Inverse Problems in Quantum Optics

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Motivation

Inverse problems of optical imaging are based on classical theories of light propagation

There are manifestly nonclassical states of light (entanglement)

Questions

Are there inverse scattering problems that exploit nonclassical states of light?

- Resolution limits
- Interaction-free measurements

Can scattering modify entanglement?

- Deterministic media
- Random media

References

- V. Markel and J. Schotland, Radiative Transport for Two-Photon Light. Phys. Rev. A. **90**, 033815 (2014)
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Overview

- Field quantization
- Scattering in quantum optics
- Transport of entanglement
- Applications

Field Quantization

Harmonic oscillator

We consider the Hamiltonian

$$H = \frac{1}{2}\left(p^2 + \omega^2 q^2\right) \ .$$

Hamilton's equations are

$$\dot{q} = \frac{\partial H}{\partial p} \ ,$$

$$\dot{p} = -\frac{\partial H}{\partial q} \ ,$$

which leads to the equation of motion

$$\ddot{q} + \omega^2 q = 0 \ .$$

To quantize the oscillator, we consider the Schrodinger equation

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle \ ,$$

where the Hamiltonian operator is given by

$$\hat{H} = \frac{1}{2} \left(\hat{p}^2 + \omega^2 \hat{q}^2 \right) \ .$$

Here the position and momentum operators obey the canonical commutation relations

$$[\hat{q}, \hat{p}] = i\hbar \ .$$

To construct the energy eigenstates, we introduce the creation and annihilation operators

$$\hat{a} = \sqrt{\frac{\omega}{2\hbar}} \left(\hat{q} + \frac{i}{\omega} \hat{p} \right) \ , \quad \hat{a}^\dagger = \sqrt{\frac{\omega}{2\hbar}} \left(\hat{q} - \frac{i}{\omega} \hat{p} \right) \ .$$

Note that $[\hat{a}, \hat{a}^\dagger] = 1$. The Hamiltonian can thus be factorized as

$$\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \ .$$

We then have

$$\hat{H} \left| n \right\rangle = E_n \left| n \right\rangle \ ,$$

where $E_n = \hbar\omega(n + 1/2)$ and

$$\left| n \right\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n \left| 0 \right\rangle \ .$$

The Heisenberg equations of motion for $\hat{q}$ and $\hat{p}$ are

$$\begin{aligned} \frac{d\hat{q}}{dt} &= \frac{1}{i\hbar}[\hat{q}, \hat{H}] = \hat{p} \ , \\ \frac{d\hat{p}}{dt} &= \frac{1}{i\hbar}[\hat{p}, \hat{H}] = -\omega^2 \hat{q} \ . \end{aligned}$$

We see that

$$\frac{d^2\hat{q}}{dt^2} + \omega^2 \hat{q} = 0$$

and thus $\hat{q}$ obeys the classical equations of motion.

Classical field theory

For simplicity, we consider a scalar model of electromagnetism with the Lagrangian

$$\mathscr{L} = \frac{1}{2}(\partial_t E)^2 - \frac{1}{2}c^2(\nabla E)^2 \ .$$

The equations of motion are derived from the variational principle

$$\frac{\delta S}{\delta E} = 0 \ ,$$

where the action S is defined by

$$S = \int d^3r \int dt \mathscr{L} \ .$$

We find that

$$c^2 \Delta E = \partial_t^2 E \ .$$

The Hamiltonian is defined as

$$H = \int d^3r \left(\Pi \dot{E} - \mathscr{L} \right) \ ,$$

where the momentum Π is given by

$$\Pi = \frac{\partial \mathscr{L}}{\partial \dot{E}} = \dot{E}$$

Thus

$$H = \int d^3r \left[\frac{1}{2} \Pi^2 + \frac{1}{2} c^2 \left(\nabla E \right)^2 \right] \ .$$

Field quantization

To quantize the field, we promote Π and E to operators and impose the equal-time commutation relations

$$
\begin{aligned}
[\hat{E}(\mathbf{r},t),\hat{\Pi}(\mathbf{r}',t)] &= i\hbar\delta(\mathbf{r}-\mathbf{r}')\ , \\
[\hat{E}(\mathbf{r},t),\hat{E}(\mathbf{r}',t)] &= 0\ , \\
[\hat{\Pi}(\mathbf{r},t),\hat{\Pi}(\mathbf{r}',t)] &= 0\ .
\end{aligned}
$$

It is convenient to work in Fourier space in a volume V with periodic boundary conditions:

$$
\hat{E}(\mathbf{r},t) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}}\hat{E}_{\mathbf{k}}(t)\ , \quad \hat{E}_{-\mathbf{k}} = \hat{E}^{\dagger}_{\mathbf{k}}\ .
$$

The commutation relations become

$$
\begin{aligned}
[\hat{E}_{\mathbf{k}},\hat{\Pi}_{\mathbf{k}'}] &= i\hbar\delta_{\mathbf{k}\mathbf{k}'}\ , \\
[\hat{E}_{\mathbf{k}},\hat{}\hat{E}_{\mathbf{k}'}] &= 0\ , \\
[\hat{\Pi}_{\mathbf{k}},\hat{\Pi}_{\mathbf{k}'}] &= 0\ .
\end{aligned}
$$

