Constructing Travel Times and Redatuming via the Boundary Control Method

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Forward Problem and Data Model

Forward Problem

Let (M, g) a smooth R.M. with bndry, $T > 0$, and $\Gamma \subset \partial M$ open. For $f \in L^2([0, 2T] \times \partial M)$, let u^f the soln. to:

$$\begin{cases} \partial_t^2 u - \Delta_g u = 0 & \text{in } (0, 2T) \times M \\ \partial_\nu u = f & \text{on } [0, 2T] \times \partial M \\ u(0, \cdot) = \partial_t u(0, \cdot) = 0 & \text{in } M \end{cases}$$

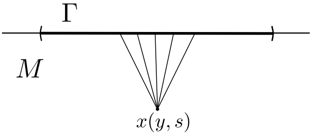
Data Model

We define the **Neumann-to-Dirichlet map**:

$$\Lambda_\Gamma^{2T} f = u^f|_{[0,2T]\times\Gamma} \text{ for } f \in L^2([0, 2T] \times \Gamma)$$

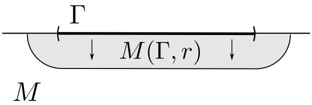
Objectives

Constructing Travel Times



Boundary distance representation (see e.g. Katchalov 2001)

Redatuming



Move data into a known near boundary region.

Brief Review of BC method

Brief (selected) historical overview

Boundary control method: a technique originally introduced to solve the inverse boundary problem for the wave equation.

- Belishev (1988): sound-speed in a domain
- Belishev and Kurylev (1992): Riemannian manifold
- Tataru (1995): Unique continuation
- Bingham et. al. (2007) : Tikhonov Reg.
- Oksanen (2011) : volumes of *domains of influence*

Some motivation for the name

Approximate Control Problem

For $\phi \in L^2(M)$, find a source f for which:

$$u^f(T, \cdot) \approx \phi$$

- We need to be able to find f blindly without knowing g, i.e. only using our boundary data

Definition

For $\tau : \overline{\Gamma} \to [0, T]$, define S_τ:

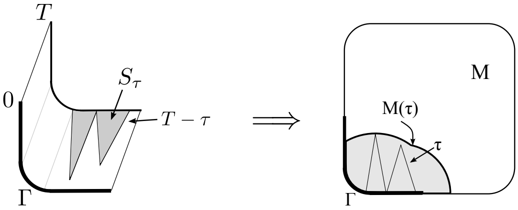
$$S_\tau := \{(t, y) \in [0, T] \times \Gamma : T - \tau(y) \leq t \leq T\}.$$

and

$$M(\tau) := \{x \in M : d(x, \Gamma) \leq \tau(y), \text{ for some } y \in \Gamma\}.$$

By finite speed of propagation:

$$\operatorname{supp}(f) \subset S_\tau \implies \operatorname{supp}(u^f(T, \cdot)) \subset M(\tau).$$



Operators for the BC method

Definition

Control Map:

$$\begin{cases} W_\tau & : \quad L^2(S_\tau) \to L^2(M(\tau)) \\ W_\tau & : \quad f \mapsto u^f(T, \cdot) \end{cases}$$

Connecting Operator:

$$\begin{cases} K_\tau & : \quad L^2(S_\tau) \to L^2(S_\tau) \\ K_\tau & : \quad f \mapsto (W_\tau)^* W_\tau f \end{cases}$$

Blagovescenskii Identity

Lemma (see e.g. Oksanen 2013)

The connecting operator can be computed by processing the data.

$$K_\tau = P_\tau (J\Lambda_\Gamma^{2T} - R\Lambda_\Gamma^{T} RJ)\Theta P_\tau$$

Where

- $Jf(t,x) := \int_t^{2T-t} f(s,x)\, ds$ for $(t,x) \in [0, T] \times \Gamma$
- $Rf(t,x) := f(T-t,x)$ for $(t,x) \in [0, T] \times \Gamma$
- $\Theta : L^2(S_\tau) \to L^2([0,2T] \times \Gamma)$, zero extension
- $P_\tau : L^2([0,2T] \times \Gamma; dt \otimes dS_g) \to L^2(S_\tau; dt \otimes dS_g)$, orthogonal projection

Solving Approximate Control Problems

Let $\phi \in L^2(M)$ and $\alpha > 0$. Find $f \in L^2(S_\tau)$ minimizing the Tikhonov functional:

$$\|W_\tau f - \phi\|^2_{L^2(M)} + \alpha \|f\|^2_{L^2(S_\tau)}$$

Lemma (various)

