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LETTER TO THE EDITOR

Inverse medium scattering for three-dimensional time harmonic Maxwell equations

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Abstract
A continuation method is developed for solving the inverse medium scattering problem of time harmonic Maxwell equations in $\mathbb{R}^3$. By using multi-frequency scattering data, our reconstruction algorithm first employs the Born approximation for an initial guess and proceeds via recursive linearization on the wavenumber k. At each linearization step, one forward and one adjoint state of the Maxwell equations are solved. Numerical examples are also presented and discussed.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Consider the time harmonic Maxwell equations in three dimensions

$$\nabla \times (\nabla \times E^{\mathrm{t}}) - k^2(1 + q(x))E^{\mathrm{t}} = 0, \tag{1.1}$$

where k is the wavenumber, and $q(x) > -1$, which has a compact support, is the scatterer. The total electric field E^{t} consists of the incident field E^{i} and the scattered field E:

$$E^{\mathrm{t}} = E^{\mathrm{i}} + E.$$

Assume that the incident field is a plane wave

$$E^{\mathrm{i}} = \vec{p}\mathrm{e}^{\mathrm{i}kx\cdot\vec{n}},$$

where $\vec{n} \in \mathbb{S}^2$ is the propagation direction, and $\vec{p} \in \mathbb{S}^2$ is the polarization vector satisfying $\vec{p}\cdot\vec{n} = 0$. Evidently, such an incident wave satisfies the homogeneous equation

$$\nabla \times (\nabla \times E^{\mathrm{i}}) - k^2 E^{\mathrm{i}} = 0. \tag{1.2}$$

It follows from the equations (1.1) and (1.2) that the scattered field satisfies

$$\nabla \times (\nabla \times E) - k^2(1 + q(x))E = k^2 q(x)E^{\mathrm{i}}. \tag{1.3}$$

In addition, the scattered field is required to satisfy the following Silver–Müller radiation condition:

$$\lim_{r\to\infty} r\left[\nabla \times E \times \frac{x}{r} - \mathrm{i}kE\right] = 0,$$

where $r = |x|$. Let Ω be a bounded domain of $\mathbb{R}^3$ with boundary Γ, which contains the compact support of the scatterer $q(x)$. Denote by ν the outward unit normal to Γ. Computationally, it is convenient to reduce the problem to a bounded domain (cell) by imposing a suitable (artificial) boundary condition on Γ. For simplicity, we employ the first-order absorbing boundary condition [10]

$$\nu \times (\nabla \times E) + \mathrm{i}k\nu \times (\nu \times E) = 0, \qquad \text{on } \Gamma. \tag{1.4}$$

Given the incident field E^{i}, the direct problem is to determine the scattered field E for the known scatterer $q(x)$, which has been well studied [4]. This work is devoted to the numerical solution of the inverse medium scattering problem, i.e., determining the scatterer $q(x)$ from the measurements of near-field current densities, $\nu \times E|_\Gamma$. Although this is a classical problem in inverse scattering theory, little is known on reconstruction methods, due to the nonlinearity, ill-posedness, and large-scale computation associated with the inverse scattering problem. We refer the reader to [1, 5, 8, 9, 14] for related results on the inverse medium problem. See [4] for an account of the recent progress on the general inverse scattering problems.

Our goal of this work is to present a recursive linearization method that solves the inverse medium scattering problem of Maxwell's equations in three dimensions. The reader is referred to [2, 3] for recursive linearization approaches for solving the inverse medium scattering problems in two dimensions. The algorithm requires multi-frequency scattering data, and the recursive linearization is obtained by a continuation method on the wavenumber k. The algorithm first solves a linear equation (Born approximation) at the lowest wavenumber k. Updates are made by using the data at higher wavenumbers k sequentially. Following the idea of the Kaczmarz method [5, 12], we use partial data and solve an underdetermined minimal norm solution at each sweep. For each iteration, one forward and one adjoint state of the Maxwell equations are solved which may be implemented by using the symmetric second-order edge elements [11, 13].