A calculation shows that the Hamiltonian becomes

$$\hat{H} = \sum_{\mathbf{k}} \left(\frac{1}{2} \hat{\Pi}_{\mathbf{k}} \hat{\Pi}_{\mathbf{k}}^{\dagger} + \frac{1}{2} \omega_k^2 \hat{E}_{\mathbf{k}} \hat{E}_{\mathbf{k}}^{\dagger} \right) \ ,$$

where $\omega_k = c|\mathbf{k}|$. Introducing creation and annihilation operators according to

$$\hat{a}_{\mathbf{k}} = \sqrt{\frac{\omega_k}{2\hbar}} \left(\hat{E}_{\mathbf{k}} + \frac{i}{\omega_k} \hat{\Pi}_{\mathbf{k}}^{\dagger} \right) \ , \quad \hat{a}_{\mathbf{k}}^{\dagger} = \sqrt{\frac{\omega_k}{2\hbar}} \left(\hat{E}_{\mathbf{k}}^{\dagger} - \frac{i}{\omega_k} \hat{\Pi}_{\mathbf{k}} \right) \ ,$$

we have $[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^{\dagger}] = \delta_{\mathbf{k}\mathbf{k}'}$ and $[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = 0$. Thus

$$\hat{H} = \sum_{\mathbf{k}} \hbar\omega_k \left(\hat{a}_{\mathbf{k}}^{\dagger} \hat{a}_{\mathbf{k}} + \frac{1}{2} \right) \ .$$

The above Hamiltonian describes a system of independent oscillators.

The electric field operator is given by

$$\hat{E}(\mathbf{r},t) = \sum_{\mathbf{k}} A_k e^{i\mathbf{k}\cdot\mathbf{r}} \hat{a}_{\mathbf{k}} + \text{h.c.} \ ,$$

where the amplitude $A_k = \sqrt{\hbar/(2\omega_k V)}$. The Heisenberg equation of motion for $\hat{a}_{\mathbf{k}}$ is

$$i\frac{d\hat{a}_{\mathbf{k}}}{dt} = \frac{1}{\hbar}[\hat{a}_{\mathbf{k}}, \hat{H}] = \omega_k \hat{a}_{\mathbf{k}} \ .$$

Thus $\hat{a}_{\mathbf{k}}(t) = \hat{a}_{\mathbf{k}}(0)\exp(-i\omega_k t)$. We immediately see that the field operator $\hat{E}$ obeys the classical equations of motion

$$\Delta \hat{E} = \frac{1}{c^2}\frac{\partial^2 \hat{E}}{\partial t^2} \ .$$

Recall that

$$\hat{H} = \sum_{\mathbf{k}} \hbar\omega_k \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} \ ,$$

where we have dropped the zero-point energy. The energy eigenstates obey

$$H \left| n_1, \ldots, n_N \right\rangle = E \left| n_1, \ldots, n_N \right\rangle \ ,$$

where

$$\begin{aligned} \left| n_1, \ldots n_N \right\rangle &= \frac{1}{\sqrt{n_1! \cdots n_N!}} (\hat{a}^\dagger_{\mathbf{k}_1})^{n_1} \cdots (\hat{a}^\dagger_{\mathbf{k}_N})^{n_N} \left| 0 \right\rangle \ , \\ E &= n_1 \hbar\omega_1 + \cdots + n_N \hbar\omega_N \ , \end{aligned}$$

and $|0\rangle$ is the vacuum state. The Hilbert space spanned by the above states is called Fock space.

The excited states of the field are called photons. There are n_k photons in the state $|n_k\rangle$, each with energy $\hbar\omega_k$. Photons are created and annihilated by the operators $\hat{a}^\dagger_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}$ according to

$$\hat{a}^\dagger_{\mathbf{k}} \left| n_k \right\rangle = \sqrt{n_k + 1} \left| n_k + 1 \right\rangle \ , \quad \hat{a}_{\mathbf{k}} \left| n_k \right\rangle = \sqrt{n_k} \left| n_k - 1 \right\rangle \ .$$

Using the commutation relation $[\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = 0$, we see that the many-body wavefunction is symmetric under exchange of particles:

$$|\mathbf{k}_1, \mathbf{k}_2\rangle = \hat{a}^\dagger_{\mathbf{k}_1}\hat{a}^\dagger_{\mathbf{k}_2}|0\rangle = \hat{a}^\dagger_{\mathbf{k}_2}\hat{a}^\dagger_{\mathbf{k}_1}|0\rangle = |\mathbf{k}_2, \mathbf{k}_1\rangle \ .$$

Thus, photons are bosons and the state $|n_k\rangle$ can have an arbitrarily large occupation number n_k.

A one-photon state is a superposition of single-mode states of the form

$$|\psi\rangle = \sum_{\mathbf{k}} c_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}}|0\rangle \ .$$

Note that the photon can occupy only one of many possible modes.

A two-photon state takes the form

$$|\psi\rangle = \sum_{\mathbf{k},\mathbf{k}'} c_{\mathbf{k}\mathbf{k}'}\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}'}|0\rangle \ .$$

Entanglement

Given two quantum systems, their joint quantum state may be entangled. The systems may be particles such as electrons. In quantum field theory, it is not the particles (photons) that form the systems, but rather the quantized field modes.

A two-photon state is said to be entangled if it cannot be expressed as a product of single-photon states. Entangled states exhibit quantum correlations. An unentangled state is factorizable. Factorizable states are analogous to independent random variables.