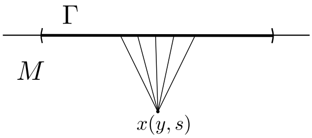
Let $\tau : \overline{\Gamma} \to [0, T]$ be either a linear combination of characteristic functions of open subsets of $\overline{\Gamma}$ or $\tau \in C(\overline{\Gamma})$ the minimization problem has a unique solution $f_\alpha \in L^2(S_\tau)$, given by solving the linear equation:

$$(K_\tau + \alpha) f_\alpha = W_\tau^* \phi$$

Moreover,

$$\lim_{\alpha \to 0^+} u^{f_\alpha}(T, \cdot) = \lim_{\alpha \to 0^+} W_\tau f_\alpha = 1_{M(\tau)} \phi$$

Constructing Travel Times (Distances)



Travel Times to points in boundary normal coordinates

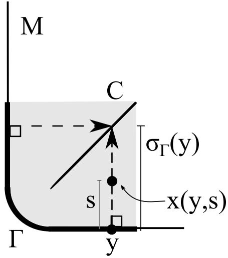
Definition

For $y \in \Gamma$, ν the inward pointing unit normal to ∂M at y define:

$$x(y,s) := \gamma_{y,\nu}(s).$$

Define:

$$\sigma_\Gamma(y) = \sup\{s : s = d(\Gamma, x(y,s))\}.$$



Goal: For $z, y \in \Gamma$ and $s < \min(\sigma_\Gamma(y), T)$, compute $d(z, x(y,s))$.

Regularized Volume Estimation

Lemma (Oksanen 2011)

For $\tau \in C(\overline{\Gamma})$, $\tau : \overline{\Gamma} \to [0, T]$, the data Λ_Γ^{2T} can be used to compute $\mathrm{Vol}_g(M(\tau))$.

- Choose $\phi \equiv 1$ in the control problem. Then:

$$W_\tau^* \phi = P_\tau b, \quad \text{where } b(t, z) := T - t \text{ for } (t, z) \in \overline{\Gamma} \times [0, T].$$

- Solve the control problem with this choice of ϕ:

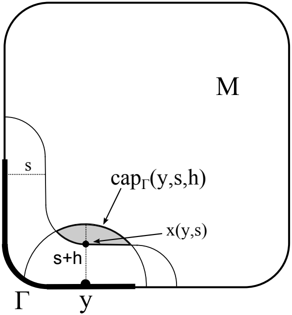
$$\langle f_\alpha, b \rangle_{L^2(S_\tau)} = \| W_\tau f_\alpha \|^2_{L^2(M)} \to \| 1_{M(\tau)} \|^2_{L^2(M)} = \mathrm{Vol}_g(M(\tau)).$$

Wave cap

Definition

For $y \in \Gamma$, $0 < s, h$, $s + h < \sigma_\Gamma(y)$, define the **wavecap** with height h, basepoint y, at distance s from Γ:

$$\mathrm{cap}_\Gamma(y, s, h) := M((s+h)1_{\{y\}}) \setminus M^o(s1_\Gamma)$$



A simplifying assumption

Assumption

The distances $d(y, z)$ are known for $y, z \in \Gamma$ with $d(y, z) < T$.

Note

- Not that big of an assumption, since Λ_Γ^{2T} determines these distances (see e.g. Dahl et. al. 2008).
- We could avoid assuming this at the expense of an additional limit in our main result.

Computing volumes of wavecaps and their intersections

Definition

For $y \in \Gamma$ and $R \in [0, \infty)$, define τ_y^R on Γ by:

$$\tau_y^R(z) := (R - d(y,z)) \vee 0 \quad \text{for } z \in \Gamma.$$

Let $y, z \in \Gamma$ and $s, h, r > 0$. Suppose that $s + h < \sigma_\Gamma(y)$ and define $\tau_1 = s1_\Gamma$, $\tau_2 = \tau_y^{s+h} \vee s$, $\tau_3 = \tau_z^{s+r} \vee s$, $\tau_4 = \tau_2 \vee \tau_3$. Then:

$$\mathrm{Vol}_g\left(\mathrm{cap}(y,s,h)\right) = \mathrm{Vol}_g(M(\tau_2)) - \mathrm{Vol}_g(M(\tau_1))$$

and

$$\begin{aligned}\mathrm{Vol}_g\left(\mathrm{cap}(y,s,h) \cap \mathrm{cap}(z,s,r)\right) = & \mathrm{Vol}_g(M(\tau_4)) - \mathrm{Vol}_g(M(\tau_3)) \\ & - \mathrm{Vol}_g(M(\tau_2)) + \mathrm{Vol}_g(M(\tau_1))\end{aligned}$$