2. Low frequency modes of the scatterer

Rewrite (1.3) as

$$\nabla \times (\nabla \times E) - k^2 E = k^2 q(x)(E^{\mathrm{i}} + E), \tag{2.1}$$

where the incident wave is taken as $E^{\mathrm{i}} = \vec{p}_1 \mathrm{e}^{\mathrm{i}kx\cdot\vec{n}_1}$. Consider a test function $F = \vec{p}_2 \mathrm{e}^{\mathrm{i}kx\cdot\vec{n}_2}$, where $\vec{p}_2, \vec{n}_2 \in \mathbb{S}^2$ satisfying $\vec{p}_2 \cdot \vec{n}_2 = 0$. Hence F satisfies (1.2).

Multiplying equation (2.1) by F, and integrating over Ω on both sides, we have

$$\int_\Omega F \cdot [\nabla \times (\nabla \times E)]\,\mathrm{d}x - k^2 \int_\Omega F \cdot E\,\mathrm{d}x = k^2 \int_\Omega q(x) F \cdot (E^{\mathrm{i}} + E)\,\mathrm{d}x.$$

Integrating by parts and noting (1.2) for F, we deduce

$$\int_\Gamma [E \times (\nabla \times F) - F \times (\nabla \times E)] \cdot \nu\,\mathrm{d}s = k^2 \int_\Omega q(x) F \cdot (E^{\mathrm{i}} + E)\,\mathrm{d}x.$$

Using the boundary condition (1.4) of the scattered field E, and the special form of the incident wave E^{i} and F, we then get

$$\int_\Gamma E \times [\mathrm{i}k(\vec{n}_2 \times \vec{p}_2)\mathrm{e}^{\mathrm{i}kx\cdot\vec{n}_2}] \cdot \nu \, \mathrm{d}s - \int_\Gamma [\mathrm{i}k\nu \times (\nu \times E)] \cdot \vec{p}_2 \mathrm{e}^{\mathrm{i}kx\cdot\vec{n}_2} \, \mathrm{d}s$$
$$= k^2 \int_\Omega q(x)\vec{p}_1 \cdot \vec{p}_2 \mathrm{e}^{\mathrm{i}kx\cdot(\vec{n}_1+\vec{n}_2)} \, \mathrm{d}x + k^2 \int_\Omega q(x)\vec{p}_2 \cdot E\mathrm{e}^{\mathrm{i}kx\cdot\vec{n}_2} \, \mathrm{d}x.$$

A simple calculation yields

$$\int_\Omega q(x)\vec{p}_1 \cdot \vec{p}_2 \mathrm{e}^{\mathrm{i}kx\cdot(\vec{n}_1+\vec{n}_2)} \, \mathrm{d}x = \frac{\mathrm{i}}{k} \int_\Gamma (\nu \times E) \cdot (\vec{n}_2 \times \vec{p}_2 + \nu \times \vec{p}_2)\mathrm{e}^{\mathrm{i}kx\cdot\vec{n}_2} \, \mathrm{d}s$$
$$- \int_\Omega q(x)\vec{p}_2 \cdot E\mathrm{e}^{\mathrm{i}kx\cdot\vec{n}_2} \, \mathrm{d}x. \tag{2.2}$$

When the wavenumber k is small, the first part of the right-hand side of the above integral equation is dominant. Dropping the nonlinear (second) term, we obtain the linear integral equation

$$\int_\Omega q(x)\mathrm{e}^{\mathrm{i}kx\cdot(\vec{n}_1+\vec{n}_2)} \, \mathrm{d}x = \frac{\mathrm{i}}{(\vec{p}_1 \cdot \vec{p}_2)k} \int_\Gamma (\nu \times E) \cdot (\vec{n}_2 \times \vec{p}_2 + \nu \times \vec{p}_2)\mathrm{e}^{\mathrm{i}kx\cdot\vec{n}_2} \, \mathrm{d}s, \tag{2.3}$$

which is the Born approximation.

Since the scatterer $q(x)$ we use a compact support, we use the notation

$$\hat{q}(\xi) = \int_\Omega q(x)\mathrm{e}^{\mathrm{i}kx\cdot(\vec{n}_1+\vec{n}_2)} \, \mathrm{d}x,$$

where $\hat{q}(\xi)$ is the Fourier transform of $q(x)$ with $\xi = k(\vec{n}_1 + \vec{n}_2)$. Choose

$$\vec{n}_i = (\sin\theta_i \cos\phi_i, \sin\theta_i \sin\phi_i, \cos\theta_i), \qquad i = 1, 2,$$

where θ_i, ϕ_i are the latitudinal and longitudinal angles respectively. It is obvious that the domain $[0, \pi] \times [0, 2\pi]$ of (θ_i, ϕ_i), $i = 1, 2$, corresponds to the ball $\{\xi : |\xi| \leqslant 2k\}$. Thus, the Fourier modes of $\hat{q}(\xi)$ in the ball $\{\xi : |\xi| \leqslant 2k\}$ can be determined. The scattering data with higher wavenumber k must be used in order to recover more modes of the true scatterer.

