Q. Since R has the strong separation property and A is a regular ideal, there exists $x \in Q \setminus P$ such that $P \subset A + (x) \subseteq Q$. It follows that x is in one of the P_i, so $P \subset A + (x) \subseteq P_i$. This contradicts the assumption that $P \in \mathcal{X}_A$ is maximal among the Krull associated primes of A. Thus $P = Q$. $\square$

Theorem 2.3. *Let R be an arithmetical ring. The following statements are equivalent.*

(i) *R has the strong separation property.*

(ii) *For each proper regular ideal B of R, $\mathcal{X}_B \subseteq \mathrm{Max}(B)$.*

Proof. By Lemma 2.2, (i) implies (ii). To prove that (ii) implies (i), suppose $P \subset Q$ are regular prime ideals of R and that a is a regular element with $a \in P$. Define $B = aP$ and note that the maximal ideals of R that contain B are precisely those that contain aR. We first show that $\mathcal{X}_B$ is the set union of $\{P\}$ and the set of maximal ideals of R that contain B but not P.

Since a is a regular element, we have $P = B : a$, so $P \in \mathrm{Ass}(B)$. We claim that $P \in \mathcal{X}_B$. Suppose $L \in \mathcal{X}_B$ and $P \subseteq L$. Let M be a maximal ideal containing L. By Lemma 1.7 of [4], $(BR_M)_{(PR_M)}$ is a P_M-primal ideal of R_M since BR_M is a finitely generated ideal of R_M. We claim $BR_M = (BR_M)_{(PR_M)}$. To verify that this is the case, we need only show that $BR_M : y = BR_M$ for all $y \in R_M \setminus P_M$. To this end, observe first that $PR_M = yPR_M$ for all $y \in R_M \setminus P_M$. This is because R_M is a chained ring, so if $y \in R_M \setminus P_M$, then $PR_M \subseteq yR_M$, and if $p \in PR_M$, then $p = yu$ for some $u \in R_M$. Since $y \notin PR_M$, it follows that $u \in PR_M$ and $p \in yP_M$. Thus $PR_M = yPR_M$. Hence, with $y \in R_M \setminus PR_M$, we have that $BR_M : y = yBR_M : y = BR_M$, since $y \notin PR_M$ implies that $PR_M \subseteq yR_M$ and hence that y is a regular element of R_M. Therefore $BR_M = (BR_M)_{PR_M}$, and we conclude that BR_M is a PR_M-primal ideal. However, by Lemma 1.1 $LR_M \in \mathrm{Ass}(BR_M)$, so this forces $LR_M \subseteq PR_M$. Hence $L \subseteq P$, and we may conclude that $P = L \in \mathcal{X}_B$.

Now suppose N is a maximal ideal of R that contains B but not P. Then $B_N = aR_N \neq 0$, so N is in $\mathcal{X}_B$ by Lemma 1.5 and Lemma 1.1. This proves that the set $\mathcal{X}_B$ contains P and the maximal ideals of R that contain B but not P. The reverse inclusion follows from the fact that any prime ideal of R that is a member of $\mathcal{X}_B$ must contain B.

Let $\{N_i\}$ be the set of members of $\mathcal{X}_B$ distinct from P. As noted, each N_i is a maximal ideal of R, and $\{N_i\}$ is precisely the set of maximal ideals of R that contain aR but not P. Since $P \in \mathcal{X}_B$ and, by (ii), $\mathcal{X}_B \subseteq \mathrm{Max}(B)$, we have $Q \not\subseteq S(B) = P \cup (\bigcup_i N_i)$, for otherwise P, as a proper subset of Q, would not be in $\mathrm{Max}(A)$. In particular, there exists $x \in Q$ such that x is not contained in P nor in any of the N_i. We have $(a, x)R \subseteq Q$, and to complete the proof we show that $P \subset (a, x)R$. It is enough by Theorem II.3.1 of [1, page 88] to show that $P_M \subset (A, x)R_M$ for all maximal ideals M of R that contain (A, x). Observe that if M is such a maximal ideal, then since by design x is not an element of any maximal ideal of R that contains Ra but not P, it must be that M contains P. Now since R_M is valuation ring, the ideals P_M and $(a, x)R_M$ are comparable, so if $P_M \not\subseteq (a, x)R_M$, then it must be that $(a, x)R_M \subseteq P_M$. But then $x \in P_M$ and

since $P \subseteq M$, it is the case that $P_{(M)} = P$, and we have $x \in P$, a contradiction to the choice of x. Thus we conclude that $P \subseteq (a, x)R \subseteq Q$, and R has the strong separation property. □

It follows from Theorem 2.3 that if R is an arithmetical ring that does not have the strong separation property, then there exist a proper regular ideal B of R and $P \in \mathcal{X}_B$ such that $P \notin \text{Max}(B)$. By definition of $\text{Max}(B)$, this means there exists $Q \in \text{Max}(B)$ such that P is properly contained in Q and by definition of $\mathcal{X}_B$, this means $Q \notin \text{Ass}(B)$. Therefore the strong separation property is a necessary condition on an arithmetical ring in order that maximal prime divisors (of regular ideals) always be Krull primes.