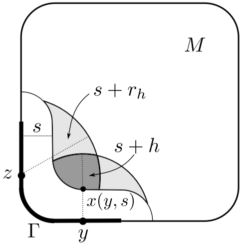
Distance Estimation

Lemma (de Hoop, K., Oksanen (2016))

Let $y, z \in \Gamma$, $s \in (0, \sigma_\Gamma(y))$, and $0 < s + h < \sigma_\Gamma(y)$. Let r_h be the solution to:

$$\mathrm{Vol}_g(\mathrm{cap}_\Gamma(y,s,h) \cap \mathrm{cap}_\Gamma(z,s,r_h)) = \frac{1}{2}\mathrm{Vol}_g(\mathrm{cap}_\Gamma(y,s,h))$$

Then, for $d_h = s + r_h$, we have that $d_h \to d(z, x(y,s))$ as $h \to 0$.



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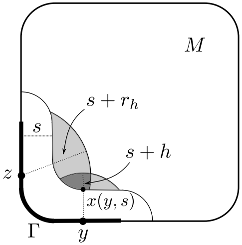

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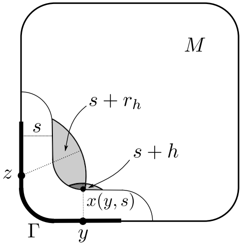

Algorithm 1 Travel time (distance) determination.

Let: $y, z \in \Gamma$ and $s > 0$ with $r_{x(y,s)}(z) < T$.
Let: $h_0 > 0$ small enough that $s + h_0 < \min\{\sigma_\Gamma(y), T\}$.
for all $0 < h < h_0$:
  for all $0 < r < T - s$:
    Let: $\tau_1 = s1_\Gamma$, $\tau_2 = \tau_y^{s+h}$, $\tau_3 = \tau_z^{s+r}$, $\tau_4 = \tau_1 \vee \tau_2 \vee \tau_3$
    for all $\alpha > 0$:
      for $i = 1, \ldots, 4$:
        Let: $f_{\alpha,i}$ be the solution to $(K_{\tau_i} + \alpha)P_{\tau_i} f = P_{\tau_i} b$
    for $i = 1, \ldots, 4$:
      Compute: $\mathrm{Vol}_g(\tau_i) = \lim_{\alpha \to 0} \langle f_{\alpha,i}, b \rangle_{L^2(S_\tau)}$
    Compute:
    $m_{\text{target cap}}(h) := \mathrm{Vol}_g(\tau_2) - \mathrm{Vol}_g(\tau_1)$
    $m_{\text{overlap}}(h, r) := \mathrm{Vol}_g(\tau_4) - \mathrm{Vol}_g(\tau_3) - \mathrm{Vol}_g(\tau_2) + \mathrm{Vol}_g(\tau_1)$
  Let: $r = r_h$ solve $m_{\text{overlap}}(h, r) = \frac{1}{2} m_{\text{target cap}}(h)$.
Compute: $d(x(y, s), z) = s + \lim_{h \to 0} r_h$.

Algorithm 2 Travel time (distance) determination.

Let: $y, z \in \Gamma$ and $s > 0$ with $r_{x(y,s)}(z) < T$.
Let: $h_0 > 0$ small enough that $s + h_0 < \min\{\sigma_\Gamma(y), T\}$.
for all $0 < h < h_0$:
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Algorithm 3 Travel time (distance) determination.

Let: $y, z \in \Gamma$ and $s > 0$ with $r_{x(y,s)}(z) < T$.
Let: $h_0 > 0$ small enough that $s + h_0 < \min\{\sigma_\Gamma(y), T\}$.
for all $0 < h < h_0$:
  for all $0 < r < T - s$:
    Let: $\tau_1 = s1_\Gamma$, $\tau_2 = \tau_y^{s+h}$, $\tau_3 = \tau_z^{s+r}$, $\tau_4 = \tau_1 \vee \tau_2 \vee \tau_3$
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Algorithm 4 Travel time (distance) determination.