In practice, the Kaczmarz method [5, 12] is used to implement the linear integral equation (2.3) in order to reduce the computational cost and instability.

3. Recursive linearization

As discussed in the previous section, when the wavenumber k is small, the Born approximation allows a reconstruction of those Fourier modes less than or equal to $2k$ for the function $q(x)$. We now describe a procedure that recursively determines q_k, an approximation of $q(x)$ at $k = k_j$ for $j = 1, 2, \ldots$, with the increasing wavenumber. Suppose now that the scatterer $q_{\tilde{k}}$ has been recovered at some $\tilde{k}$, and that $k > 0$ is slightly larger than $\tilde{k}$. We wish to determine q_k or, equivalently, to determine the perturbation

$$\delta q = q_k - q_{\tilde{k}}.$$

For the reconstructed scatterer $q_{\tilde{k}}$, we can solve at the frequency k the forward scattering problem

$$\nabla \times (\nabla \times \tilde{E}) - k^2(1 + q_{\tilde{k}})\tilde{E} = k^2 q_{\tilde{k}} E^{\mathrm{i}}, \qquad x \in \Omega, \tag{3.1}$$
$$\nu \times (\nabla \times \tilde{E}) + \mathrm{i}k\nu \times (\nu \times \tilde{E}) = 0, \qquad \text{on } \Gamma. \tag{3.2}$$

For the scatterer q_k, we have

$$\nabla \times (\nabla \times E) - k^2(1 + q_k)E = k^2 q_k E^{\mathrm{i}}, \qquad x \in \Omega, \tag{3.3}$$
$$\nu \times (\nabla \times E) + \mathrm{i}k\nu \times (\nu \times E) = 0, \qquad \text{on } \Gamma. \tag{3.4}$$

Subtracting (3.1), (3.2) from (3.3), (3.4), and omitting the second-order smallness in δq and in $\delta E = E - \tilde{E}$, we obtain

$$\nabla \times (\nabla \times \delta E) - k^2(1 + q_{\tilde{k}})\delta E = k^2 \delta q(E^{\mathrm{i}} + \tilde{E}), \qquad x \in \Omega, \tag{3.5}$$

$$\nu \times (\nabla \times \delta E) + \mathrm{i}k\nu \times (\nu \times \delta E) = 0, \qquad \text{on } \Gamma. \tag{3.6}$$

For the scatterer q_k, and the incident wave E^{i}, we define the map $S(q_k, E^{\mathrm{i}})$ by

$$S(q_k, E^{\mathrm{i}}) = E,$$

where E is the solution of (3.3), (3.4). Let γ be the trace operator to the boundary Γ of the bounded domain Ω. Define the scattering map

$$M(q_k, E^{\mathrm{i}}) = \gamma S(q_k, E^{\mathrm{i}}).$$

For simplicity, denote $M(q_k, E^{\mathrm{i}})$ by $M(q_k)$. By the definition of trace operator, we have

$$M(q_k) = \nu \times E|_{\Gamma}.$$

We next examine the boundary data $\nu \times E(x; \theta_1, \phi_1; k)$. Here, the variable x is the observation point which has two degrees of freedom because it is on the artificial boundary Γ. The terms θ_1, ϕ_1 are latitudinal and longitudinal angles of the incident wave E^{i}, respectively. At each frequency, we have four degrees of freedom, and thus again data redundancy, which may be addressed by fixing one of the incident angles, say θ_1.