Discussion 2.4. Some important examples of Prüfer domains do not have the separation property, and so by Theorem 2.3, these Prüfer domains have ideals B such that $\mathcal{X}_B \nsubseteq \text{Max}(B)$. Thus there exist maximal prime divisors of B that are not in $\text{Ass}(B)$. For example, neither the ring of entire functions nor the ring of integer-valued polynomials has the separation property. Indeed, both these rings are completely integrally closed integral domains of Krull dimension greater than one. If P is a nonzero nonmaximal prime ideal of such a Prüfer domain R, then $\text{End}(P) = R$ and P is not maximal in $\text{End}(P)$. Hence by Lemma 2.1 (ii), such domains do not have the separation property. On the other hand, every Prüfer domain having Noetherian maximal spectrum has the separation property (apply Theorem 4.2.39 of [3] and [22]).

3. The case $\mathcal{X}_A = \text{Max}(A)$

Our main objective in this section is to investigate conditions on an arithmetical ring R in order that $\mathcal{X}_A = \text{Max}(A)$ for each regular ideal A of R. We recall that an integral domain R is a *QR-domain* if every overring of R is a localization of R with respect to some multiplicatively closed subset of R. It is well-known that QR-domains are necessarily Prüfer, and that a Prüfer domain with torsion Picard group (e.g. a Bézout domain) is a QR-domain. More generally, a Prüfer domain R is a QR-domain if and only if the radical of every finitely generated ideal of R is the radical of a principal ideal of R [21]. There exist QR-domains having non-torsion Picard group [9]. Thus the condition on a Prüfer domain R that the radical of every finitely generated ideal is the radical of a principal ideal does not imply R has torsion Picard group. However, it is clear that a QR-domain that has the separation property also has the strong separation property.

Lemma 3.1. *Let A be an ideal of a Prüfer domain R. Suppose Q is a prime ideal of R that contains A, and P is a prime ideal such that $\operatorname{End}(A)_Q = R_P$. If $P \in \operatorname{Ass}(A)$, then $\operatorname{End}(A)_Q = \operatorname{End}(A_Q)$.*

Proof. Since $P \in \operatorname{Ass}(A)$, $A_{(P)}$ is a primal ideal with adjoint prime P, and it follows that A_P is a P_P-primal ideal. By Lemma 1.4, $\operatorname{End}(A_P) = R_P$. Thus $\operatorname{End}(A_P) = \operatorname{End}(A)_Q$, so $A \operatorname{End}(A_P) = A \operatorname{End}(A)_Q$ implies $A_P = A_Q$. Consequently, $\operatorname{End}(A_Q) = \operatorname{End}(A_P) = R_P = \operatorname{End}(A)_Q$. □

Lemma 3.2. *Let R be a Prüfer domain with field of fractions F, let X be an R-submodule of F, and let M be a maximal ideal of R. Then $\operatorname{End}(X)_M = R_P$ for some $P \in \operatorname{Spec} R$ with $P \subseteq M$. Assume that P is the union of prime ideals P_i, where each P_i is the radical of a finitely generated ideal. Then $\operatorname{End}(X)_Q = \operatorname{End}(X_Q)$ for all prime ideals Q such that $P \subseteq Q \subseteq M$.*

Proof. Since $R_M \subseteq \operatorname{End}(X)_M$ and R_M is a valuation domain, $\operatorname{End}(X)_M = R_P$ for some prime ideal $P \subseteq M$. If $\operatorname{End}(X)_M = F$, then clearly $\operatorname{End}(X)_M = \operatorname{End}(X_M)$, so we assume $\operatorname{End}(X)_M \neq F$ and thus $P \neq (0)$. Let Q be a prime ideal of R such that $P \subseteq Q \subseteq M$. Since $\operatorname{End}(X)_M = R_P$, we have $\operatorname{End}(X)_Q = R_P$. Now $R_P = \operatorname{End}(X)_Q \subseteq \operatorname{End}(X_Q) \subseteq \operatorname{End}(X_P)$, so to prove Lemma 3.2, it suffices to show that $\operatorname{End}(X_P) \subseteq R_P$.