Examples. Consider two quantized field modes. The Fock space of the system is spanned by $|0,0\rangle$, $|0,1\rangle$, $|1,0\rangle$ and $|1,1\rangle$.

The state $|\psi\rangle = |1,1\rangle = |1\rangle\,|1\rangle$ is unentangled.

The state $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0,0\rangle + |1,1\rangle\right)$ is entangled.

Entangled states have remarkable properties. According to the rules of quantum mechanics, measurement of one member of an entangled pair modifies instantaneously the state of its partner, an effect which occurs independent of the distance between the particles. The apparent conflict between this result and the predictions of relativity is a startling feature of quantum mechanics.

From the point of view of information theory, an entangled state contains more information than the sum of the information contained in each subsystem. We will see that the entropy is a useful measure of entanglement.

Observables

It will prove convenient to write the electric field operator as $\hat{E} = \hat{E}^{+} + \hat{E}^{-}$, where

$$\hat{E}^{+}(\mathbf{r}, t) = \sum_{\mathbf{k}} A_k e^{i\mathbf{k}\cdot\mathbf{r}} \hat{a}_{\mathbf{k}}$$

and $\hat{E}^{-} = (\hat{E}^{+})^{\dagger}$. We interpret $\hat{E}^{+}(\mathbf{r}, t)$ as an operator that annihilates a photon at the point $\mathbf{r}$ at time t. Similarly, $\hat{E}^{-}(\mathbf{r}, t)$ is interpreted as a photon creation operator. It follows that if the field is initially in the state $|\psi\rangle$, the probability of detecting a photon at the point $\mathbf{r}$ at time t is

$$P(\mathbf{r}, t) = \langle\psi| \hat{E}^{+}(\mathbf{r}, t)\hat{E}^{-}(\mathbf{r}, t) |\psi\rangle \ .$$

Likewise, using two detectors, the joint probability of detecting a photon at the point $\mathbf{r}$ at time t and the point $\mathbf{r}'$ at time t' is

$$P(\mathbf{r}, t; \mathbf{r}', t') = \langle\psi| \hat{E}^{+}(\mathbf{r}, t)\hat{E}^{+}(\mathbf{r}', t)\hat{E}^{-}(\mathbf{r}, t)\hat{E}^{-}(\mathbf{r}', t') |\psi\rangle \ .$$

Scattering in Quantum Optics

Scattering

We consider the propagation of light in a material medium with dielectric permittivity $\varepsilon(\mathbf{r})$. The positive part of the electric field operator $\widehat{E}^+$ obeys

$$\Delta \widehat{E}^+ = \frac{\varepsilon(\mathbf{r})}{c^2} \frac{\partial^2 \widehat{E}^+}{\partial t^2} \ .$$

Here the medium is taken to be nonabsorbing, so that ε is purely real and positive. Now, let $|\psi\rangle$ be a two-photon state. Recall that the joint probability of detecting one photon at $\mathbf{r}$ at time t and a second photon at $\mathbf{r}'$ at time t' is given by

$$P(\mathbf{r}, t; \mathbf{r}', t') = \langle\psi| \, \widehat{E}^-(\mathbf{r}, t)\widehat{E}^-(\mathbf{r}', t')\widehat{E}^+(\mathbf{r}', t')\widehat{E}^+(\mathbf{r}, t) \, |\psi\rangle \ .$$

It can be seen that P factorizes as

$$P(\mathbf{r}, t; \mathbf{r}', t') = |A(\mathbf{r}, t; \mathbf{r}', t')|^2 \ .$$

Here $|0\rangle$ is the vacuum state and the two-photon amplitude A is defined by

$$A(\mathbf{r}, t; \mathbf{r}', t') = \langle 0| \, \widehat{E}^+(\mathbf{r}, t)\widehat{E}^+(\mathbf{r}', t') \, |\psi\rangle \ .$$

Evidently, the two-photon amplitude A satisfies the pair of wave equations

$$\begin{aligned} \Delta_{\mathbf{r}} A &= \frac{\varepsilon(\mathbf{r})}{c^2} \frac{\partial^2 A}{\partial t^2} \ , \\ \Delta_{\mathbf{r}'} A &= \frac{\varepsilon(\mathbf{r}')}{c^2} \frac{\partial^2 A}{\partial t'^2} \ . \end{aligned}$$

We will find it convenient to introduce the Fourier transform of A:

$$A(\mathbf{r}, \omega; \mathbf{r}', \omega') = \int dt dt' e^{i(\omega t + \omega' t')} A(\mathbf{r}, t; \mathbf{r}', t') \ .$$

Thus

$$\begin{aligned} \Delta_{\mathbf{r}} A + k^2 \varepsilon(\mathbf{r}) A &= 0 \ , \\ \Delta_{\mathbf{r}'} A + k'^2 \varepsilon(\mathbf{r}') A &= 0 \ , \end{aligned}$$

where $k = \omega / c$ and $k' = \omega' / c$.

We now develop the scattering theory for the two-photon amplitude. To proceed, we consider the Helmholtz equation

$$\Delta u + k^2 \varepsilon(\mathbf{r}) u = 0 \ .$$

The field u is taken to consist of incident and scattered parts. The scattered field u_s is given by

$$u_s(\mathbf{r}) = \int d^3 r_1 d^3 r_2 G(\mathbf{r}, \mathbf{r}_1) T(\mathbf{r}_1, \mathbf{r}_2) u_i(\mathbf{r}_2) \ ,$$

where u_i is the incident field and G is the Green's function. The T-matrix obeys the integral equation

$$T(\mathbf{r}, \mathbf{r}') = k^2 \eta(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}') + k^2 \eta(\mathbf{r}) \int d^3 r'' G(\mathbf{r}, \mathbf{r}'') T(\mathbf{r}'', \mathbf{r}') \ ,$$

where the susceptiblity η is defined by the relation $\varepsilon = 1 + 4\pi\eta$.

