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Analysis of Direct and Inverse Cavity Scattering Problems

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Abstract. Consider the scattering of a time-harmonic electromagnetic plane wave by an arbitrarily shaped and filled cavity embedded in a perfect electrically conducting infinite ground plane. A method of symmetric coupling of finite element and boundary integral equations is presented for the solutions of electromagnetic scattering in both transverse electric and magnetic polarization cases. Given the incident field, the direct problem is to determine the field distribution from the known shape of the cavity; while the inverse problem is to determine the shape of the cavity from the measurement of the field on an artificial boundary enclosing the cavity. In this paper, both the direct and inverse scattering problems are discussed based on a symmetric coupling method. Variational formulations for the direct scattering problem are presented, existence and uniqueness of weak solutions are studied, and the domain derivatives of the field with respect to the cavity shape are derived. Uniqueness and local stability results are established in terms of the inverse problem.

AMS subject classifications: 78A46, 78M30

Key words: electromagnetic cavity, direct problem, inverse problem, finite element methods, boundary integral equations.

1. Introduction

The radar cross section (RCS) is a measure of the detectability of a target by radar system. Deliberate control in the form of enhancement or reduction of the RCS of a target is of no less importance than many radar applications. The cavity RCS caused by jet engine inlet ducts or cavity-backed antennas can dominate the total RCS. A thorough understanding of the electromagnetic scattering characteristic of a target, particularly a cavity, is necessary for successful implementation of any desired control of its RCS, and is of high interest to the scientific and engineering community.

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Time-harmonic analysis of cavity-backed apertures with penetrable material filling the cavity interior has been examined by numerous researchers in the engineering community, such as Jin [18], Jin and Volakis [19], Liu and Jin [21], Wood and Wood [30], and references cited therein. Mathematical treatment of the direct scattering problems involving cavities can be found in Ammari *et al* [2,3], Bao and Sun [7], Van and Wood [28], where a non-local transparent boundary condition, based on the Fourier transform, is proposed on the open aperture of the cavity. It is a common assumption that the cavity opening coincides with the aperture on an infinite ground plane, and hence simplifying the modeling of the exterior (to the cavity) domain. This limits the application of these methods since many cavity openings are not planar. Recently, Wood [29] has developed a technique that is capable of characterizing the scattering by over-filled cavities in the frequency domain, where an artificial boundary condition, based on Fourier series, is introduced on a semicircle enclosing the cavity. The solution domain is the cavity plus the interior region enclosed by the semicircle, which may be large and thus computationally demanding if the aperture of the cavity is wide. This paper aims to develop an efficient alternative for dealing with both regular and over-filled cavities with arbitrary shape, and analyze the associated direct and inverse electromagnetic scattering problems.

Specifically, we consider a time-harmonic electromagnetic plane wave incident on an open cavity embedded in an infinite ground plane. The ground plane and the wall of the cavity are perfect electric conductors, and the open cavity is filled with a nonmagnetic material which may be inhomogeneous. The infinite upper half-space above the ground plane and the cavity is composed of a homogeneous medium characterized by its permittivity ε_0 and permeability μ_0. Two fundamental polarizations, transverse electric (TE) and transverse magnetic (TM), are considered for the direct and inverse electromagnetic scattering from the cavity.

Given a time-harmonic plane incident wave and the shape of the cavity, the direct scattering problem is to predict the field distribution away from the cavity. We present a method of symmetric coupling of finite element and boundary integral equations. Computationally, the symmetric coupling leads to a complex symmetric coefficient matrix which can be efficiently stored and solved, especially for three-dimensional problems. In this method, the unbounded region is first divided into an interior region and an exterior region through an artificial boundary. The field in the interior region is formulated using the finite element method, and the field in the exterior region is formulated via the boundary integral method. The interior and exterior fields are subsequently coupled by the continuity conditions at the boundary separating the two regions. Therefore, the boundary integral equation essentially provides a transparent boundary condition on the boundary of the truncated domain to avoid artificial wave reflection. The position of the artificial boundary is rather flexible and can chosen to greatly reduce the computational effort: it will just be the aperture on the ground plane for a regular cavity; it can be put as close as possible to the opening of the overfilled cavity. In the general two-dimensional setting, we study the well-posedness of the direct scattering problem based on variational approaches and show the differentiability of the field with respect to the cavity shape.

Given a time-harmonic plane incident wave, the inverse scattering problem is to de-

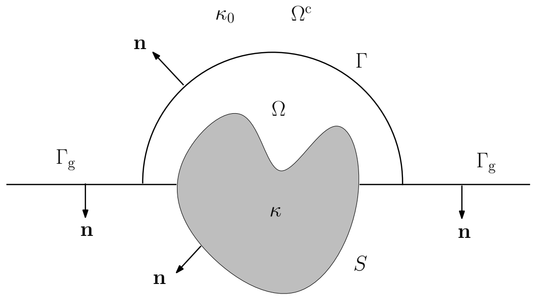

Figure 1: Problem geometry. A cavity with the wall S is placed on a perfectly conducting ground Γ_g. The medium inside the cavity is inhomogeneous with a variable wavenumber κ and the medium above the cavity and ground is homogeneous with a constant wavenumber κ_0. A Lipschitz continuous curve Γ encloses the cavity and divides the physical domain into Ω, bounded by S and Γ, and its complementary Ω^c in the upper half-space.

termine the shape of the cavity from the measurement of the total field on the artificial boundary. Regarding the inverse problem, this paper is concerned with the uniqueness and the local stability questions: what information can we extract about the shape, i.e., the cavity wall, from measurements of the electromagnetic field on the artificial boundary? We prove the uniqueness of the inverse problem: the cavity shape is uniquely determined by the boundary measurement of the total field. Based on the domain derivative of the field, we establish a local stability result, which indicates that if the measurements are “close" to the true fields, then the resulting cavity is also “close" to the true cavity. The proof is motivated by the technique in Bao [6] for the diffraction grating problem, where a main ingredient was to estimate a quotient difference function due to the perturbation of the grating profile. Noticing the quotient difference function is an approximation to the domain derivative, we directly investigate the domain derivative and provide the stability proof for both the TE and TM polarization cases in a similar manner. See [12] for the uniqueness and local stability of a regular cavity filled with homogeneous medium in TE case.

The paper is outlined as follows. In Section 2, the Maxwell equations are presented and reduced into two fundamental modes in two dimensions: TE polarization and TM polarization. Section 3 is devoted to the TE polarization case. A mathematical model is described; variational formulations for a symmetric coupling of a finite element method in the inhomogeneous cavity with a boundary integral method on the artificial boundary is presented; the well-posedness of the variational problem is studied; the domain derivative of the field with respect to the cavity shape is derived; uniqueness and local stability results are obtained. Parallel results for the case of TM polarization are given in Section 4. The paper is concluded with some general remarks and directions for future research in Section 5.

2. Maxwell's equations

Throughout, the media are assumed to be non-magnetic, and a constant magnetic permeability, $\mu = \mu_0$, is assumed everywhere. Then the electromagnetic wave propagation is governed by the time-harmonic Maxwell equations (time dependence $e^{-i\omega t}$):

$$\operatorname{curl}\mathbf{E} = i\omega\mathbf{B}, \tag{2.1}$$

$$\operatorname{curl}\mathbf{H} = -i\omega\mathbf{D} + \mathbf{J}, \tag{2.2}$$

where $\mathbf{E}$ is the electric field, $\mathbf{H}$ is the magnetic field, $\mathbf{B}$ is the magnetic flux density, $\mathbf{D}$ is the electric flux density, $\mathbf{J}$ is the electric current density, and ω is the angular frequency. The constitutive relations, describing the macroscopic properties of the medium, are taken as

$$\mathbf{B} = \mu\mathbf{H}, \quad \mathbf{D} = \varepsilon\mathbf{E}, \quad \text{and} \quad \mathbf{J} = \sigma\mathbf{E},$$

where the constitutive parameters μ, ε, and σ denote, respectively, the magnetic permeability, the electric permittivity, and the conductivity of the medium. Throughout, we assume $\mathrm{Re}\varepsilon \geq \varepsilon_0 > 0$, $\sigma > 0$ accounting for a loss medium and excludes the possible existence of eigen-frequencies, and $\varepsilon \in L^\infty(\mathbb{R}^3), \sigma \in L^\infty(\mathbb{R}^3)$. Substituting the constitutive relations into Eqs. (2.1) and (2.2) gives a coupled system for the electric and magnetic fields

$$\operatorname{curl}\mathbf{E} = i\omega\mu\mathbf{H}, \tag{2.3}$$

$$\operatorname{curl}\mathbf{H} = (-i\omega\varepsilon + \sigma)\mathbf{E}. \tag{2.4}$$

In addition, standard jump conditions are satisfied for the fields across an interface.