Let $S = \operatorname{End}(X)$. Now $PS \subseteq PR_P$, so $PS \neq S$. Since S is an overring of the Prüfer domain R, S is a flat extension of R, so PS is a prime ideal of S and $S_{PS} = R_P$. Also, PS is the union of the prime ideals P_iS, and each P_iS is the radical of a finitely generated ideal of S.

Let L be a prime ideal of S such that $L \subseteq PS$ and such that $L = \sqrt{I}$, where I is a finitely generated ideal of S. We prove there exists a nonzero $q \in F$ such that qX_L is an ideal of S_L that is primary for L_L. The invertible ideal I^2 of S is an intersection of principal fractional ideals of S. Since $\operatorname{End}(X) = S$, each principal fractional ideal of S is an intersection of S-submodules of F of the form qX, $q \in F$. Since $I^2 \subseteq L$, I^2 is an intersection of ideals of S of the form $L \cap qX$, where $q \in F$. Since $I^2 \subsetneq I \subseteq L$, there exists $q \in F$ such that $I^2 \subseteq L \cap qX \subsetneq L$. Hence there exists a maximal ideal N of S with $I^2 \subseteq N$ such that $I_N^2 \subseteq L_N \cap qX_N \subsetneq L_N$. Since S_N is a valuation domain, the S_N-modules qX_N and L_N are comparable and $I_N^2 \subseteq L_N \cap qX_N \subsetneq L_N$ implies $I_N^2 \subseteq qX_N \subsetneq L_N$. Now $\sqrt{I^2} = \sqrt{I} = L$ and $I^2 \subseteq N$ implies $L \subseteq N$. Thus $I_L^2 \subseteq qX_L \subseteq L_L$, and we conclude that $\sqrt{qX_L} = L_L$.

We observe next that $X_P \neq F$. Since $P \neq 0$, there exists a nonzero $L = P_iS \subseteq PS$ such that $L = \sqrt{I}$, where I is finitely generated. As we have established in the

paragraph above, there exists a nonzero $q \in F$ such that qX_L is an ideal of S_L. Thus $qX_P \subseteq qX_L \subseteq S_L$, so it is not possible that $X_P = F$.

Fix some member L of the chain $\{P_iS\}$. Since $X_P \neq F$, $L \subseteq PS$ and R_P is a valuation domain, there exists a nonzero element s of S such that $sX \subseteq L_L$. Since $\mathrm{End}(X_P) = \mathrm{End}(sX_P)$ and we wish to show that $\mathrm{End}(X_P) \subseteq R_P$ we may assume without loss of generality that $s = 1$; that is, we assume for the rest of the proof that $X \subseteq L_L$. Define $A = X \cap S$. Then A is an ideal of S. Moreover A is contained in L since $A_L \subseteq X_L \subseteq L_L$.

With the aim of applying Lemma 3.1, we observe that $PS \in \mathrm{Ass}(A)$. For each i define $L_i = P_iS$. It suffices to show each L_i with $L \subseteq L_i \subseteq PS$ is in $\mathrm{Ass}(A)$, since this implies that $PS = \bigcup_{L_i \supseteq L} L_i$ is a union of members of $\mathrm{Ass}(A)$. Let i be such that $L \subseteq L_i$. Since L_i is the radical of a finitely generated ideal of S, there exists (as we have established above) a nonzero $q \in F$ such that qX_{L_i} is an ideal of S_{L_i} that is primary for $(L_i)_{L_i}$. Now $A_{L_i} = X_{L_i} \cap S_{L_i}$. Since S_{L_i} is a valuation domain, $A_{L_i} = X_{L_i}$ or $S_{L_i} \subseteq X_{L_i}$. By assumption, $X \subseteq L_L$. Since $L \subseteq L_i$, $X_{L_i} \subseteq L_L$, and it is impossible that $S_{L_i} \subseteq X_{L_i}$. Thus $A_{L_i} = X_{L_i}$. Consequently, $qX_{L_i} = qA_{L_i}$ and qA_{L_i} is an ideal of S_{L_i} that is primary for $(L_i)_{L_i}$. Since S_{L_i} is a valuation domain, it follows that $qA_{L_i} = A_{L_i} : s$ for some $s \in S$. Thus $(L_i)_{L_i} \in \mathrm{Ass}(A_{L_i})$, so by Lemma 1.1, $L_i \in \mathrm{Ass}(A)$. This proves $PS \in \mathrm{Ass}(A)$.