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for all $0 < h < h_0$:
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Algorithm 5 Travel time (distance) determination.

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for all $0 < h < h_0$:
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Algorithm 6 Travel time (distance) determination.

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for all $0 < h < h_0$:

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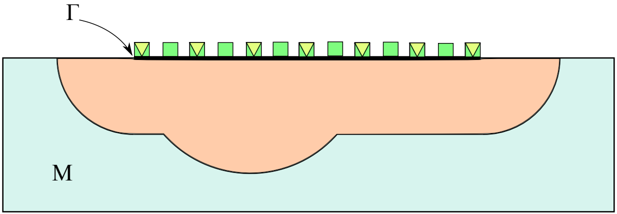
Numerical Set-Up

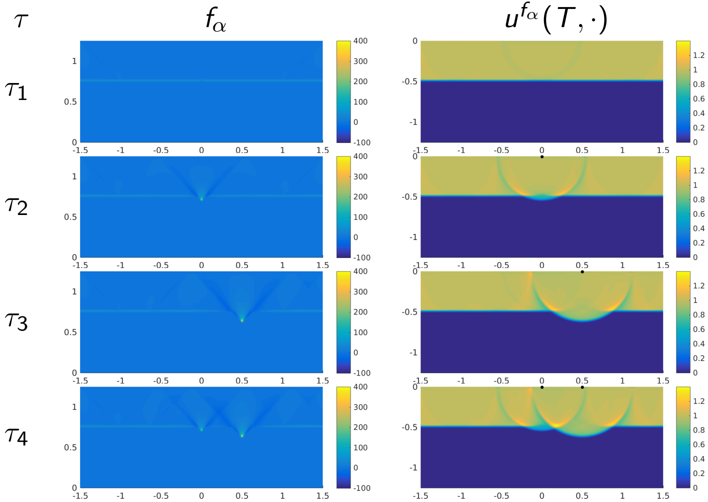


- M the lower half-space $c = 1$ or $c = 1/(1 - x^2)$
- $\Gamma = [-2.05, 2.05] \times \{0\}$, $T = 1.25$
- **Basis of sources:**

$$\varphi_{i,j}(t, x) = C \exp\left(-a_t(t - t_i)^2 - a_x(x - x_j)^2\right)$$

  with $a_t = a_x = 5.562 \cdot 10^3$
- **Data:**

$$\left\{\Lambda^{2T}(\varphi_{i,j})(t_{r,l}, x_{r,k}) = u^{\varphi_{i,j}}(t_{r,l}, x_{r,k}) : i, j, k, l\right\}$$


τ
f_α
u^{f_α}(T, ·)
τ1
τ2
τ3
τ4

Distance Estimation Results

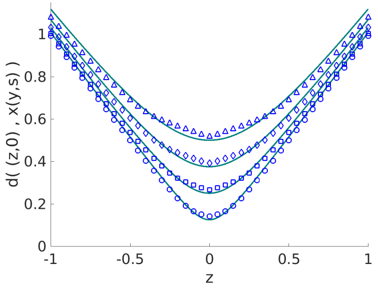


Figure: Euclidean case,
$\Delta_{g,\mu} = c^2\Delta$, $c = 1.0$

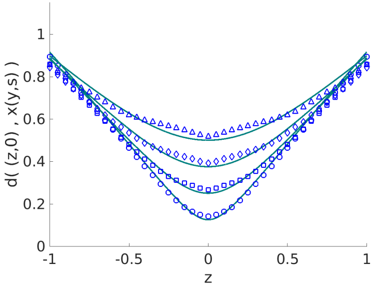
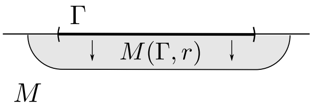


Figure: Hyperbolic case,
$\Delta_{g,\mu} = c^2\Delta$, $c = 1/(1 - x^2)$

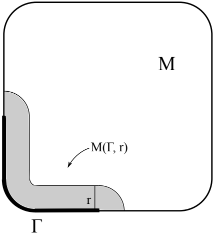
Redatuming



Redatuming

Assumption

We suppose that in addition to Λ_Γ^{2T} we now know, for some $r > 0$ with $T/2 \geq r$ the manifold $(M(\Gamma, r), g)$ where $M(\Gamma, r) := M(r1_\Gamma)$.