Let a plane wave $(\mathbf{E}^{\mathrm{i}}, \mathbf{H}^{\mathrm{i}})$ be incident on an electromagnetic cavity, as shown in Fig. 1. A cavity with the wall S is placed on a perfectly conducting ground Γ_g. The medium inside the cavity is inhomogeneous with a dielectric coefficient ε. Above the cavity and the ground, the medium is assumed to be homogeneous with a constant dielectric permittivity ε_0. Choose a Lipschitz continuous curve Γ to enclose the cavity, denote by Ω the region bounded from S and Γ, and let $\Omega^{\mathrm{c}} = \mathbb{R}^2_+ \setminus \overline{\Omega}$ be the complementary set of Ω in the upper half-space $\mathbb{R}^2_+ = \{\mathbf{x} \in \mathbb{R}^2 : x_2 > 0\}$. On the surface of the perfectly conducting medium, the following boundary condition is satisfied for the electric field

$$\mathbf{n} \times \mathbf{E} = 0 \quad \text{on}\, \Gamma_{\mathrm{g}} \cup S, \tag{2.5}$$

where $\mathbf{n}$ is the unit outward normal to the boundary.

Taking the curl of Eq. (2.3) and eliminating the magnetic field from Eq. (2.4), we obtain the equation for the electric field

$$\operatorname{curl}\operatorname{curl}\mathbf{E} - \kappa^2\mathbf{E} = 0, \tag{2.6}$$

where $\kappa^2 = \omega^2\mu_0\varepsilon + i\omega\mu_0\sigma$ and κ is known as the wavenumber. Similarly, we may derive the equation for the magnetic field by eliminating the electric field

$$\operatorname{curl}\kappa^{-2}\operatorname{curl}\mathbf{H} - \mathbf{H} = 0. \tag{2.7}$$

For the TE polarization case, the incident wave has the electric field parallel to the x_3-axis, which is also the infinite axis of the aperture. Since both the incident field and medium are uniform along the x_3-axis, i.e., no variation of any kind with respect to x_3, the scattered electric field, and thus the total electric field, are also parallel to the x_3-axis, i.e., $\mathbf{E} = [0, 0, u]^\top$. It is therefore convenient to formulate the problem in terms of the electric field since it has only one component. It deduces from Eqs. (2.5) and (2.6) that the total electric field satisfies

$$\Delta u + \kappa^2 u = 0 \quad \text{above}\,\Gamma_g \cup S, \tag{2.8}$$

$$u = 0 \quad \text{on}\,\Gamma_g \cup S. \tag{2.9}$$

For the case of TM polarization, the magnetic field has only a x_3-component, i.e., $\mathbf{H} = [0, 0, u]^\top$, and therefore it is convenient to formulate the problem in terms of the magnetic field. It follows from Eqs. (2.5) and (2.7) that the total magnetic field satisfies

$$\nabla \cdot \left(\kappa^{-2} \nabla u\right) + u = 0 \quad \text{above}\,\Gamma_g \cup S, \tag{2.10}$$

$$\partial_{\mathbf{n}} u = 0 \quad \text{on}\,\Gamma_g \cup S. \tag{2.11}$$

We shall study the direct and inverse problems only for TE and TM polarized solutions. The more complicated three-dimensional problem will be considered in a separate work. We refer to [1, 4, 5] for the method of coupling finite element and boundary element for solving some related scattering problems involving the full Maxwell equations.

To apply numerical methods, the open domain needs to be truncated into a bounded domain. Therefore, a suitable boundary condition has to be imposed on the boundary of the bounded domain so that no artificial wave reflection occurs. Here we present a method of symmetric coupling of a boundary integral method for the truncated domain combined with a finite element method in the inhomogeneous cavity. In this method, it is necessary to enclose the cavity with inhomogeneous medium by a fictitious surface to separate the finite element region from the exterior region where the boundary integral equations apply. The field inside the surface is formulated using the finite element method, whereas those exterior to the surface can be expressed in terms of surface integrals. The interior and exterior are finally coupled via the field continuity conditions, leading to a complete system for the solution of interior and surface fields. This technique is particularly attractive for open-region problems involving complex structures and inhomogeneous materials, e.g., Brezzi and Johnson [8], Costable and Stephan [11], Gatica and Hsiao [13], Hsiao [17], Johnson and Nédélec [22], Li [24], Meddahi *et al* [26].

3. TE polarization

In this section, we shall introduce a variational formulation for the direct problem using a symmetric coupling of finite element and boundary integral methods and study the well-posedness of the variational problem, and present a local stability result for the inverse problem based on the domain derivative of the field with respect to the shape of the cavity. As the discussion for the TE polarization and TM polarization are parallel, we shall concentrate on the TE polarization first, and state the corresponding results on the TM polarization and give the proofs when necessary.

3.1. The direct problem

Let an incoming plane wave $u^{\mathrm{i}} = \exp(i\alpha x_1 - i\beta x_2)$ be incident on the perfect electrically conducting surface $\Gamma_{\mathrm{g}} \cup S$ from above, where $\alpha = \kappa_0 \sin\theta, \beta = \kappa_0 \cos\theta, \theta \in (-\pi/2, \pi/2)$ is the angle of incidence with respect to the positive x_2-axis, and $\kappa_0 = \omega\sqrt{\mu_0\varepsilon_0}$ is the wavenumber of the free space.

Denote the reference field u^{ref} as the solution of the homogeneous equation in the upper half space:

$$\Delta u^{\mathrm{ref}} + \kappa_0^2 u^{\mathrm{ref}} = 0 \quad \text{in}\,\mathbb{R}^2_+$$

together with the boundary condition

$$u^{\mathrm{ref}} = 0 \quad \text{on}\,\{x_2 = 0\}.$$

It can be shown that the reference field consists of the incident field u^{i} and the reflected field u^{r}:

$$u^{\mathrm{ref}} = u^{\mathrm{i}} + u^{\mathrm{r}},$$

where $u^{\mathrm{r}} = -\exp(i\alpha x_1 + i\beta x_2)$.

The total field u is composed of the reference field u^{ref} and the scattered field u^{s}:

$$u = u^{\mathrm{ref}} + u^{\mathrm{s}}.$$

It can be verified that the scattered field satisfies

$$\Delta u^{\mathrm{s}} + \kappa^2 u^{\mathrm{s}} = -(\kappa^2 - \kappa_0^2)u^{\mathrm{ref}} \quad \text{above}\,\Gamma_{\mathrm{g}} \cup S, \tag{3.1}$$

$$u^{\mathrm{s}} = -u^{\mathrm{ref}} \quad \text{on}\,\Gamma_{\mathrm{g}} \cup S. \tag{3.2}$$

In addition, the scattered field is required to satisfy the radiation condition

$$\lim_{\rho\to\infty} \sqrt{\rho}\left(\frac{\partial u^{\mathrm{s}}}{\partial \rho} - i\kappa_0 u^{\mathrm{s}}\right) = 0, \quad \rho = |\mathbf{x}|. \tag{3.3}$$

Given the reference field u^{ref}, the direct scattering problem is to determine the field distribution satisfying Eqs. (3.1)–(3.3). The following lemma is concerned with the uniqueness of the direct scattering problem, while the existence will be given in the next section.