Since $A = X \cap S$ is an ideal of S, $S \subseteq \mathrm{End}(A)$. For each maximal ideal N of S, either $A_N = X_N$ or $A_N = S_N$ It follows that $\mathrm{End}(A) \subseteq \mathrm{End}(X) = S$, so $\mathrm{End}(A) = S$. Thus $\mathrm{End}(A)_P = S_P = R_P$, and by Lemma 3.1, $\mathrm{End}(A_P) = R_P$. (We have used here that $S_{SP} = R_P$.) Now $A_P = X_P \cap S_P = X_P \cap R_P$. Since R_P is a valuation domain, $A_P = X_P$ or $R_P \subseteq X_P$. The latter case is impossible since $X_P \subseteq X_L \subseteq L_L$. Thus $A_P = X_P$. We conclude that $\mathrm{End}(X_P) = \mathrm{End}(A_P) = R_P$. $\qquad \square$

A prime ideal P is *branched* if there exists a P-primary ideal different from P. If P is a nonzero prime ideal of a Prüfer domain R, then P is branched if and only if P is not the union of the prime ideals properly contained in P, and in this case if P fails to be branched, then the valuation domain R_P is infinite dimensional and there is no maximal element among the prime ideals properly contained in P.

Lemma 3.3. *Let R be a Prüfer domain having the separation property. If there exists a finitely generated ideal J of R having infinitely many minimal primes, then there exists a submodule X of the field of fractions F of R and a maximal ideal M containing J such that $\mathrm{End}(X)_M \neq \mathrm{End}(X_M)$.*

Proof. Let $P_1, P_2, P_3, \ldots$ be countably many distinct minimal primes of J. Define $S = \bigcap_i R_{P_i}$. Then S is a Prüfer overring of R and the minimal primes of JS are of the form QS, where $Q \in \operatorname{Spec} R$ is a minimal prime of J. In particular, $P_i S$ is a minimal prime of JS for each $i \geq 1$. Since R has the separation property, each $P_i S$ is a maximal ideal of S. This is because if j is any positive integer, then $\operatorname{End}(P_j) = R_{P_j} \cap (\bigcap_N R_N)$ by Theorem 3.2.6 of [3], so $\operatorname{End}(P_j) \subseteq S$ since the P_i's are comaximal. By Lemma 2.1(ii) P_j is a maximal ideal of $\operatorname{End}(P_j)$, and since R is a Prüfer domain, either P_j extends to a maximal ideal SP_j of S or $SP_j = S$. The latter case is impossible since $S \subseteq R_{P_j}$. Thus SP_j is a maximal ideal of S.

Since J is a nonzero finitely generated ideal, each minimal prime of J is branched. Since JS has infinitely many minimal primes, Theorem 1.6 of [7] implies there exists a minimal prime QS of JS that is not the radical of a finitely generated ideal. If $Q = P_j$ for some j, then by [6, Theorem 2], we have $S = \bigcap_{i \neq j} R_{P_i}$. Therefore, by relabeling if necessary, we may assume that $Q \notin \{P_i\}_{i=1}^{\infty}$.

Define $A = JR_Q \cap R$. Then $AS = JR_Q \cap S$ is QS-primary. In particular, QS is the unique minimal prime of AS and $A \nsubseteq P_i S \cap R = P_i$ for each $i \geq 1$.

Let Q_1 be the prime ideal of R just below Q (such a prime Q_1 exists because Q is branched). Since R has the separation property, there is a finitely generated ideal B of R such that $Q_1 \subset B \subseteq Q$. Local verification shows that $Q_1 \subset B^n \subseteq Q$ for each positive integer n.

For each $i \geq 1$, let

$$x_i \in A^i \setminus (P_1 \cup \cdots \cup P_i \cup A^{i+1}).$$

(By Prime Avoidance (see for example [2, Lemma 3.3]) such an element x_i exists.) Define

$$J_i = B^{i+1} + x_i R.$$

Then $Q_1 \subseteq J_i$ and $J_i \nsubseteq (P_1 \cup \cdots \cup P_i)$. Define $X = \sum_{i>1} J_i^{-1} S$.

Let $N = QS$ and for each $i > 0$, define $N_i = P_i S$. As previously demonstrated, N and the ideals N_i, $i > 0$, are maximal ideals of S. We show first that $\operatorname{End}(X_N) \neq \operatorname{End}(X)_N$.