Propagating each coordinate of the two-photon amplitude separately, we obtain

$$A_s(\mathbf{r},\mathbf{r}') = \int d^3r_1 d^3r_2 d^3r_1' d^3r_2' G(\mathbf{r},\mathbf{r}_1) T(\mathbf{r}_1,\mathbf{r}_2) G'(\mathbf{r}',\mathbf{r}_1') T'(\mathbf{r}_1',\mathbf{r}_2') A_i(\mathbf{r}_2,\mathbf{r}_2') \ ,$$

where A_i is the two-photon amplitude of the incident field.

If $A_s(\mathbf{r},\mathbf{r}')$ factorizes into a product of two functions which depend upon $\mathbf{r}$ and $\mathbf{r}'$ separately, we will say that the two-photon state $|\psi\rangle$ is not entangled. In contrast, an entangled state is not separable.

It follows directly from the above that if A_i is separable, then A_s is separable. That is, if A_s is entangled then A_i is entangled. Thus, entanglement cannot be created by scattering an unentangled incident state.

Now consider the far-field asymptotics of the two-photon amplitude $A_s(\mathbf{r}, \mathbf{r}')$. We find that

$$A_s(\hat{\mathbf{k}}, \hat{\mathbf{k}}') = \int d\hat{\mathbf{k}}_1 d\hat{\mathbf{k}}_2 \langle \mathbf{k}| T |\mathbf{k}_1\rangle \langle \mathbf{k}'| T' |\mathbf{k}_2\rangle A_i(\hat{\mathbf{k}}_1, \hat{\mathbf{k}}_2) ,$$

where $\mathbf{k} = k\hat{\mathbf{r}}$ and $\mathbf{k}' = k'\hat{\mathbf{r}}'$. We note that the above T-matrices are on-shell. The momentum-space T-matrix elements are defined by

$$\langle \mathbf{k}| T |\mathbf{k}'\rangle = \int d^3r d^3r' e^{-i(\mathbf{k}\cdot\mathbf{r} - \mathbf{k}'\cdot\mathbf{r}')} T(\mathbf{r}, \mathbf{r}') ,$$

where $|\mathbf{k}| = |\mathbf{k}'| = k$. From now on, we consider a fully entangled incident state with $A_i(\hat{\mathbf{k}}_1, \hat{\mathbf{k}}_2) = \delta(\hat{\mathbf{k}}_1 - \hat{\mathbf{k}}_2)$. Thus

$$\mathcal{A}_s(\hat{\mathbf{k}}, \hat{\mathbf{k}}') = \int d\hat{\mathbf{k}}'' \langle \mathbf{k}| T |\mathbf{k}''\rangle \langle \mathbf{k}'| T' |\mathbf{k}''\rangle ,$$

where $|\mathbf{k}| = k$ and $|\mathbf{k}'| = k'$.

Small scatterer

We now compute A_s for a collection of small scatterers. We begin with the case of a small spherical scatter of radius a, where $ka \ll 1$. The T-matrix is then given by

$$\langle \mathbf{k} | \, T \, | \mathbf{k}' \rangle = t(k) e^{i(\mathbf{k}-\mathbf{k}')\cdot \mathbf{r}_0} \ ,$$

where $\mathbf{r}_0$ is the position of the scatterer and $t(k) = \alpha k^2$. The renormalized polarizability α is defined by

$$\alpha = \frac{\alpha_0}{1 - 3\alpha_0 k^2/(2a) - i\alpha_0 k^3} \ ,$$

where α_0 is the polarizability. We find that

$$A_s(\hat{\mathbf{k}}, \hat{\mathbf{k}}') = 4\pi t(k) t(k') e^{i(k\hat{\mathbf{k}} + k'\hat{\mathbf{k}}')\cdot \mathbf{r}_0} \mathrm{sinc}\left(|(k + k')\mathbf{r}_0| \right) \ .$$

We see at once that A_s is a separable function of $\hat{\mathbf{k}}$ and $\hat{\mathbf{k}}'$. Thus the scattered field is unentangled, even when the incident field is entangled.

Collection of small scatterers

Next, we consider a collection of identical small scatterers. The T-matrix is given by

$$\langle \mathbf{k} | \, T \, | \mathbf{k}' \rangle = \sum_{a,b} t_{ab}(k) e^{i(\mathbf{k}\cdot\mathbf{r}_a - \mathbf{k}'\cdot\mathbf{r}_b)} ,$$

where $\{\mathbf{r}_a\}$ are the positions of the scatterers and $t_{ab} = \alpha_0 k^2 M_{ab}^{-1}$. The matrix M is defined by $M_{ab} = \delta_{ab} - \alpha_0 k^2 G_{ab}$, where G_{ab} is the renormalized Green's function. We find that A_s is given by

$$A_s(\hat{\mathbf{k}}, \hat{\mathbf{k}}') = 4\pi \sum_{a,b} \sum_{a',b'} t_{ab}(k) t_{a'b'}(k') e^{i(k\hat{\mathbf{k}}\cdot\mathbf{r}_a + k'\hat{\mathbf{k}}'\cdot\mathbf{r}_{a'})} \mathrm{sinc}(|k\mathbf{r}_b + k'\mathbf{r}_{b'}|) .$$

We note that A is nonseparable; thus the scattered field is entangled.