Lemma 3.1. *The direct problem (3.1)–(3.3) has at most one solution.*

Proof. It suffices to show that $u^{\mathrm{s}} = 0$ if no source is present. Let D_ρ be the union of the domain Ω and the semi-disc with radius ρ and boundary Γ_ρ in the upper half-space. We have from the radiation condition (3.3) that

$$\left|\frac{\partial u^{\mathrm{s}}}{\partial \rho} - i\kappa_0 u^{\mathrm{s}}\right|^2 = \left|\frac{\partial u^{\mathrm{s}}}{\partial \rho}\right|^2 + \kappa_0^2|u^{\mathrm{s}}|^2 + 2\kappa_0 \mathrm{Im}\left(\partial_{\mathbf{n}}\overline{u^{\mathrm{s}}}\, u^{\mathrm{s}}\right) = o(\rho^{-1}) \quad \text{as}\,\rho \to \infty. \tag{3.4}$$

Multiplying the complex conjugate of u^s and integrating by parts for Eq. (3.1), we have from Green's theorem that

$$\int_{D_\rho} \left(|\nabla u^s|^2 - \bar{\kappa}^2 |u^s|^2\right) = \int_{\Gamma_\rho} \partial_{\mathbf{n}} \overline{u^s}\, u^s,$$

which yields

$$\mathrm{Im} \int_{\Gamma_\rho} \partial_{\mathbf{n}} \overline{u^s}\, u^s = \omega\mu_0 \int_{D_\rho} \sigma |u^s|^2. \tag{3.5}$$

Combining Eqs. (3.4) and (3.5) give

$$\int_{\Gamma_\rho} \left(\left| \frac{\partial u^s}{\partial \rho} \right|^2 + \kappa_0^2 |u^s|^2 \right) + 2\kappa_0 \omega\mu_0 \int_{D_\rho} \sigma |u^s|^2 \to 0 \quad \text{as}\, \rho \to \infty.$$

Hence u^s must be identically zero in $\mathbb{R}^2$. $\square$

Lemma 3.1 is an extension of the result of Rellich for obstacle scattering in free space, see Colton and Kress [10], to the case of the cavity scattering problem in half-space. When the medium inside the cavity is lossless, i.e., $\sigma = 0$, there may exist eigen-frequencies for the direct scattering problem. Therefore, we exclude a possibly set of discrete wavenumber κ_0, corresponding to the eigen-frequencies, and assume that the direct problem always has a unique solution.

3.2. Variational formulation

In this section, a variational formulation for the scattering problem will be derived by using a symmetric coupling of the finite element and boundary integral methods, and the well-posedness of variational problem will be studied.

We shall use the following notations: for a bounded region Ω in $\mathbb{R}^2$ with boundary Γ, $H^s(\Omega)$ and $H^s(\Gamma)$ will denote the usual Sobolev spaces with norm $\|\cdot\|_{H^s(\Omega)}$ and $\|\cdot\|_{H^s(\Gamma)}$, respectively. Define the following spaces:

$$\begin{aligned} L^2(\Gamma) &:= \{u|_\Gamma : u \in L^2(\partial\Omega)\}, \\ H^{1/2}(\Gamma) &:= \{u|_\Gamma : u \in H^{1/2}(\partial\Omega)\}, \\ \widetilde{H}^{1/2}(\Gamma) &:= \{u \in H^{1/2}(\Gamma) : \operatorname{supp} u \subset \overline{\Gamma}\}. \end{aligned}$$

In other words, $\widetilde{H}^{1/2}(\Gamma)$ contains functions $u \in H^{1/2}(\Gamma)$ such that their extension by zero to the whole boundary Ω is in $H^{1/2}(\Gamma)$. Now we denote by $H^{-1/2}(\Gamma)$ the dual space of $\widetilde{H}^{1/2}(\Gamma)$ and by $\widetilde{H}^{-1/2}(\Gamma)$ the dual space of $H^{1/2}(\Gamma)$. We refer to [9, 25] for detailed discussions on these spaces.

In Ω, Eqs. (2.8) and (2.9) have an equivalent variational form: find $u \in H_S^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } S\}$ such that

$$\int_\Omega \nabla u \cdot \nabla \overline{v} - \int_\Omega \kappa^2 u\, \overline{v} - \langle \phi, v \rangle = 0 \quad \text{for all}\, v \in H_S^1(\Omega), \tag{3.6}$$

where ϕ is the normal derivative of the total field u on Γ, i.e., $\phi = \partial_{\mathbf{n}} u$, and $\langle \cdot, \cdot \rangle$ denotes the duality between $H^{-1/2}(\Gamma)$ and $\widetilde{H}^{1/2}(\Gamma)$.

In Ω^c, based on Eq. (3.1) and the radiation condition (3.3), it follows from Green's theorem that we obtain the following integral representation

$$u^s(\mathbf{x}) = \int_\Gamma \partial_{\mathbf{n}_y} G_{\mathrm{TE}}(\mathbf{x},\mathbf{y}) u^s(\mathbf{y}) ds_{\mathbf{y}} - \int_\Gamma G_{\mathrm{TE}}(\mathbf{x},\mathbf{y}) \partial_{\mathbf{n}} u^s(\mathbf{y}) ds_{\mathbf{y}}, \quad \mathbf{x} \in \Omega^c. \tag{3.7}$$

Taking the limit of u^s by letting $\mathbf{x}$ go to Γ and using jump relations for surface potentials yield

$$u^s(\mathbf{x}) = \frac{1}{2} u^s(\mathbf{x}) + \int_\Gamma \partial_{\mathbf{n}_y} G_{\mathrm{TE}}(\mathbf{x},\mathbf{y}) u^s(\mathbf{y}) ds_{\mathbf{y}} - \int_\Gamma G_{\mathrm{TE}}(\mathbf{x},\mathbf{y}) \partial_{\mathbf{n}} u^s(\mathbf{y}) ds_{\mathbf{y}}, \quad \mathbf{x} \in \Gamma, \tag{3.8}$$

where

$$G_{\mathrm{TE}}(\mathbf{x},\mathbf{y}) = \frac{i}{4} H_0^{(1)}(\kappa_0 |\mathbf{x}-\mathbf{y}|) - \frac{i}{4} H_0^{(1)}(\kappa_0 |\mathbf{x}-\mathbf{y}'|).$$

Here $\mathbf{y}' = (y_1, -y_2)$ and $H_0^{(1)}$ is the Hankel function of first kind with order zero.

Regarding the reference field in $\Omega \cap \{x_2 \geq 0\}$, we have again from Green's theorem that

$$u^{\mathrm{ref}}(\mathbf{x}) = -\int_\Gamma \partial_{\mathbf{n}_y} G_{\mathrm{TE}}(\mathbf{x},\mathbf{y}) u^{\mathrm{ref}}(\mathbf{y}) ds_{\mathbf{y}} + \int_\Gamma G_{\mathrm{TE}}(\mathbf{x},\mathbf{y}) \partial_{\mathbf{y}} u^{\mathrm{ref}}(\mathbf{y}) ds_{\mathbf{y}}, \quad \mathbf{x} \in \Omega. \tag{3.9}$$

It follows from the taking the limit of u^{ref} by letting $\mathbf{x}$ go to Γ and using jump relations for surface potentials that

$$u^{\mathrm{ref}}(\mathbf{x}) = \frac{1}{2} u^{\mathrm{ref}}(\mathbf{x}) - \int_\Gamma \partial_{\mathbf{n}_y} G_{\mathrm{TE}}(\mathbf{x},\mathbf{y}) u^{\mathrm{ref}}(\mathbf{y}) ds_{\mathbf{y}} + \int_\Gamma G_{\mathrm{TE}}(\mathbf{x},\mathbf{y}) \partial_{\mathbf{y}} u^{\mathrm{ref}}(\mathbf{y}) ds_{\mathbf{y}}, \quad \mathbf{x} \in \Gamma. \tag{3.10}$$

Combining (3.8) and (3.10) leads to an integral equation for the total field on the boundary Γ:

$$u(\mathbf{x}) = \frac{1}{2} u(\mathbf{x}) + \int_\Gamma \partial_{\mathbf{n}_y} G_{\mathrm{TE}}(\mathbf{x},\mathbf{y}) u(\mathbf{y}) ds_{\mathbf{y}} - \int_\Gamma G_{\mathrm{TE}}(\mathbf{x},\mathbf{y}) \phi(\mathbf{y}) ds_{\mathbf{y}} + u^{\mathrm{ref}}(\mathbf{x}). \tag{3.11}$$

By taking the normal derivatives in the representation formulas (3.7) and (3.9) on both sides, and letting $\mathbf{x}$ go to Γ for $\mathbf{x}$ in D, we obtain

$$\phi(\mathbf{x}) = \frac{1}{2} \phi(\mathbf{x}) + \int_\Gamma \partial_{\mathbf{n}_x} \partial_{\mathbf{n}_y} G_{\mathrm{TE}}(\mathbf{x},\mathbf{y}) u(\mathbf{y}) ds_{\mathbf{y}} - \int_\Gamma \partial_{\mathbf{n}_x} G_{\mathrm{TE}}(\mathbf{x},\mathbf{y}) \phi(\mathbf{y}) ds_{\mathbf{y}} + \partial_{\mathbf{n}} u^{\mathrm{ref}}(\mathbf{x}). \tag{3.12}$$