For each $i \geq 1$,

$$X_{N_i} = \sum_{k=1}^{i-1} J_k^{-1} S_{N_i},$$

since by design, $J_k^{-1} S_{N_k} = S_{N_k}$ for all $k \geq i$. So as a finitely generated fractional ideal of S_{N_i}, X_{N_i} is isomorphic to S_{N_i}. Thus $\operatorname{End}(X_{N_i}) = S_{N_i}$ for each i. We have

that for each i, $S_{N_i} = R_{P_i}$; hence

$$\text{End}(X) \subseteq \bigcap_i \text{End}(X_{N_i}) = S.$$

Therefore, $\text{End}(X) = S$. In particular, $\text{End}(X)_N = S_N$.

Thus to prove the claim that $\text{End}(X)_N \neq \text{End}(X_N)$, it suffices to show that $N \text{End}(X_N) = \text{End}(X_N)$. To this end we verify that $X_N = NX_N$. Since $N = SQ$, we have $S_N = R_Q$, so for each $i \geq 1$, $A^{i+1}S_N \subset A^i S_N$. In particular, $x_{i+1}S_N \subset x_i S_N$ since $x_i \in A^i \setminus A^{i+1}$ and $A^{i+1}S_N \cap R = A^{i+1}R_Q \cap R = A^{i+1}$. It follows that $(J_{i+1})S_N \subset (J_i)S_N$. Since J_i is a finitely generated ideal of S, it follows that $(J_{i+1})S_N \subseteq N(J_i)S_N$. Consequently, $J_i^{-1} \subseteq N(J_{i+1}^{-1})S_N$ for each $i \geq 1$. Therefore $X \subseteq NX_N$, as claimed.

For S-submodules Y and W of F, let $[W : Y] = \{q \in F : qY \subseteq W\}$. Since $X_N = NX_N$ and $\text{End}(N_N) = S_N$, we have

$$[S_N : X_N] = [N_N : NX_N] = [N_N : X_N].$$

Moreover, $[S_N : X_N] = \bigcap_i J_i S_N$ and

$$Q_1 S_N \subseteq \bigcap_i J_i S_N \subseteq \bigcap_i A^i S_N = Q_1 S_N.$$

The latter equality follows from the fact that since S_N is a valuation domain and AS_N is a principal ideal of S_N, the intersection $\bigcap_i A^i S_N$ is a prime ideal. Since Q_1 is the largest prime ideal of R properly contained in Q, $A \subseteq Q$ and $A \nsubseteq Q_1$, we have $\bigcap_i A^i S_N = Q_1 S_N$.

It follows that $[N_N : X_N] = [S_N : X_N] = (SQ_1)_N$. Since R_N is a valuation domain, the maximal ideal N_N of S_N is m-canonical (see [10]), that is,

$$X_N = [N_N : [N_N : X_N]] = [N_N : (SQ_1)_N].$$

Since SQ_1 is a nonmaximal prime ideal of S, $[N_N : (SQ_1)_N] = S_{SQ_1}$ [3, Theorem 4.1.21]. Thus $X_N = S_{SQ_1}$, and we have $\text{End}(X_N) = S_{SQ_1} = R_{Q_1}$. In particular, $\text{End}(X_N) \neq R_Q$. Now $X_N = X_{SQ} = X_Q$, and if M is a maximal ideal of R containing Q, then $S_{SQ} = S_M$, so $X_M = X_Q$. Thus

$$R_{Q_1} = \text{End}(X_N) = \text{End}(X_M),$$

while

$$\text{End}(X)_M = S_M = R_Q.$$

Therefore $\text{End}(X_M) \neq \text{End}(X)_M$. $\qquad\square$

Theorem 3.4. *Let R be a Prüfer domain with field of fractions F. The following statements are equivalent.*

(i) *For each finitely generated ideal A of R, every weak-Bourbaki associated prime of A is a Zariski-Samuel associated prime of A.*

(ii) *Every finitely generated ideal of R has only finitely many minimal prime ideals.*

(iii) *Every principal ideal of R has only finitely many minimal prime ideals.*

(iv) *Every branched prime ideal of R is the radical of a finitely generated ideal.*

(v) *For each R-submodule X of F, $\mathrm{End}(X_M) = \mathrm{End}(X)_M$, for every maximal ideal M of R.*

Proof. (i) $\Rightarrow$ (ii) Let A be a finitely generated ideal of R. Every minimal prime P of A is a weak-Bourbaki prime of A and therefore by (i) a Zariski-Samuel prime of A and hence of the form $\sqrt{(A : x)}$. Since R is Prüfer, $(A : x)$ is finitely generated, see, for example, [6, Lemma 2]. Thus each minimal prime of R/A is the radical of a finitely generated ideal of R/A. By Theorem 1.6 in [7], R/A has finitely many minimal primes. This proves (ii).