Spherical scatterer

Finally, we consider a homogeneous spherical scatterer of radius a centered at the origin with index of refraction n. The T-matrix is of the form

$$\langle \mathbf{k} | \, T \, | \mathbf{k}' \rangle = \sum_l (2l+1) M_l(k) P_l(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}') \ ,$$

where the Mie coefficient M_l is defined as

$$M_l(k) = \frac{1}{ik} \frac{j_l(nka) j_l'(ka) - n j_l(ka) j_l'(nka)}{n h_l^{(1)}(ka) j_l'(nka) - h_l^{(1)\prime}(ka) j_l(nka)} \ .$$

We find that A_s is given by

$$A_s(\hat{\mathbf{k}}, \hat{\mathbf{k}}') = 4\pi \sum_l (2l+1) A_l(k) A_l(k') P_l(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}') \ .$$

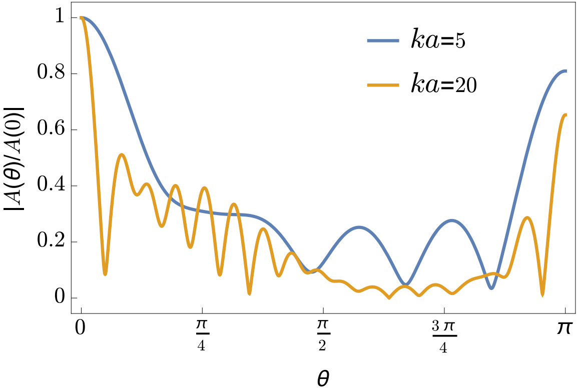


Figure: Two-photon amplitude $A(\hat{\mathbf{k}}, \hat{\mathbf{k}}')$ of a spherical scatterer of radius a as a function of the angle θ between $\hat{\mathbf{k}}$ and $\hat{\mathbf{k}}'$. The index of refraction of the sphere is $n = 1.5$.

Entanglement entropy

We consider the singular value decomposition of the two-photon amplitude, viewed as an operator with kernel $A_s(\hat{\mathbf{k}}, \hat{\mathbf{k}}')$. We find that $A_s(\hat{\mathbf{k}}, \hat{\mathbf{k}}')$ can be decomposed into a superposition of separable terms of the form

$$A_s(\hat{\mathbf{k}}, \hat{\mathbf{k}}') = \sum_n \sigma_n u_n(\hat{\mathbf{k}}) v_n^*(\hat{\mathbf{k}}') \ ,$$

where each term corresponds to a unentangled. Here the singular values σ_n are real-valued and the singular functions obey

$$\begin{aligned} \int (A_s{}^* A_s)(\hat{\mathbf{k}}, \hat{\mathbf{k}}') v_n(\hat{\mathbf{k}}') d\hat{\mathbf{k}}' &= \sigma_n^2 v_n(\hat{\mathbf{k}}) \ , \\ \int (A_s A_s{}^*)(\hat{\mathbf{k}}, \hat{\mathbf{k}}') u_n(\hat{\mathbf{k}}') d\hat{\mathbf{k}}' &= \sigma_n^2 u_n(\hat{\mathbf{k}}) \ . \end{aligned}$$

A measure of the degree of entanglement is the entropy S, which is defined by

$$S = -\sum_n \sigma_n \log \sigma_n \ .$$

Spherical scatterer

We can compute the entanglement entropy of a spherical scatterer in terms of Mie coefficients. The singular values are given by

$$\sigma_l = (4\pi)^2 |M_l(k) M_l(k')|$$

and

$$S = -\sum_l (2l+1)\sigma_l \log \sigma_l \ .$$

In the limit where the radius tends to zero, the entropy vanishes, consistent with the separability of the two-photon amplitude for the case of a point scatterer in (26). There are oscillations in the entropy related to the presence of scattering resonances.

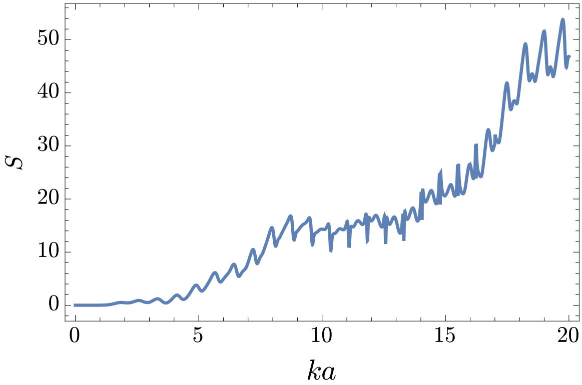


Figure: Entropy of entanglement of a spherical scatterer as a function of radius a. The index of refraction of the sphere is $n = 1.5$.