To study the boundary integral equations, we introduce the single-layer potential operator V_{TE}, the hypersingular integral operator D_{TE}, the double-layer potential operator K_{TE}

and its adjoint operator K^*_{TE}, which are defined as

$$(V_{\mathrm{TE}}u)(\mathbf{x}) = \int_\Gamma G_{\mathrm{TE}}(\mathbf{x},\mathbf{y})u(\mathbf{y})ds_{\mathbf{y}},$$
$$(D_{\mathrm{TE}}u)(\mathbf{x}) = -\int_\Gamma \partial_{\mathbf{n}_\mathbf{x}}\partial_{\mathbf{n}_\mathbf{y}} G_{\mathrm{TE}}(\mathbf{x},\mathbf{y})u(\mathbf{y})ds_{\mathbf{y}},$$
$$(K_{\mathrm{TE}}u)(\mathbf{x}) = \int_\Gamma \partial_{\mathbf{n}_\mathbf{y}} G_{\mathrm{TE}}(\mathbf{x},\mathbf{y})u(\mathbf{y})ds_{\mathbf{y}},$$
$$(K^*_{\mathrm{TE}}u)(\mathbf{x}) = \int_\Gamma \partial_{\mathbf{n}_\mathbf{x}} G_{\mathrm{TE}}(\mathbf{x},\mathbf{y})u(\mathbf{y})ds_{\mathbf{y}}.$$

Using these operators, Eqs. (3.11) and (3.12) can be written as

$$u = (\frac{1}{2}I + K_{\mathrm{TE}})u - V_{\mathrm{TE}}\phi + f, \tag{3.13}$$
$$\phi = -D_{\mathrm{TE}}u + (\frac{1}{2}I - K^*_{\mathrm{TE}})\phi + g, \tag{3.14}$$

where $f = u^{\mathrm{ref}}, g = \partial_{\mathbf{n}} u^{\mathrm{ref}}$, and I is the identity operator.

Substituting ϕ in Eq. (3.14) into Eq. (3.6) gives

$$\int_\Omega \nabla u\cdot\nabla\overline{v} - \int_\Omega \kappa^2 u\overline{v} + \langle D_{\mathrm{TE}}u, v\rangle - \langle(\frac{1}{2}I - K^*_{\mathrm{TE}})\phi, v\rangle = \langle g, v\rangle \quad \text{for all}\, v\in H^1_S(\Omega). \tag{3.15}$$

Multiplying Eq. (3.13) by the complex conjugate of ψ and integrating over Γ arrives

$$\langle(\frac{1}{2}I - K_{\mathrm{TE}})u, \psi\rangle + \langle V_{\mathrm{TE}}\phi, \psi\rangle = \langle f, \psi\rangle \quad \text{for all}\, \psi\in H^{-1/2}(\Gamma). \tag{3.16}$$

Eqs. (3.15) and (3.16) consist of the variational formulation for the symmetric coupling of the finite element and boundary integral methods for the direct cavity scattering problem.

As this paper is concerned with the Helmholtz equation, the associated bilinear form is not elliptic. Therefore, a generalized notion of coercivity is considered for our purpose. A bilinear form $a : \mathscr{V}\times\mathscr{V}\to\mathbb{C}$ on a Hilbert space $\mathscr{V}$ is said to be coercive if it satisfies a generalized Gårding inequality of the form

$$\mathrm{Re}[a(\mathbf{u},\mathbf{u}) + c(\mathbf{u},\mathbf{u})] \geq C||\mathbf{u}||^2_{\mathscr{V}} \quad \text{for all}\, \mathbf{u}\in\mathscr{V},$$

where $C > 0, c : \mathscr{V}\times\mathscr{V}\to\mathbb{C}$ is a compact bilinear form.

Before presenting the well-posedness result for the variational problem, we state a useful lemma. The reader is referred to [25] for detailed discussions and proofs.

Lemma 3.2. *The single-layer potential operators V_{TE} is compact from $H^{-1/2}(\Gamma)$ into $\widetilde{H}^{1/2}(\Gamma)$, the double-layer potential operator K_{TE} and its adjoint K^*_{TE} are compact from $\widetilde{H}^{1/2}(\Gamma)$ into*

$\widetilde{H}^{1/2}(\Gamma)$ and from $H^{-1/2}(\Gamma)$ into $H^{-1/2}(\Gamma)$, respectively, and the hypersingular integral operator $D_{\rm TE}$ is compact from $\widetilde{H}^{1/2}(\Gamma)$ into $H^{-1/2}(\Gamma)$. Furthermore, the sinle-layer potential operator $V_{\rm TE}$ the and hypersingular integral operator $D_{\rm TE}$ are coercive in $H^{-1/2}(\Gamma)$ and $\widetilde{H}^{1/2}(\Gamma)$, i.e., there exit compact operators V_0 and D_0 such that

$$\begin{aligned}\operatorname{Re}[\langle V_{\rm TE}\phi,\phi\rangle + \langle V_0\phi,\phi\rangle] &\geq C \parallel \phi \parallel_{H^{-1/2}(\Gamma)} \quad \textit{for all}\, \phi \in H^{-1/2}(\Gamma),\\ \operatorname{Re}[\langle D_{\rm TE}u,u\rangle + \langle D_0 u,u\rangle] &\geq C \parallel u \parallel_{\widetilde{H}^{1/2}(\Gamma)} \quad \textit{for all}\, u \in \widetilde{H}^{1/2}(\Gamma).\end{aligned}$$

Denote $\mathscr{V}_{\rm TE} = H^1_S(\Omega)\times H^{-1/2}(\Gamma)$ and the norm in $\mathscr{V}_{\rm TE}$ is naturally defined for any $\mathbf{u} = [u,\phi] \in \mathscr{V}_{\rm TE}$

$$\parallel \mathbf{u} \parallel^2_{\mathscr{V}_{\rm TE}} = \parallel u \parallel^2_{H^1(\Omega)} + ||\phi||^2_{H^{-1/2}(\Gamma)}.$$

Theorem 3.1. *The variational problem (3.15)–(3.16) admits a unique solution $[u,\phi] \in \mathscr{V}_{\rm TE}$.*

Proof. The variational problem (3.15)–(3.16) is equivalent to: find $\mathbf{u} = [u,\phi] \in \mathscr{V}_{\rm TE}$ such that

$$a_{\rm TE}(\mathbf{u},\mathbf{v}) = \langle g, v\rangle + \langle f,\psi\rangle \quad \text{for all}\, \mathbf{v} = [v,\psi] \in \mathscr{V}_{\rm TE}, \tag{3.17}$$

where the bilinear form $a_{\rm TE}$ is defined by

$$\begin{aligned}a_{\rm TE}(\mathbf{u},\mathbf{v}) = \int_\Omega \nabla u\cdot\nabla\overline{v} - \int_\Omega \kappa^2 u\,\overline{v} + \langle D_{\rm TE}u, v\rangle + \langle V_{\rm TE}\phi,\psi\rangle \\ -\langle(\frac{1}{2}I - K^*_{\rm TE})\phi, v\rangle + \langle(\frac{1}{2}I - K_{\rm TE})u,\psi\rangle.\end{aligned} \tag{3.18}$$

To prove the theorem, it suffices to check the coercivity of the bilinear form a. Define another bilinear form c by

$$c(\mathbf{u},\mathbf{v}) = \int_\Omega \kappa^2 u\,\overline{v} + \langle D_0 u, v\rangle + \langle V_0\phi,\psi\rangle.$$

It follows from the compactness of the operators D_0 and V_0 that the bilinear form c is compact on $\mathscr{V}_{\rm TE}\times\mathscr{V}_{\rm TE}$.

Evidently we have

$$\begin{aligned}\operatorname{Re}[a_{\rm TE}(\mathbf{u},\mathbf{u}) + c(\mathbf{u},\mathbf{u})] &= \int_\Omega |\nabla u|^2 + \operatorname{Re}[\langle D_{\rm TE}u,u\rangle + \langle D_0u,u\rangle] + \operatorname{Re}[\langle V_{\rm TE}\phi,\phi\rangle + \langle V_0\phi,\phi\rangle]\\ &\geq C\left[\parallel u\parallel^2_{H^1(\Omega)} + \parallel \phi \parallel^2_{H^{-1/2}(\Gamma)}\right] = C \parallel \mathbf{u} \parallel^2_{\mathscr{V}_{\rm TE}},\end{aligned}$$

where the Poincaré inequality is applied. It follows from the Fredholm alternative and Lemma 3.1 regarding the uniqueness that the variational problem (3.15)–(3.16) admits a unique solution $\mathbf{u} = [u,\phi] \in H^1_S(\Omega)\times H^{-1/2}(\Gamma)$. $\square$

3.3. Uniqueness of the inverse problem

In this section, we study the uniqueness of the inverse cavity scattering problem: to determine the cavity wall S from the total field u measured on Γ.