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (iv) Let P be a branched prime of R. Then there exists a prime ideal Q of R such $Q \subset P$ and there are no other prime ideals between Q and P. Let $x \in P \setminus Q$. Since the prime ideals contained in P are linearly ordered, P is a minimal prime ideal of xR. By (iii), xR has only finitely many minimal prime ideals. Since R is Prüfer, if B is the intersection of the other minimal primes of xR, then $B + P = R$. Hence there exists $y \in P$ such that $B + yR = R$. It follows that P is the radical of $(x, y)R$. This proves (iv).

(iv) $\Rightarrow$ (v) Since every nonzero prime ideal of R is the union of branched prime ideals, this follows from Lemma 3.2.

(v) $\Rightarrow$ (i) Let A be a finitely generated ideal of R and let P be a weak-Bourbaki associated prime of A. Then P is a minimal prime of $(A : x)$ for some $x \in R$. Since R is Prüfer, $(A : x)$ is finitely generated. Lemma 3.3 implies that $(A : x)$ has only finitely many minimal primes. Thus by Lemma 5.10 of [4] each minimal prime of $(A : x)$ is a Zariski-Samuel associated prime of $(A : x)$, and therefore a Zariski-Samuel associated prime of A. This proves (i). $\qquad\square$

There is an interesting connection between the conditions of Theorem 3.4 and "trace properties" of Prüfer domains. T. Lucas shows in [16, Theorem 23] that a Prüfer domain R satisfies (iv) of Theorem 3.4 if and only if R has the *radical trace property*, namely, for every ideal A of R, AA^{-1} is either R or a radical ideal of R.

Corollary 3.5. *The following statements are equivalent for a Prüfer domain R.*

(i) *Every branched prime ideal of R of height greater than one is the radical of a finitely generated ideal of R.*

(ii) *For each nonzero ideal A of R, $\operatorname{End}(A_M) = \operatorname{End}(A)_M$ for all maximal ideals M of R.*

(iii) *For each ideal A of R, $\operatorname{Ass}(A)$ is precisely the set of prime ideals P such that $A \subseteq P$ and $P \operatorname{End}(A) \neq \operatorname{End}(A)$.*

Proof. (i) $\Rightarrow$ (ii) Suppose (i) holds, and let A be a nonzero ideal of R. Let M be a maximal ideal of R. If M has height one then R_M is a one-dimensional valuation domain, so $\operatorname{End}(A_M) = R_M$. Consequently, $\operatorname{End}(A)_M = \operatorname{End}(A_M)$. If M has height greater than 1, then M is the union of branched prime ideals of height greater than 1, so by (i) and Lemma 3.2, $\operatorname{End}(A_M) = \operatorname{End}(A)_M$. This proves (ii).

(ii) $\Rightarrow$ (i) Let P be a branched prime ideal of R such that there is a nonzero prime ideal Q of R with $Q \subset P$. From (ii) and Lemma 2.1, it follows that R has the separation property. Thus Q is a maximal ideal of $\operatorname{End}(Q)$ and $\operatorname{End}(Q)/Q$ is the quotient field of R/Q. From (ii) it follows that if X is an R-submodule of $\operatorname{End}(Q)$ with $Q \subseteq X$, then

$$\operatorname{End}((X/Q)_{M/Q}) = \operatorname{End}(X/Q)_{M/Q}$$

for all maximal ideals M of R containing Q. By Theorem 3.4, the branched prime ideal P/Q of R/Q is the radical of a finitely generated ideal. Hence P is the radical of an ideal $I + Q$, where I is a finitely generated ideal of R. Since R has the separation property, there is a finitely generated ideal J such that $Q \subset J \subseteq P$. Thus P is the radical of the finitely generated ideal $I + J$.

(ii) $\Rightarrow$ (iii) Let A be an ideal of R, and suppose $P \in \operatorname{Ass}(A)$. Then there exists $Q \in \mathcal{X}_A$ such that $P \subseteq Q$. By Lemma 1.4, $R_Q = \operatorname{End}(A_M)$ for some maximal ideal M of R. Thus by (ii), $R_Q = \operatorname{End}(A)_M$, and we have $Q \operatorname{End}(A) \neq \operatorname{End}(A)$. It follows that $P \operatorname{End}(A) \neq \operatorname{End}(A)$.