Use the notation $(\phi_1)_j = (j - 1) * 2\pi/m$, $j = 1, \ldots, m$, and the residual operator

$$R_j(q_{\tilde{k}}) = \nu \times E(x; \theta_1, (\phi_1)_j; k)|_{\Gamma} - \nu \times \tilde{E}(x; \theta_1, (\phi_1)_j; k)|_{\Gamma},$$

where $\tilde{E}(x; \theta_1, (\phi_1)_j; k)$ is the solution of (3.1), (3.2) with the incident longitudinal angle $(\phi_1)_j$, and the scatterer $q_{\tilde{k}}$. For each j, consider the minimal norm solution of the following problem:

$$DM_j(q_{\tilde{k}})\delta q_j = R_j(q_{\tilde{k}}),$$

which has the form

$$\delta q_j = DM_j^*(q_{\tilde{k}})[DM_j(q_{\tilde{k}})DM_j^*(q_{\tilde{k}})]^{-1} R_j(q_{\tilde{k}}).$$

In practice, some regularization [6] would also be needed.

Lemma 3.1. *Given residual $R_j(q_{\tilde{k}})$, there exists a function F_j such that the adjoint Fréchet operator $DM_j^*(q_{\tilde{k}})$ satisfies*

$$[DM_j^*(q_{\tilde{k}})R_j(q_{\tilde{k}})](x) = k^2(\overline{E_j^{\mathrm{i}}}(x) + \overline{\tilde{E}}_j(x)) \cdot F_j(x),$$

where the bar denotes the complex conjugate, E_j^{i} is the incident wave with the longitudinal angle $(\phi_1)_j$, and $\tilde{E}_j$ is the solution of (3.1), (3.2) with the incident wave E_j^{i}.

From the above lemma, it follows easily that

$$\delta q_j = \frac{k^2}{\beta_k}(\overline{E_j^{\mathrm{i}}}(x) + \overline{\tilde{E}}_j(x)) \cdot F_j(x). \tag{3.7}$$

Therefore, for each incident wave with longitudinal angle $(\phi_1)_j$, it is necessary to solve one forward and one adjoint problem for the Maxwell equations. Observe that the adjoint problem in fact takes a similar variational form to the forward problem. Essentially, we need to compute two forward problems at each sweep. Once δq_j is determined, $q_{\tilde{k}}$ is updated by $q_{\tilde{k}} + \delta q_j$. After the mth sweep is completed, we get the reconstructed scatterer q_k at the wavenumber k.