Theorem 3.2. *Let $[u_j,\phi_j]$ be the solution of (3.13)–(3.14) in Ω_j with $\partial\Omega_j = \Gamma\cup S_j$ for $j=1,2$. If $u_1=u_2$ on Γ, then $S_1=S_2$.*

Proof. Assume that $S_1\neq S_2$. Then $\Omega_1\setminus(\Omega_1\cap\Omega_2)$ or $\Omega_2\setminus(\Omega_1\cap\Omega_2)$ is a non-empty set. Without loss of generality, we assume that $D=\Omega_1\setminus(\Omega_1\cap\Omega_2)\neq\emptyset$. Denote ∂D by $C_1\cup C_2$ with $C_j\subset S_j$ for $j=1,2$.

Since $u_1-u_2=0$ on Γ, it follows from the injectivity of the single-layer potential operator V_{TE} that $\phi_1-\phi_2=0$ on Γ. Noticing $u_1=u_2=0$ on $\Gamma_g\cup\Gamma$ and the radiation condition, we have $u_1-u_2=0$ above $\Gamma_g\cup\Gamma$. By unique continuation, we get $u_1-u_2=0$ in $\overline{\Omega_1\cap\Omega_2}$ and especially $u_1-u_2=0$ on C_2. It follows from $u_2=0$ on C_2 that we have $u_1=0$ on C_2 and the problem

$$\Delta u_1+\kappa^2u_1=0 \quad \text{in}\,\Omega_1\setminus(\Omega_1\cap\Omega_2), \tag{3.19}$$

$$u_1=0 \quad \text{on}\,C_1\cup C_2. \tag{3.20}$$

Recalling the expression of wavenumber, we have from (3.19)–(3.20) that

$$\int_D|\nabla u_1|^2-\omega^2\mu_0\int_D\varepsilon|u_1|^2-i\omega\mu_0\int_D\sigma|u_1|^2=0,$$

which yields $u_1=0$ in D. An application of the unique continuation again gives $u_1=0$ in Ω_1. But this contradicts the transparent boundary condition (3.13) since f is a nonzero function involving the incoming plane wave. $\square$

The above uniqueness theorem relies on the fact that the wavenumber has nonzero imaginary part. For real wavenumber or lossless medium, i.e., $\sigma=0$, the uniqueness result is still valid under the following two cases: (1) if $u_1=u_2$ on Γ for all wavenumbers in some open interval; (2) if $u_1=u_2$ on Γ for one wavenumber κ satisfying $\kappa^2\in(0,\lambda_0)$, where λ_0 is the first Dirichlet eigenvalue of negative Laplacian in a given ball B_0 containing Ω_1 and Ω_2. The proof is based on the exclusion of the possibility of the existence of eigenvalues for (3.19)–(3.20). In fact, (1) u_1 solves (3.19) and (3.20) for all wavenumber in an open interval. Since the Dirichlet eigenvalue of negative Laplacian in $\Omega_1\setminus(\Omega_1\cap\Omega_2)$ is discrete, there exists some κ_0 belonging to the interval but not being the Dirichlet eigenvalue. We have for this κ_0 that $u_1=0$ in $\Omega_1\setminus(\Omega_1\cap\Omega_2)$ from (3.19) and (3.20). The application of the unique continuation yields $u_1=0$ in D_1, which is a contradiction to non-zero boundary condition for u_1 on Γ; (2) denote by τ_n and λ_n the n-Dirichlet eigenvalue of negative Laplacian in $\Omega_1\setminus(\Omega_1\cap\Omega_2)$ and B_0, respectively, where the eigenvalues are arranged in ascending order with respect to the magnitude. It follows from the strong monotonicity property of negative Laplacian with Dirichlet boundary (see Colton and Kress [10]) that $\lambda_n<\tau_n$ for $n=0,1,\dots$ due to $\Omega_1\setminus(\Omega_1\cap\Omega_2)\subset B_0$. Therefore κ^2 is not the Dirichlet eigenvalue of negative Laplacian for $\kappa^2<\lambda_0\leq\lambda_n<\tau_n$ for $n=0,1,\dots$ So we also derive $u_1=0$ in $\Omega_1\setminus(\Omega_1\cap\Omega_2)$.

3.4. Domain derivative

The calculation of domain derivative, or more generally of the Fréchet derivative of the wave field with respect to the perturbation to the boundary of the medium, is an essential step for inverse scattering problems. We will investigate the domain derivative of the total field with respect to the perturbation of the cavity wall, which plays an important role in the stability analysis. The domain derivatives for the inverse obstacle scattering problem with different boundary conditions have been discussed by a number of authors, e.g, Hettlich [16], Kirsch [23], and Haddar and Kress [15]. See Liu [20] for the domain derivative for regular cavities based on a nonlocal transparent boundary condition from Fourier transform by Ammari *et al* [2,3].

Introduce a domain Ω_h bounded by S_h and Γ, where

$$S_h = \{\mathbf{x} + h\mathbf{p}(\mathbf{x}) : \mathbf{x} = (x_1, x_2) \in S\},$$

where the cavity wall S is assumed to be in C^2, the constant $h > 0$, and the function $\mathbf{p} = (p_1(\mathbf{x}), p_2(\mathbf{x})) \in C^2(S, \mathbb{R}^2)$ satisfying $\mathbf{p}(\mathbf{x}) = 0$ at the two end points of S. Obviously, if h is small enough then $S_h \in C^2$ is a perturbation of S.

According to a standard continuity argument for elliptic boundary value problems, there exists a unique solution $\mathbf{u}_h = [u_h, \phi_h]$ to (3.15)–(3.16) corresponding to the domain Ω_h for any small enough h. Define a nonlinear map

$$M_{\rm TE} : S_h \to \mathbf{u}_h|_\Gamma.$$

The domain derivative of the operator $M_{\rm TE}$ on the boundary S along the direction $\mathbf{p}$ is define by

$$M'_{\rm TE}(S, \mathbf{p}) := \lim_{h\to 0} \frac{\mathbf{u}_h|_\Gamma - \mathbf{u}|_\Gamma}{h}.$$

The weak formulation for $\mathbf{u}_h$ is

$$a^h_{\rm TE}(\mathbf{u}_h, \mathbf{v}_h) = \langle g, v_h\rangle + \langle f, \psi_h\rangle \quad \text{for all}\, \mathbf{v}_h \in H^1_{S_h}(\Omega) \times H^{-1/2}(\Gamma), \tag{3.21}$$

where $a^h_{\rm TE}$ is the bilinear form defined in (3.18) over Ω_h.

Motivated by the technique applied by Kirsch [23], we make change of variables to convert the integral in Ω_h into Ω. For $\mathbf{p} \in C^2(S, \mathbb{R}^2)$, we extend the definition of function $\mathbf{p}(\mathbf{x})$ to $\overline{\Omega}$ satisfying: $\mathbf{p}(\mathbf{x}) \in C^2(\Omega, \mathbb{R}^2) \cap C(\overline{\Omega})$; $\mathbf{p}(\mathbf{x}) = 0$ on Γ; $\mathbf{y} = \xi^h(\mathbf{x}) = \mathbf{x} + h\mathbf{p}(\mathbf{x})$ maps $\Omega \to \Omega_h$. In this way, ξ^h for h small enough is a diffeomorphism from Ω to Ω_h. Denote by $\eta^h(\mathbf{y}) = (\eta^h_1(\mathbf{y}), \eta^h_2(\mathbf{y})) : \Omega_h \to \Omega$ the inverse map of ξ^h.

For $\mathbf{y} \in \Omega_h$, make the change of variable $\mathbf{y} = \xi^h(\mathbf{x}) : \mathbf{x} \in \Omega$ and define $\breve{u}_h(\mathbf{x}) := u_h \circ \xi^h$. The integrals over Ω_h in (3.18) can be converted into ones over Ω:

$$\int_{\Omega_h} \nabla u_h \cdot \nabla \overline{v}_h - \int_{\Omega_h} \kappa^2 u_h \overline{v}_h = \int_\Omega \left[\nabla \breve{u}_h J_{\eta^h} J^\top_{\eta^h} \nabla \overline{\breve{v}}_h - \kappa^2 \breve{u}_h \overline{\breve{v}}_h\right] \det(J_{\xi^h}), \tag{3.22}$$

where $\breve{v}_h = v_h \circ \xi^h$, J_{η^h} and J_{ξ^h} are the Jacobian matrices of the transform η^h and ξ^h, respectively.