Collection of small scatterers

To construct the singular value decomposition of A_s, we expand the singular functions u_n into spherical harmonics of the form

$$u_n(\hat{\mathbf{k}}) = \sum_{l,m} u_{lm}^{(n)} Y_{lm}(\hat{\mathbf{k}}) \ ,$$

where the coefficients $u_{lm}^{(n)}$ are to be determined. We find that the $u_{lm}^{(n)}$ can be obtained from the solution to the eigenproblem

$$\sum_{l',m'} A_{lm}^{l'm'} u_{l'm'}^{(n)} = \sigma_n^2 u_{lm}^{(n)} \ ,$$

$$A_{l'm'}^{lm} = \sum_{l'',m''} C_{l''m''}^{lm*} C_{l'm'}^{l''m''} \ ,$$

$$C_{l'm'}^{lm} = \sum_{\substack{a,a' \\ b,b'}} i^{l+l'} t_{ab} t_{a'b'} \operatorname{sinc}\left(|k\mathbf{r}_a + k'\mathbf{r}_{a'}|\right) j_l(kr_b) j_{l'}(k'r_{b'}) Y_{lm}^*(\hat{\mathbf{r}}_a) Y_{l'm'}(\hat{\mathbf{r}}_{a'}) \ .$$

The entropy is computed from the singular values σ_n.

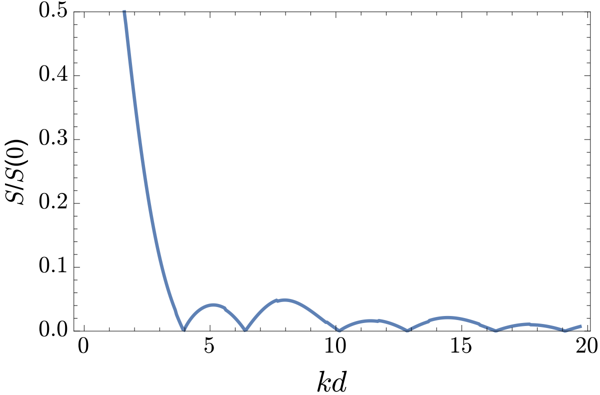


Figure: The entropy of entanglement of a pair of scatterers separated by a distance d. The radii of the scatterers is $ka = 0.2$ and their index of refraction is $n = 1.5$.

Transport of Entanglement

Quantum optics in random media

The propagation of light in disordered media is generally considered within the framework of classical optics. However, recent experiments have demonstrated the existence of novel quantum effects in multiple light scattering.

- Spatial correlations in multiply-scattered squeezed light.
- Statistics of two-photon speckles. Alterations in speckle patterns can be observed, depending on the degree of spatial entanglement of the initial state.
- Quantum interference survives averaging over disorder and is manifest as photon correlations.

In general terms, there is an interplay between quantum interference and interference due to multiple scattering.

Single-photon RTE

Consider the single-photon state $|\psi\rangle$. The first-order coherence function is defined as the normally-ordered expectation of field operators:

$$\Gamma^{(1)}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = \langle\psi|\, \widehat{E}^-(\mathbf{r}_1, t_1)\widehat{E}^+(\mathbf{r}_2, t_2)\,|\psi\rangle \ ,$$

where $\widehat{E}^-$ and $\widehat{E}^+$ are the negative- and positive-frequency components of the electric-field operator with $\widehat{E}^- = [\widehat{E}^+]^\dagger$. In a material medium with dielectric permittivity ε, the field operator $\widehat{E}^+$ obeys the wave equation

$$\Delta\widehat{E}^+ - \frac{\varepsilon(\mathbf{r})}{c^2}\frac{\partial^2\widehat{E}^+}{\partial t^2} = 0 \ .$$

The medium is taken to be nonabsorbing, so that ε is purely real. The first-order coherence function $\Gamma^{(1)}$ obeys the wave equations

$$\Delta_{\mathbf{r}_1}\Gamma^{(1)} - \frac{\varepsilon(\mathbf{r}_1)}{c^2}\frac{\partial^2\Gamma^{(1)}}{\partial t_1^2} = 0 \quad ,$$

$$\Delta_{\mathbf{r}_2}\Gamma^{(1)} - \frac{\varepsilon(\mathbf{r}_2)}{c^2}\frac{\partial^2\Gamma^{(1)}}{\partial t_2^2} = 0 \quad .$$

If the field is stationary, then $\Gamma^{(1)}$ depends upon the difference $t_1 - t_2$. We thus define the Fourier transform of $\Gamma^{(1)}$ according to

$$G^{(1)}(\mathbf{r}_1, \mathbf{r}_2; \omega) = \int d(t_1 - t_2) e^{i\omega(t_1 - t_2)} \Gamma^{(1)}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) \ .$$

Evidently, $G^{(1)}$ satisfies the time-independent wave equations

$$\begin{aligned} \Delta_{\mathbf{r}_1} G^{(1)} + k_0^2 \varepsilon(\mathbf{r}_1) G^{(1)} = 0 \quad , \\ \Delta_{\mathbf{r}_2} G^{(1)} + k_0^2 \varepsilon(\mathbf{r}_2) G^{(1)} = 0 \quad , \end{aligned}$$

where $k_0 = \omega / c$.
We introduce the Wigner distribution of $G^{(1)}$ which is defined as

$$W(\mathbf{r}, \mathbf{k}) = \int d^3 r' e^{i\mathbf{k}\cdot\mathbf{r}'} G^{(1)}(\mathbf{r} - \mathbf{r}'/2, \mathbf{r} + \mathbf{r}'/2) \ .$$