Now suppose P is a prime ideal containing A such that $P \operatorname{End}(A) \neq \operatorname{End}(A)$. Then $\operatorname{End}(A)_M \neq P \operatorname{End}(A)_M$ for some maximal ideal M of R, and it follows that $\operatorname{End}(A)_M \subseteq R_P$. By (ii), $\operatorname{End}(A_M) \subseteq R_P$. If $\operatorname{End}(A_M) = R_Q$ for some prime ideal Q, then by Lemma 1.2 and 1.4, it follows that $Q \in \operatorname{Ass}(A)$. Since $A \subseteq P \subseteq Q$, Lemma 1.2 implies $P \in \operatorname{Ass}(A)$. This proves (iii).

(iii) $\Rightarrow$ (ii) Let A be a proper ideal of R and M be a maximal ideal of R containing A. If $\operatorname{End}(A)_M = R_P$ for some prime ideal P of R, then $P \operatorname{End}(A) \neq \operatorname{End}(A)$, so by (iii), $P \in \operatorname{Ass}(A)$. By Lemma 3.1, $\operatorname{End}(A)_M = \operatorname{End}(A_M)$. This proves (ii). $\qquad \square$

Remark 3.6. Theorem 3.4 shows that conditions for good behavior with respect to localization for endomorphism rings of submodules of the fraction field involve all the primes of R. On the other hand, Corollary 3.5 shows that the corresponding property for ideals involves only conditions on the primes of height greater than one. This phenomenon is illustrated in the article Goeters-Olberding [8], where the statement of Corollary 3.5 that $\text{End}(A)$ commutes with localizations is of central importance. Our proof of Lemma 3.3 is a refinement of the argument given in Theorem 3.7 of [8].

Theorem 3.7. *Consider the following statements for an arithmetical ring R.*

(i) $\mathcal{X}_A = \text{Max}(A)$ *for each proper regular ideal A of R.*

(ii) *Every branched prime ideal of R that properly contains a regular prime ideal is the radical of a finitely generated ideal.*

(iii) *Every branched prime ideal of R that properly contains a regular prime ideal is the radical of a principal ideal.*

Then (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii). Moreover, if R is a QR-domain, then all three statements are equivalent.

Proof. (i) $\Rightarrow$ (ii) We consider first the case where R is a domain. Then to prove (ii), it suffices by Corollary 3.5 to show that for every ideal A of R and maximal ideal M containing A, $\text{End}(A)_M = \text{End}(A_M)$. Let A be a proper nonzero ideal of R and let M be a maximal ideal such that $A \subseteq M$. Since R_M is a valuation domain, there exists a prime ideal P of R such that $\text{End}(A)_M = R_P$. We claim first that the elements of P are not prime to A. Let $\{M_\alpha\}$ denote the set of maximal ideals of R that contain A, and let $\{N_\beta\}$ denote the set of maximal ideals of R that do not contain A. For each α, there exists a prime ideal P_α such that $\text{End}(A_{M_\alpha}) = R_{P_\alpha}$. Moreover, by Lemmas 1.2 and 1.4, we have $\mathcal{X}_A \subseteq \{P_\alpha\} \subseteq \text{Ass}(A)$. Since $\text{End}(A) = (\bigcap_\alpha \text{End}(A_{M_\alpha})) \cap (\bigcap_\beta \text{End}(A_{N_\beta}))$, we have

$$R_P = \text{End}(A)_M = (\bigcap_\alpha R_{P_\alpha})R_M \cap (\bigcap_\beta R_{N_\beta})R_M.$$

Since R has the separation property, Lemma 2.3(iv) implies that $\bigcap_\beta R_{N_\beta} \not\subseteq R_P$. Thus because R_P is a valuation domain, it must be that $\bigcap_\alpha R_{P_\alpha} \subseteq R_P$, and it follows that $P \subseteq \bigcup_\alpha P_\alpha$. Since $S(A) = \cup_\alpha P_\alpha$, the elements of P are not prime to A, as claimed. Therefore there exists $Q \in \text{Max}(A)$ such that $P \subseteq Q$. By (i), $Q \in \mathcal{X}_A$, so by Lemma 1.2, $P \in \text{Ass}(A)$ since $A \subseteq P \subseteq Q$. Thus by Lemma 3.1, $\text{End}(A)_M = \text{End}(A_M)$. This proves that (i) implies (ii) in the case R is a domain.