For an arbitrary test function v_h in domain Ω_h, the function $\breve{v}_h$ is a test function for domain Ω according to the transform. Therefore, the bilinear form a^h_{TE} in (3.21) can be rewritten as

$$a^h_{\mathrm{TE}}(\breve{\mathbf{u}}_h,\mathbf{v}) = \int_\Omega \left[\nabla\breve{u}_h J_{\eta^h} J^\top_{\eta^h}\nabla\overline{v} - \kappa^2\breve{u}_h\,\overline{v}\right]\det(J_{\xi^h}) + \langle D_{\mathrm{TE}}\breve{u}_h, v\rangle + \langle V_{\mathrm{TE}}\breve{\phi}_h,\psi\rangle$$
$$-\langle(\frac{1}{2}I - K'_{\mathrm{TE}})\breve{\phi}_h, v\rangle + \langle(\frac{1}{2}I - K_{\mathrm{TE}})\breve{u}_h,\psi\rangle, \qquad (3.23)$$

which leads to an equivalent variational formulation to (3.21):

$$a^h_{\mathrm{TE}}(\breve{\mathbf{u}}_h,\mathbf{v}) = \langle g, v\rangle + \langle f,\psi\rangle \quad \text{for all}\,\mathbf{v}\in H^1_S(\Omega)\times H^{-1/2}(\Gamma). \qquad (3.24)$$

Simple calculations yield

$$a_{\mathrm{TE}}(\breve{\mathbf{u}}_h-\mathbf{u},\mathbf{v}) = a_{\mathrm{TE}}(\breve{\mathbf{u}}_h,\mathbf{v}) - \langle g,v\rangle - \langle f,\psi\rangle = a_{\mathrm{TE}}(\breve{\mathbf{u}}_h,\mathbf{v}) - a^h_{\mathrm{TE}}(\breve{\mathbf{u}}_h,\mathbf{v})$$
$$= -\int_\Omega \nabla\breve{u}_h(J_{\eta^h}J^\top_{\eta^h}\det(J_{\xi^h}) - I)\nabla\overline{v} - \kappa^2(\det(J_{\xi^h})-1)\breve{u}_h\,\overline{v}. \qquad (3.25)$$

Define a matrix $J_{\mathbf{p}} = \left[\frac{\partial p_j}{\partial x_i}\right]_{i,j=1,2}$. Following from the definition of Jacobian matrix that

$$\det(J_{\xi^h}) = 1 + h\nabla\cdot\mathbf{p} + O(h^2) \qquad (3.26)$$

and $J_{\eta^h} = J^{-1}_{\xi^h}\circ\eta^h = I - hJ_{\mathbf{p}} + O(h^2)$, we can verify that

$$J_{\eta^h}J^\top_{\eta^h}\det(J_{\xi^h}) = I - h(J_{\mathbf{p}} + J^\top_{\mathbf{p}}) + h\nabla\cdot\mathbf{p}I + O(h^2). \qquad (3.27)$$

Substituting (3.26) and (3.27) into (3.25), and dividing by h yield

$$a_{\mathrm{TE}}\left(\frac{\breve{\mathbf{u}}_h-\mathbf{u}}{h},\mathbf{v}\right) = \int_\Omega \nabla\breve{u}_h[J_{\mathbf{p}} + J^\top_{\mathbf{p}} - \nabla\cdot\mathbf{p}I + O(h)]\nabla\overline{v} + \int_\Omega[\kappa^2\nabla\cdot\mathbf{p} + O(h)]\breve{u}_h\,\overline{v}. \quad (3.28)$$

Based on this variational form, we have the following result for the domain derivative.

Theorem 3.3. *Let $[u,\phi]$ be the solution of (3.15)–(3.16) in Ω and $\mathbf{n}$ the outward normal to S. Then the domain derivative can be expressed as $M'_{\mathrm{TE}}(S,\mathbf{p}) = [u'|_\Gamma,\phi']$, where $[u',\phi']\in H^1(\Omega)\times H^{-1/2}(\Gamma)$ is the weak solution of the following boundary value problem*

$$\Delta u' + \kappa^2 u' = 0 \quad in\,\Omega, \qquad (3.29)$$
$$u' = (\frac{1}{2}I + K_{\mathrm{TE}})u' - V_{\mathrm{TE}}\phi' \quad on\,\Gamma, \qquad (3.30)$$
$$\phi' = -D_{\mathrm{TE}}u' + (\frac{1}{2}I - K^*_{\mathrm{TE}})\phi' \quad on\,\Gamma, \qquad (3.31)$$
$$u' = -(\mathbf{p}\cdot\mathbf{n})\phi \quad on\,S, \qquad (3.32)$$

where $\phi' = \partial_{\mathbf{n}}u'$ on Γ.

Proof. Given $\mathbf{p}$, it follows from the well-posedness of variational formulation in domain Ω that $\breve{\mathbf{u}}_h \to \mathbf{u}$ in $H^1_S(\Omega)\times H^{-1/2}(\Gamma)$ as $h\to 0$. Taking $h\to 0$ in (3.28) gives

$$a_{\rm TE}\left(\lim_{h\to 0}\frac{\breve{\mathbf{u}}_h-\mathbf{u}}{h},\mathbf{v}\right)=\int_\Omega \nabla u(J_{\mathbf{p}}+J_{\mathbf{p}}^\top-\nabla\cdot\mathbf{p}I)\nabla\overline{v}+\kappa^2(\nabla\cdot\mathbf{p})u\,\overline{v}. \tag{3.33}$$

Therefore $(\breve{\mathbf{u}}_h-\mathbf{u})/h$ is convergent in $H^1_S(\Omega)\times H^{-1/2}(\Gamma)$ as $h\to 0$. Denote by $\mathbf{w}$ this limit and rewrite (3.33) as

$$a_{\rm TE}(\mathbf{w},\mathbf{v})=l(\mathbf{p})(u,v). \tag{3.34}$$

Next we compute $l(\mathbf{p})(u,v)$. Using the fact that $\mathbf{p}=0$ on Γ and the identity

$$\begin{aligned}\nabla u(J_{\mathbf{p}}+J_{\mathbf{p}}^\top-\nabla\cdot\mathbf{p}I)\nabla\overline{v}=\nabla\cdot[(\mathbf{p}\cdot\nabla u)\nabla\overline{v}+(\mathbf{p}\cdot\nabla\overline{v})\nabla u-(\nabla u\cdot\nabla\overline{v})\mathbf{p}]\\ -(\mathbf{p}\cdot\nabla u)\Delta\overline{v}-(\mathbf{p}\cdot\nabla\overline{v})\Delta u,\end{aligned}$$

we obtain from the divergence theorem that

$$\begin{aligned}l(\mathbf{p})(u,v)=&\int_\Omega \kappa^2(\nabla\cdot\mathbf{p})u\,\overline{v}-(\mathbf{p}\cdot\nabla u)\Delta\overline{v}-(\mathbf{p}\cdot\nabla\overline{v})\Delta u\\ &+\int_S[(\mathbf{p}\cdot\nabla u)\nabla\overline{v}+(\mathbf{p}\cdot\nabla\overline{v})\nabla u-(\nabla u\cdot\nabla\overline{v})\mathbf{p}]\cdot\mathbf{n},\end{aligned} \tag{3.35}$$

for any test function $v\in H^1_S(\Omega)\cap H^2(\Omega)$.