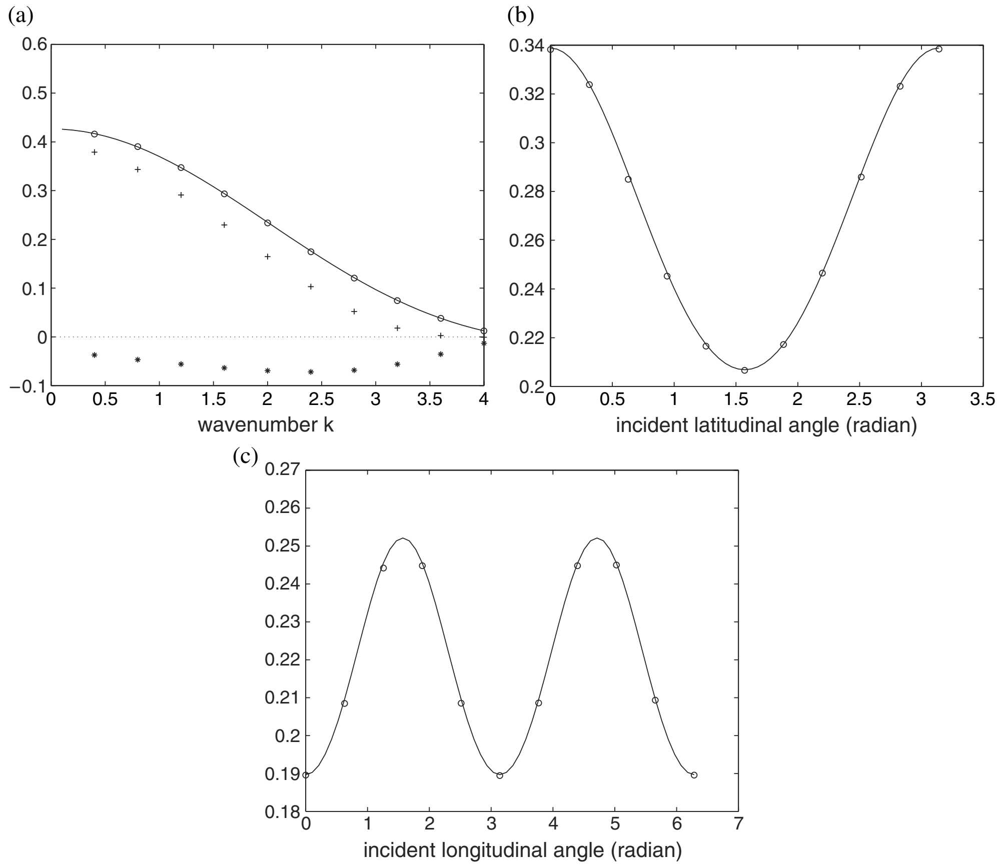

Figure 1. (a) Integrals with different wavenumbers k for fixed incident angle $\theta = \pi/3$, $\phi = \pi/3$. Solid curve: the exact integral value of the left-hand side of (2.2); +: the numerical integral of the first term of the right-hand side of (2.2); $*$: the numerical integral of the second term of the right-hand side of (2.2); $\circ$: the numerical integral of the right-hand side of (2.2). (b) The numerical integral with different θ for fixed wavenumber $k = 2.0$ and $\phi = \pi/3$. Solid curve: the exact integral value of the left-hand side of (2.2); $\circ$: the numerical integral of the right-hand side of (2.2) (c) The numerical integral with different ϕ for fixed wavenumber $k = 2.0$ and $\theta = \pi/3$. Solid curve: the exact integral of the left-hand side of (2.2); $\circ$: the numerical integral of the right-hand side of (2.2).

4. Numerical experiments

In this section, we discuss the treatment of the forward scattering problem, and computational aspects of the recursive linearization algorithm.

For the forward solver, we adopt the symmetric second-order tetrahedral edge elements [11]. The reverse Cuthill–McKee ordering [7], the compressed row storage format, and the quasi-minimal residual algorithm with diagonal preconditioning are used in the assembly of unknowns, the storage of the coefficient matrix, and solving the linear system, respectively.

Consider a test problem with the exact scatterer

$$q(x, y, z) = \begin{cases} 1 - \sin\left[\dfrac{\pi}{2}\left(\dfrac{x^2}{1^2} + \dfrac{y^2}{0.8^2} + \dfrac{z^2}{0.5^2}\right)\right], & \text{for } \dfrac{x^2}{1^2} + \dfrac{y^2}{0.8^2} + \dfrac{z^2}{0.5^2} \leqslant 1. \\ 0, & \text{otherwise.} \end{cases}$$

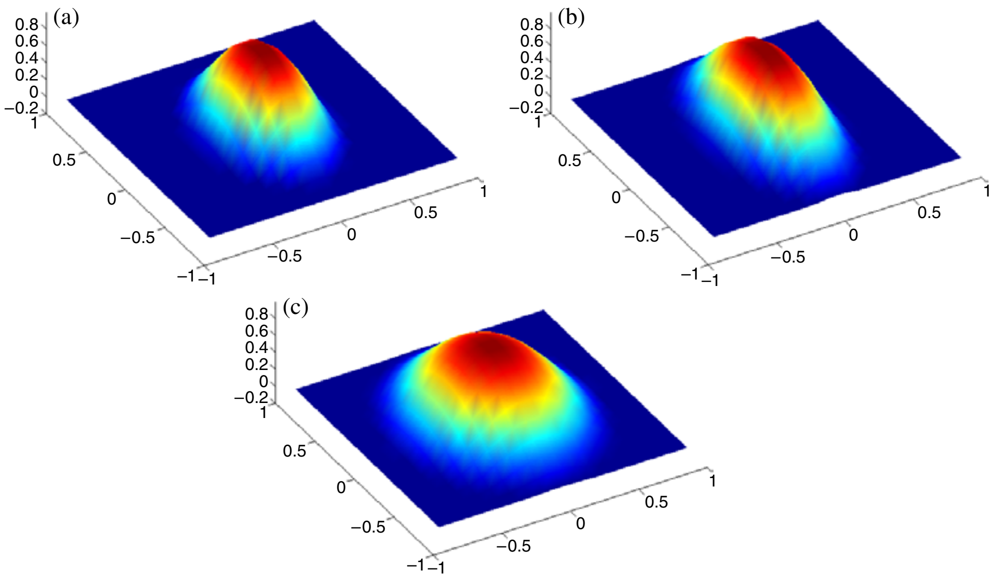

Figure 2. (a) The true scatterer, slice $x = 0$; (b) the true scatterer, slice $y = 0$; (c) the true scatterer, slice $z = 0$.