Now consider the general case where R is not necessarily a domain. Observe that since statement (i) holds for R, statement (i) holds for R/P for all regular

prime ideals P of R. This follows from the observation that if A is an ideal of R that contains P then for all $x \in R$,

$$(A/P :_{R/P} x + P) = ((A : x) + P)/P.$$

In particular, $\mathrm{Ass}(A/P) = \{Q/P : Q \in \mathrm{Ass}(A), P \subseteq Q\}$. Also, $\mathrm{Max}(A/P) = \{Q/P : Q \in \mathrm{Max}(A), P \subseteq Q\}$. Thus statement (ii) holds for R/P for all regular prime ideals P of R. Let Q be a branched prime ideal of R containing at least one regular prime ideal. Then there exists a regular prime ideal P of R such that $P \subset Q$. By Theorem 2.3, there exists a finitely generated ideal B of R such that $P \subseteq B \subset Q$. Moreover, since (ii) holds for R/P, there exists a finitely generated ideal C of R such that Q is the radical of $C + P$. It follows that Q is the radical of the finitely generated ideal $C + B$.

(iii) $\Rightarrow$ (i) Let A be a proper regular ideal of R. For each $P \in \mathcal{X}_A$ there exists $M \in \mathrm{Max}(A)$ such that $P \subseteq M$ and then $P = M$ if and only if $M \in \mathrm{Ass}(A)$. Thus it suffices to prove that $M \in \mathrm{Max}(A)$ implies $M \in \mathrm{Ass}(A)$. This is clear if M/A has height one as prime ideal of R/A. Assume that $\mathrm{ht}(M/A) \geq 2$. Then M is the union of a chain of branched prime ideals P_i of R, where $A \subseteq P_i$ and $\mathrm{ht}(P_i/A) \geq 2$. Since A is a regular ideal, so is each P_i. Thus by (iii), each P_i is the radical of a principal ideal of R. Since $P_i \subseteq M$, the elements in P_i are non-prime to A. Since P_i is the radical of a principal ideal and $S(A) = \bigcup_{Q \in \mathcal{X}_A} Q$, P_i is contained in some member of $\mathcal{X}_A$. By Lemma 1.2, each $P_i \in \mathrm{Ass}(A)$. Therefore $\bigcup_i P_i = M \in \mathrm{Ass}(A)$.

Finally, we conclude that statements (i),(ii) and (iii) are equivalent for a QR-domain R since in a QR-domain every prime ideal that is the radical of a finitely generated ideal is the radical of a principal ideal. $\square$

Corollary 3.8. *If R is a Prüfer domain having the property that every maximal prime divisor of an ideal A of R is a Krull associated prime of A, then every branched prime ideal of R of height greater than one is the radical of a finitely generated ideal.*

Proof. The corollary is a consequence of Theorem 3.7 and Corollary 1.3. $\square$

Example 3.9 illustrates the fact that it is possible for a 2-dimensional Prüfer domain to satisfy the hypotheses of Theorem 2.3, but not Theorem 3.7.

Example 3.9. *There exists a Prüfer domain R that has the strong separation property and yet contains an ideal A such that $\mathcal{X}_A$ is a proper subset of $\mathrm{Max}(A)$. Moreover, R does not satisfy any of the conditions (i), (ii) or (iii) of Theorem 3.7. Let D be an almost Dedekind domain (that is, D_M is a DVR for all maximal ideals*

M of D) that is not Dedekind. (See [3, page 281] for an example of such a domain.) Let F denote the field of fractions of D, let x be an indeterminate over F, and let $P = xF[x]_{(x)}$. Define $R = D + P$. Then R is a Prüfer domain and P is the unique nonzero nonmaximal prime ideal of R. Moreover, if M is a maximal ideal of R, then M contains P and if $r \in M \setminus P$, then $P \subseteq Rr \subseteq M$, so R has the strong separation property. In particular, by Theorem 2.3, every ideal A of R has the property that $\mathcal{X}_A \subseteq \mathrm{Max}(A)$. Since D is almost Dedekind, but not Dedekind, there exists a maximal ideal Q of D that is not the radical of a finitely generated ideal. Consequently, the (height 2) maximal ideal $Q + P$ of R is not the radical of a finitely generated ideal of R. By Theorem 3.7, there exists an ideal A of R such that $\mathcal{X}_A$ is a proper subset of $\mathrm{Max}(A)$.

In Example 2.9 of [4], we construct a 2-dimensional Prüfer domain with Noetherian spectrum having an ideal A for which there exists a maximal ideal M such that M is in $\mathrm{Max}(A)$ but M is not a Krull associated prime of A. Since this domain R has Noetherian spectrum, R satisfies statement (ii) of Theorem 3.7. Hence condition (ii) of Theorem 3.7 does not imply condition (i), even when R is a 2-dimensional Prüfer domain. This leaves the question of whether in general (i) implies (iii) in Theorem 3.7:

Question 3.10. *Does there exist an arithmetical ring R such that R satisfies statement (i) but not statement (iii) of Theorem 3.7?*

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