Noticing $\Delta u+\kappa^2u=0$ in Ω, we have from Green's formula that

$$l(\mathbf{p})(u,v)=\int_\Omega \kappa^2(\nabla\cdot\mathbf{p})u\overline{v}+\kappa^2(\mathbf{p}\cdot\nabla\overline{v})u+\nabla\overline{v}\cdot\nabla(\mathbf{p}\cdot\nabla u)+\int_S \mathbf{l}(\mathbf{p})(u)\cdot\nabla\overline{v},$$

where

$$\mathbf{l}(\mathbf{p})(u)=(\nabla u\cdot\mathbf{n})\mathbf{p}-(\mathbf{p}\cdot\mathbf{n})\nabla u. \tag{3.36}$$

Noticing $\mathbf{p}=0$ on Γ and $u=0$ on S and applying

$$(\nabla\cdot\mathbf{p})u\overline{v}+(\mathbf{p}\cdot\nabla\overline{v})u=\nabla\cdot(u\overline{v}\mathbf{p})-(\mathbf{p}\cdot\nabla u)\overline{v},$$

we have from the Gauss theorem that

$$l(\mathbf{p})(u,v)=\int_\Omega \nabla(\mathbf{p}\cdot\nabla u)\cdot\nabla\overline{v}-\kappa^2(\mathbf{p}\cdot\nabla u)\overline{v}+\int_S \mathbf{l}(\mathbf{p})(u)\cdot\nabla\overline{v}.$$

It follows from the definition of the bilinear form a and the fact $\mathbf{p}=0$ on Γ that $\mathbf{u}'=[u',\phi']:=\mathbf{w}-[\mathbf{p}\cdot\nabla u,0]$ satisfies

$$a_{\rm TE}(\mathbf{u}',\mathbf{v})=\int_S \mathbf{l}(\mathbf{p})(u)\cdot\nabla\overline{v}\quad\text{for all}\,\mathbf{v}\in H^1_S(\Omega)\times H^{-1/2}(\Gamma). \tag{3.37}$$

On the other hand, we have on Γ that

$$\mathbf{u}' = \mathbf{w} = \lim_{h\to 0} \frac{\breve{\mathbf{u}}_h - \mathbf{u}}{h} = \lim_{h\to 0} \frac{\mathbf{u}_h - \mathbf{u}}{h} = M'_{\mathrm{TE}}(S, \mathbf{p}). \tag{3.38}$$

Finally we consider the right-hand side of (3.37). Since $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$, it is enough to consider $u, v \in C^1(\overline{\Omega})$ in this limit procedure. The spatial gradient and the surface gradient has the relation for v and u:

$$\nabla v = \nabla_S v + (\partial_{\mathbf{n}} v)\mathbf{n} \quad \text{and} \quad \nabla u = \nabla_S u + (\partial_{\mathbf{n}} u)\mathbf{n},$$

where ∇_S is the surface gradient on curve S and $\mathbf{n}$ is its unit outward normal vector. As $h \to 0$, we get $\nabla_S u = \nabla_S v = 0$ on S since $v = u = 0$ on S. Therefore

$$\begin{aligned}\int_S \mathbf{l}(\mathbf{p})(u)\cdot\nabla v &= \int_S [(\nabla u\cdot\mathbf{n})\mathbf{p} - (\mathbf{p}\cdot\mathbf{n})\nabla u]\cdot\nabla v \\ &= \int_S [(\partial_{\mathbf{n}} u)\mathbf{p} - (\mathbf{p}\cdot\mathbf{n})(\partial_{\mathbf{n}} u)\mathbf{n}]\partial_{\mathbf{n}} v\mathbf{n} = 0.\end{aligned}$$

It follows from (3.37) that

$$\begin{aligned}&\int_\Omega \nabla u'\cdot\nabla\overline{v} - \int_\Omega \kappa^2 u'\overline{v} + \langle D_{\mathrm{TE}} u', v\rangle + \langle V_{\mathrm{TE}}\phi', \psi\rangle \\ &\quad -\langle(\frac{1}{2}I - K^*_{\mathrm{TE}})\phi', v\rangle + \langle(\frac{1}{2}I - K_{\mathrm{TE}})u', \psi\rangle = 0.\end{aligned} \tag{3.39}$$

for all $\mathbf{v} = [v, \psi] \in H^1_S(\Omega) \times H^{-1/2}(\Gamma)$, which is the weak formulation of the problem (3.29)-(3.31).

To verify the boundary condition of u' on S, we recall the definition of $\mathbf{u}'$ and have

$$u' = \lim_{h\to 0}\frac{\breve{u}_h - u}{h} - \mathbf{p}\cdot\nabla u = -\mathbf{p}\cdot\nabla u = -(\mathbf{p}\cdot\mathbf{n})\partial_{\mathbf{n}} u \quad \text{on } S, \tag{3.40}$$

since $\breve{u}_h - u = u_h - u = 0$ on S. The proof is complete by combining (3.39) and (3.40). □

3.5. Local stability of the inverse problem

In applications, it is impossible to make exact measurements. Stability is crucial in the practical reconstruction of cavity walls since it contains necessary information to determine to what extent the data can be trusted.

For any two domains D_1 and D_2 in $\mathbb{R}^2$, define $d(D_1, D_2)$ the Hausdorff distance between them by

$$d(D_1, D_2) = \max\{\rho(D_1, D_2), \rho(D_2, D_1)\}$$

where

$$\rho(D_m, D_n) = \sup_{\mathbf{x}\in D_m}\inf_{\mathbf{y}\in D_n} |\mathbf{x} - \mathbf{y}|.$$

Introduce domains Ω_h bounded by S_h and Γ, where

$$S_h : \mathbf{x} + hp(\mathbf{x})\mathbf{n},$$

where $p \in C^2(S, \mathbb{R})$. It is easily seen that the Hausdorff distance between Ω and Ω_h is of the order h, i.e., $d(\Omega_h, \Omega) = O(h)$.

We have the following local stability result.

Theorem 3.4. *If $p \in C^2(S, \mathbb{R})$ and $h > 0$ is sufficiently small, then*

$$d(\Omega_h, \Omega) \leq C \parallel u_h - u \parallel_{\widetilde{H}^{1/2}(\Gamma)} \tag{3.41}$$

where C is a positive constant independent of h.

Proof. We prove it by contradiction. Suppose now that the assertion is not true, for any given $p \in C^2(S, \mathbb{R})$, there exists a subsequence from $\{u_h\}$, which is still denoted as $\{u_h\}$ for simplicity, such that

$$\left\| \frac{u_h - u}{h} \right\|_{\widetilde{H}^{1/2}(\Gamma)} \to \parallel u' \parallel_{\widetilde{H}^{1/2}(\Gamma)} = 0 \quad \text{as } h \to 0, \tag{3.42}$$

which yields $u' = 0$ on Γ. Based on Theorem 3.3, it follows from the boundary condition of u' on Γ in (3.30) and the injectivity of the single-layer potential operator V_{TE} that $\phi' = 0$ on Γ. We infer by unique continuation that $u' = 0$ in Ω. The boundary condition of u' in (3.32) gives $(\mathbf{p} \cdot \mathbf{n})\phi = p\phi = 0$ on S. Since p is arbitrary, $\phi = 0$ on S. Recalling that $u = 0$ on S, we infer by unique continuation once again that $u = 0$ in Ω, which is a contradiction to (3.17). □

The result indicates that for small h, if the boundary measurements are $O(h)$ close to the wave field in the $\widetilde{H}^{1/2}$ norm, then Ω_h is $O(h)$ close to Ω in the Hausdorff distance.

4. TM polarization

In this section we study the direct and inverse problems for the TM polarization case. We will state some parallel results for the direct problem and show the differentiability of the field with respect to the cavity shape and prove a local stability result.

4.1. The direct problem

Consider the same problem geometry as that in TE polarization case, let an incoming plane wave $u^{\mathrm{i}} = \exp(i\alpha x_1 - i\beta x_2)$ be incident on the perfect electrically conducting surface $\Gamma_{\mathrm{g}} \cup S$ from above.

Denote the reference field u^{ref} as the solution of the homogeneous equation in the upper half space:

$$\Delta u^{\mathrm{ref}} + \kappa_0^2 u^{\mathrm{ref}} = 0 \quad \text{in } \mathbb{R}^2_+$$

together with the boundary condition

$$\partial_{\mathbf{n}} u^{\mathrm{ref}} = 0 \quad \text{on } \{x_2 = 0\}.$$

It can be shown that the reference field consists of the incident field u^{i} and the reflected field u^{r}:

$$u^{\mathrm{ref}} = u^{\mathrm{i}} + u^{\mathrm{r}},$$

where $u^{\mathrm{r}} = \exp(i\alpha x_1 + i\beta x_2)$.