Interior Source Problem

For $F \in L^2([0, T/2] \times M(\Gamma, r))$, let w^F solve:

$$\begin{cases} \partial_t^2 w(t,x) - \Delta_g w(t,x) = F(t,x), & (t,x) \in (0,T) \times M \\ \partial_\nu w(t,x) = 0, & (t,x) \in [0,T] \times \Gamma \\ w(0,\cdot) = 0, \quad \partial_t w(0,\cdot) = 0, & x \in M. \end{cases}$$

Moving Receivers into $M(\Gamma, r)$

Lemma (essentially Bingham et. al. (2008))

The map, $L : L^2([0,T] \times \Gamma) \to L^2([0,T] \times M(\Gamma, r))$ given by,

$$L : h \mapsto u^h|_{M(\Gamma,r)\times(0,T)}$$

is bounded and can be computed from Λ_Γ^{2T} and $(M(\Gamma, r), g)$.

- Let $h \in C_0^\infty([0,T] \times \Gamma)$ and $0 \leq t \leq T$.
- To compute $Lh(t) = u^h(t, \cdot)|_{M(\Gamma,r)}$ consider control prob. with $\tau = r1_\Gamma$ and $\phi = u^h(t, \cdot)$
- $W_\tau^* \phi = P_\tau K_T \delta_t h$, where $\delta_t h$ is the source h delayed by $T - t$ time units

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is bounded and can be computed from Λ_Γ^{2T} and $(M(\Gamma, r), g)$.

- The solns to the regularized control probs satisfy:

$$\lim_{\alpha\to0^+} u^{f_\alpha}(T, \cdot) = \lim_{\alpha\to0^+} W_\tau f_\alpha = 1_{M(\Gamma,r)} \cdot u^h(t, \cdot)$$

- Since $f_\alpha \in L^2(S_\tau) = L^2([T - r, T] \times \Gamma)$ we can compute $u^{f_\alpha}(T, \cdot)$ by solving the wave eqn in $M(\Gamma, r)$.

Moving Receivers: Example

$M = \mathbb{R} \times [-1, 0]$, an infinitely long rectangle with height 1 and wavespeed $c = 1/(1-y)$, $\Gamma = [-3.1, 3.1] \times 0$, $r = 0.5$, and $T = 2.0$.

$u^h(t)$	$u^{f_\alpha,t}(T)$
(Half Space)	

Blagovescenskii-type Identity

Lemma (de Hoop, K., Oksanen)

For $F \in L^2([0, T/2] \times M(\Gamma, r))$, $h \in L^2([0, T/2] \times \Gamma)$,

$$(w^F(T/2), u^h(T/2))_{L^2(M)} = (F, \mathcal{K}h)_{L^2([0,T/2]\times M(\Gamma,r))}.$$

Where, $\mathcal{K}$ is bounded and can be computed by,

$$\mathcal{K} := J^{T/2} L \Theta^{T/2} - \rho^{T/2} R^T L R^T \Theta^{T/2} J^{T/2} \Theta^{T/2}.$$

Moving Sources into $M(\Gamma, r)$

Theorem (de Hoop, K., Oksanen)

The map, $\mathcal{L} : L^2([0, T/2] \times M(\Gamma, r)) \to L^2([0, T/2] \times M(\Gamma, r))$ given by,

$$\mathcal{L} : F \mapsto w^F|_{M(\Gamma,r)\times(0,T/2)}$$

is bounded and can be computed using L.

- Let $F \in C_0^\infty([0, T/2] \times M(\Gamma, r))$. To compute $\mathcal{L}F(t)$ consider $\tau = r1_\Gamma$ and $\phi = w^F(t, \cdot)$
- We have $\left(W_\tau^{T/2}\right)^* \phi = P_\tau^{T/2} \mathcal{K}^* \delta_t F$, where $\delta_t F$ is the source F delayed by $T/2 - t$ time units
- Then the solns to the regularized control probs satisfy:

$$\lim_{\alpha \to 0^+} u^{f_\alpha}(T/2, \cdot) = \lim_{\alpha \to 0^+} W_r^{T/2} f_\alpha = 1_{M(\Gamma,r)} \cdot w^F(t, \cdot)$$

Moving Sources: Example

$M = \mathbb{R} \times [-1, 0]$, an infinitely long rectangle with height 1 and wavespeed $c = 1/(1 - x^2)$, $\Gamma = [-3.1, 3.1] \times \{0\}$, $r = 0.5$, and $T = 2.0$.

$w^F(t)$	$u^{f_\alpha, t}(T/2)$
(Half Space)	

Thank you for listening!