The Wigner distribution is a phase-space energy density. It is real-valued and is related to the intensity I and energy current by

$$I = \frac{c}{4\pi} \int \frac{d^3k}{(2\pi)^3} W(\mathbf{r}, \mathbf{k}) \ , \quad \mathbf{J} = \int \frac{d^3k}{(2\pi)^3} \ \mathbf{k} W(\mathbf{r}, \mathbf{k}) \ .$$

We find that W obeys the deterministic equation

$$\mathbf{k} \cdot \nabla_{\mathbf{r}} W + \frac{i}{2} k_0^2 \int \frac{d^3p}{(2\pi)^3} e^{-i\mathbf{p}\cdot\mathbf{r}} \widetilde{\varepsilon}(\mathbf{p}) \left[W(\mathbf{r}, \mathbf{k} + \mathbf{p}/2) - W(\mathbf{r}, \mathbf{k} - \mathbf{p}/2) \right] = 0 \ .$$

In a random medium, averaging of the above transport equation leads to the RTE

$$\hat{\mathbf{k}} \cdot \nabla_{\mathbf{r}} \overline{W} = \mu_s \int d^2k' \left[p(\hat{\mathbf{k}}', \hat{\mathbf{k}}) \overline{W}(\mathbf{r}, \hat{\mathbf{k}}') - p(\hat{\mathbf{k}}, \hat{\mathbf{k}}') \overline{W}(\mathbf{r}, \hat{\mathbf{k}}) \right] \ ,$$

where $\overline{W} = \langle W \rangle$ is the statistical average over the disorder. The phase function p and scattering coefficient μ_s are related to correlations in the random medium. The averaging can be carried by multiscale asymptotics. As may be expected, we obtain the classical result.

Two-photon light

Consider the two-photon state $|\psi\rangle$. The second-order coherence function is defined as

$$\Gamma^{(2)}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = \langle\psi|\, \widehat{E}^{-}(\mathbf{r}_1, t_2)\widehat{E}^{-}(\mathbf{r}_2, t_2)\widehat{E}^{+}(\mathbf{r}_2, t_2)\widehat{E}^{+}(\mathbf{r}_1, t_1)\,|\psi\rangle \ .$$

The quantity $\Gamma^{(2)}$ is proportional to the probability of detecting one photon at $(\mathbf{r}_1, t_1)$ and a second photon at $(\mathbf{r}_2, t_2)$. For the two-photon state $|\psi\rangle$, we have seen that $\Gamma^{(2)}$ factorizes as

$$\Gamma^{(2)}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = |A(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2)|^2 \ ,$$

where the two-photon probability amplitude A is defined by

$$A(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = \langle 0|\widehat{E}^{+}(\mathbf{r}_1, t_1)\widehat{E}^{+}(\mathbf{r}_2, t_2)|\psi\rangle \ .$$

Two-photon RTE

We now consider the Wigner distribution of A which is defined by

$$W(\mathbf{r},\mathbf{k}) = \int d^3 r' e^{i\mathbf{k}\cdot\mathbf{r}'} A(\mathbf{r}-\mathbf{r}'/2,\omega_1;\mathbf{r}+\mathbf{r}'/2,\omega_2) \ .$$

It can be seen that W obeys the equation

$$\mathbf{k}\cdot\nabla_{\mathbf{r}} W + \frac{i}{2}\int \frac{d^3 p}{(2\pi)^3} e^{-i\mathbf{p}\cdot\mathbf{r}} \widetilde{\varepsilon}(\mathbf{p}) \left[k_1^2 W(\mathbf{r},\mathbf{k}+\mathbf{p}/2) - k_2^2 W(\mathbf{r},\mathbf{k}-\mathbf{p}/2)\right] = 0 \ .$$

This is an exact result which describes the propagation of the Wigner distribution for two-photon light in a material medium.
We now proceed to derive the RTE for two-photon light. We consider a statistically homogeneous random medium and assume that the susceptibility η is a Gaussian random field with correlations

$$\langle \eta(\mathbf{r}) \rangle = 0 \ , \quad \langle \eta(\mathbf{r})\eta(\mathbf{r}') \rangle = C(|\mathbf{r}-\mathbf{r}'|) \ .$$

Here η is related to the dielectric permittivity by $\varepsilon = 1 + 4\pi\eta$, C is the two-point correlation function and $\langle \cdots \rangle$ denotes statistical averaging.

It can be seen that $\mathcal{I} = \langle W \rangle$ obeys the equation

$$\hat{\mathbf{k}} \cdot \nabla_{\mathbf{r}} \mathcal{I}(\mathbf{r}, \hat{\mathbf{k}}) + (\sigma_a + \sigma_s)\mathcal{I}(\mathbf{r}, \hat{\mathbf{k}}) = \sigma_s \int d^2 k' f(\hat{\mathbf{k}}, \hat{\mathbf{k}}')\mathcal{I}(\mathbf{r}, \hat{\mathbf{k}}') \ .$$

Here the coefficients σ_a, σ_s and the scattering kernel f are defined by

$$\begin{aligned} \sigma_a &= i\gamma \ , \\ \sigma_s &= k_1^2 k_2^2 \int \widetilde{C}(k(\hat{\mathbf{k}} - \hat{\mathbf{k}}'))d^2 k' \ , \\ f(\hat{\mathbf{k}}, \hat{\mathbf{k}}') &= \frac{\widetilde{C}(k(\hat{\mathbf{k}} - \hat{\mathbf{k}}'))}{\int d^2 k' \widetilde{C}(k(\hat{\mathbf{k}} - \hat{\mathbf{k}}'))} \ . \end{aligned}$$

We will refer to the above as the two-photon RTE. In contrast to the specific intensity, the quantity $\mathcal{I}$ is not real-valued and is not directly measurable. Nevertheless, $\mathcal{I}$ is related to the average two-photon probability amplitude which allows for the calculation of the coherence function $\Gamma^{(2)}$.