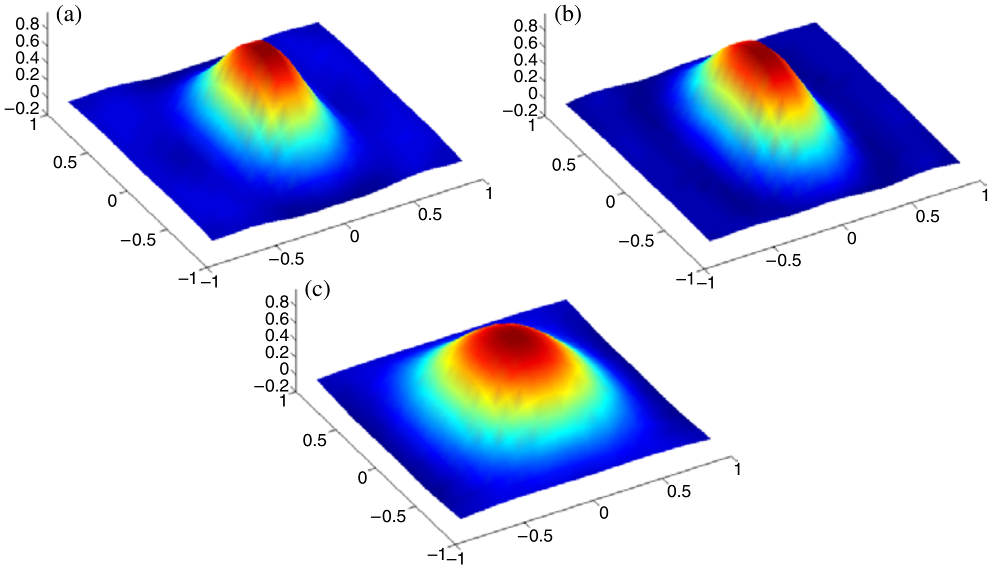

Figure 3. (a) The reconstruction, slice $x = 0$; (b) the reconstruction, slice $y = 0$; (c) the reconstruction, slice $z = 0$.

The compact support of this scatterer is an ellipsoid contained in the unit ball. For simplicity, we take $\vec{n}_1 = \vec{n}_2$, and $\vec{p}_1 = \vec{p}_2$ in the test of the forward solver. Numerical results are shown in figure 1. In figure 1(a), for fixed incident latitudinal angle $\theta = \pi/3$, and the longitudinal angle $\phi = \pi/3$, the direct problem is solved at different wavenumbers k. In figures 1(b), and (c), the numerical results are shown with different latitudinal angles $\theta \in [0, \pi]$ (fix $\phi = \pi/3$), and

Table 1. Relative error at different wavenumbers k.

k	1	2	3	4	5
e_2	0.6313	0.5180	0.3757	0.2299	0.1350

with different longitudinal angles $\phi \in [0, 2\pi]$ (fix $\theta = \pi/3$), respectively. As is easily seen from figure 1(a), the first term of the right-hand side of the integral equation (2.2) is indeed dominant compared with the nonlinear second term when the wavenumber k is small.

For a simple stability analysis, some relative random noise is added to the data, e.g., the tangential component of the electric field is updated to

$$\nu \times E|_\Gamma := (1 + \sigma \,\text{rand}) \cdot (\nu \times E|_\Gamma),$$

where rand gives normally distributed random numbers in $[-1, 1]$ and σ is an error parameter. In our numerical experiments, the latitudinal angle of the incident wave $\theta_1 = \pi/2$ is fixed, $\sigma = 0.02$, the sweep number $m = 20$, and some appropriately chosen regularization parameters $\beta_k = 0.8/k$, where k is the wavenumber. Define the relative error by

$$e_2 = \frac{\left(\sum_{i,j,k} |q_{ijk} - \bar{q}_{ijk}|^2\right)^{\frac{1}{2}}}{\left(\sum_{i,j,k} |q_{ijk}|^2\right)^{\frac{1}{2}}},$$

where $\bar{q}$ is the reconstructed scatterer, and q is the true scatterer. Figure 2 shows the slices of true scatterer, and figure 3 gives the reconstruction at the wavenumber $k = 5$. The relative errors are shown in table 1 at different wavenumbers k.

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