Transport of entanglement

Consider an infinite medium in which the averaged two-photon Wigner distribution is known on the plane $z = 0$. That is, we suppose that

$$\mathcal{I}_0(\mathbf{r}, \mathbf{k}) = \mathcal{I}(\mathbf{r}, \mathbf{k})|_{z=0} = A\delta(k - k_0) \ ,$$

where A is constant. On the planes $z_1 = z_2 = 0$ the two-photon wavefunction

$$\widetilde{\Phi} = 4\pi k_0^2 A \frac{\sin\left(k_0|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|\right)}{k_0|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|}$$

corresponds to a entangled two-photon state, where $\mathbf{r}_1 = (\boldsymbol{\rho}_1, z_1)$ and $\mathbf{r}_2 = (\boldsymbol{\rho}_2, z_2)$. To propagate $\mathcal{I}$ into the $z > 0$ half-space, we make use of the formula

$$\mathcal{I}(\mathbf{r}, \mathbf{k}) = \int d^3k' \int_{z'=0} d^2r' \ |\hat{\mathbf{n}} \cdot \mathbf{k}'| G(\mathbf{r}, \mathbf{k}; \mathbf{r}', \mathbf{k}')\mathcal{I}_0(\mathbf{r}', \mathbf{k}') \ .$$

In the limit of strong scattering and at large distances from the source, the Green's function is given by

$$G(\mathbf{r}, \mathbf{r}') = \frac{e^{-\kappa|\mathbf{r}-\mathbf{r}'|}}{4\pi D|\mathbf{r}-\mathbf{r}'|} \ .$$

Here $\kappa = \sqrt{\sigma_a/D}$, $D = 1/[3(\sigma_a + (1-g)\sigma_s)]$ and $g = \int \hat{\mathbf{k}} \cdot \hat{\mathbf{k}}' f(\hat{\mathbf{k}}, \hat{\mathbf{k}}') d^2k'$. We find that

$$\langle \widetilde{\Phi}(\mathbf{r}_1, \mathbf{r}_2) \rangle = \frac{aAk_0}{2D(2\pi)^2} \frac{\sin(k_0|\mathbf{r}_1 - \mathbf{r}_2|)}{|\mathbf{r}_1 - \mathbf{r}_2|} \times \int_0^\infty \frac{dq}{\sqrt{q^2+\kappa^2}} J_1(qa) J_0(q|\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2|/2) e^{-\sqrt{q^2+\kappa^2}(z_1+z_2)/2} \ .$$

In the on-axis configuration, in the absence of absorption, we have

$$\langle \widetilde{\Phi}(0, z; 0, z) \rangle = \frac{Ak_0}{2D(2\pi)^2} \left[\sqrt{z^2 + a^2} - z \right] \ .$$

The entanglement of the photon pair is destroyed with propagation.

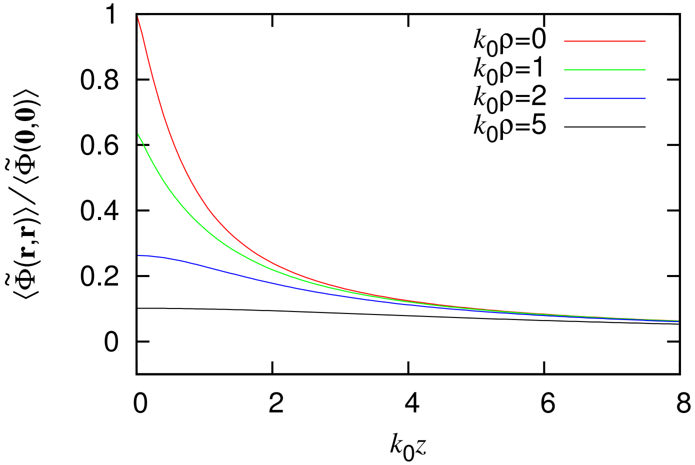
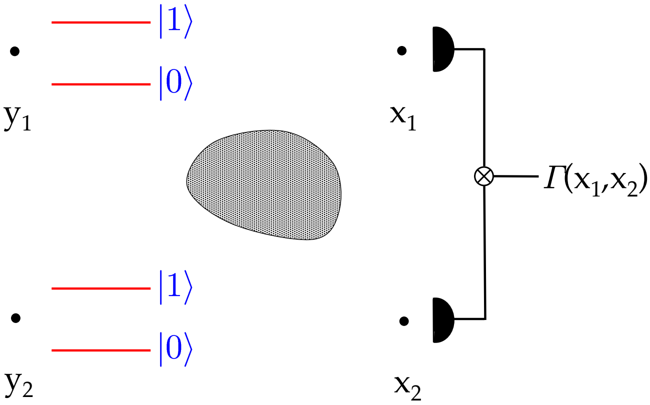


Figure: Dependence of A on the distance of propagation z for $k_0\rho = 0, 1, 2, 5$, from top to bottom.

Applications

Two-photon imaging



THANK YOU!