The total field u is composed of the reference field u^{ref} and the scattered field u^{s}:

$$u = u^{\mathrm{ref}} + u^{\mathrm{s}}.$$

It is easy to verify that the scattered field satisfies

$$\begin{aligned} \nabla \cdot (\kappa^{-2} \nabla u^{\mathrm{s}}) + u^{\mathrm{s}} &= -\nabla \cdot [(\kappa^{-2} - \kappa_0^{-2}) \nabla u^{\mathrm{ref}}] \quad \text{above } \Gamma_{\mathrm{g}} \cup S, \\ \partial_{\mathbf{n}} u^{\mathrm{s}} &= -\partial_{\mathbf{n}} u^{\mathrm{ref}} \quad \text{on } \Gamma_{\mathrm{g}} \cup S. \end{aligned}$$

In addition, the scattered field is required to satisfy the radiation condition

$$\lim_{\rho \to \infty} \sqrt{\rho} \left(\frac{\partial u^{\mathrm{s}}}{\partial \rho} - i\kappa_0 u^{\mathrm{s}} \right) = 0, \quad \rho = |\mathbf{x}|.$$

Regarding Eqs. (2.10) and (2.11) for the total field, the direct problem has an equivalent variational form: find $u \in H^1(\Omega)$ such that

$$(\kappa^{-2} \nabla u, \nabla v) - (u, v) - \kappa_0^{-2} \langle \phi, v \rangle = 0 \quad \text{for all } v \in H^1(\Omega), \tag{4.1}$$

where $\phi = \partial_{\mathbf{n}} u$ is normal derivative of the magnetic field on the boundary Γ.

Following a similar procedure, we may also derive a boundary integral equation for the total field on Γ:

$$u(\mathbf{x}) = \frac{1}{2} u(\mathbf{x}) + \int_\Gamma \partial_{\mathbf{n}_{\mathbf{y}}} G_{\mathrm{TM}}(\mathbf{x}, \mathbf{y}) u(\mathbf{y}) ds_{\mathbf{y}} - \int_\Gamma G_{\mathrm{TM}}(\mathbf{x}, \mathbf{y}) \phi(\mathbf{y}) ds_{\mathbf{y}} + u^{\mathrm{ref}}(\mathbf{x}), \tag{4.2}$$

$$\phi(\mathbf{x}) = \frac{1}{2} \phi(\mathbf{x}) + \int_\Gamma \partial_{\mathbf{n}_{\mathbf{x}}} \partial_{\mathbf{n}_{\mathbf{y}}} G_{\mathrm{TM}}(\mathbf{x}, \mathbf{y}) u(\mathbf{y}) ds_{\mathbf{y}} - \int_\Gamma \partial_{\mathbf{n}_{\mathbf{x}}} G_{\mathrm{TM}}(\mathbf{x}, \mathbf{y}) \phi(\mathbf{y}) ds_{\mathbf{y}} + \partial_{\mathbf{n}} u^{\mathrm{ref}}(\mathbf{x}), \tag{4.3}$$

where

$$G_{\mathrm{TM}}(\mathbf{x}, \mathbf{y}) = \frac{i}{4} H_0^{(1)}(\kappa_0 |\mathbf{x} - \mathbf{y}|) + \frac{i}{4} H_0^{(1)}(\kappa_0 |\mathbf{x} - \mathbf{y}'|).$$

To study the boundary integral equations, we similarly introduce the single-layer potential operator V_{TM}, the hypersingular integral operator D_{TM}, the double-layer potential operator $K_{\rm TM}$ and its adjoint operator $K^*_{\rm TM}$, which are defined as

$$(V_{\rm TM}u)(\mathbf{x}) = \int_\Gamma G_{\rm TM}(\mathbf{x},\mathbf{y})u(\mathbf{y})ds_{\mathbf{y}},$$

$$(D_{\rm TM}u)(\mathbf{x}) = -\int_\Gamma \partial_{\mathbf{n}_{\rm x}}\partial_{\mathbf{n}_{\rm y}} G_{\rm TM}(\mathbf{x},\mathbf{y})u(\mathbf{y})ds_{\mathbf{y}},$$

$$(K_{\rm TM}u)(\mathbf{x}) = \int_\Gamma \partial_{\mathbf{n}_{\rm y}} G_{\rm TM}(\mathbf{x},\mathbf{y})u(\mathbf{y})ds_{\mathbf{y}},$$

$$(K^*_{\rm TM}u)(\mathbf{x}) = \int_\Gamma \partial_{\mathbf{n}_{\rm x}} G_{\rm TM}(\mathbf{x},\mathbf{y})u(\mathbf{y})ds_{\mathbf{y}}.$$

Similarly, the proofs of the following lemma can be found in [25].

Lemma 4.1. *The single-layer potential operators $V_{\rm TM}$ is compact from $\widetilde{H}^{-1/2}(\Gamma)$ into $H^{1/2}(\Gamma)$, the double-layer potential operator $K_{\rm TM}$ and its adjoint $K^*_{\rm TM}$ are compact from $H^{1/2}(\Gamma)$ into $H^{1/2}(\Gamma)$ and from $\widetilde{H}^{-1/2}(\Gamma)$ into $\widetilde{H}^{-1/2}(\Gamma)$, respectively, and the hypersingular integral operator $D_{\rm TM}$ is compact from $H^{1/2}(\Gamma)$ into $\widetilde{H}^{-1/2}(\Gamma)$. Furthermore, the sinle-layer potential operator $V_{\rm TM}$ the and hypersingular integral operator $D_{\rm TM}$ are coercive in $\widetilde{H}^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, i.e., there exit compact operators V_1 and D_1 such that*

$$\mathrm{Re}[\langle V_{\rm TM}\phi,\phi\rangle + \langle V_1\phi,\phi\rangle] \ge C \parallel \phi \parallel_{\widetilde{H}^{-1/2}(\Gamma)} \quad \textit{for all}\, \phi \in \widetilde{H}^{-1/2}(\Gamma),$$
$$\mathrm{Re}[\langle D_{\rm TM}u,u\rangle + \langle D_1 u,u\rangle] \ge C \parallel u \parallel_{H^{1/2}(\Gamma)} \quad \textit{for all}\, u \in H^{1/2}(\Gamma).$$

Using these operators, boundary integral equations can be written as

$$u = (\frac{1}{2}I + K_{\rm TM})u - V_{\rm TM}\phi + f, \tag{4.4}$$

$$\phi = -D_{\rm TM}u + (\frac{1}{2}I - K^*_{\rm TM})\phi + g, \tag{4.5}$$

where $f = u^{\rm ref}$, $g = \partial_{\mathbf{n}}u^{\rm ref}$, and I is the identity operator.

Substituting (4.5) into (4.1) and multiplying a test function on (4.4), we arrive the variational problem for the symmetric coupling of finite element and boundary integral method

$$\int_\Omega \kappa^{-2}\nabla u\cdot\nabla\overline{v} - \int_\Omega u\,\overline{v} + \kappa_0^{-2}\langle D_{\rm TM}u,v\rangle - \langle(\frac{1}{2}I - K^*_{\rm TM})\phi, v\rangle$$
$$= \langle g, v\rangle \quad \text{for all}\, v\in H^1(\Omega), \tag{4.6}$$

$$\langle(\frac{1}{2}I - K_{\rm TM})u,\psi\rangle + \langle V_{\rm TM}\phi,\psi\rangle = \langle f,\psi\rangle \quad \text{for all}\, \psi \in \widetilde{H}^{-1/2}(\Gamma). \tag{4.7}$$

Introducing a bilinear form

$$a_{\rm TM}(\mathbf{u},\mathbf{v}) = \int_\Omega \kappa^{-2}\nabla u\cdot\nabla\overline{v} - \int_\Omega u\,\overline{v} + \kappa_0^{-2}\langle D_{\rm TM}u,v\rangle + \langle V_{\rm TM}\phi,\psi\rangle$$
$$-\langle(\frac{1}{2}I - K^*_{\rm TM})\phi, v\rangle + \langle(\frac{1}{2}I - K_{\rm TM})u,\psi\rangle, \tag{4.8}$$

we have the variational formulation for the coupling of the finite element and boundary integral methods for the direct scattering problem: find $\mathbf{u} = [u, \phi] \in \mathscr{V}_{\mathrm{TM}} = H^1(\Omega) \times \widetilde{H}^{-1/2}(\Gamma)$ such that

$$a_{\mathrm{TM}}(\mathbf{u}, \mathbf{v}) = \langle g, v\rangle + \langle f, \psi\rangle \quad \text{for all}\, \mathbf{v} \in \mathscr{V}_{\mathrm{TM}}. \tag{4.9}$$

The following theorem is an analogue of Theorem 3.1.

Theorem 4.1. *The variational problem (4.9) admits a unique solution* $[u, \varphi] \in \mathscr{V}_{\mathrm{TM}}$.

4.2. The inverse problem

In this section, we investigate the uniqueness, domain derivative, and the local stability. We will briefly prove the uniqueness but elaborate the domain derivative and the local stability since the arguments vary quite a lot from those for the TE case.

Theorem 4.2. *Let* $[u_j, \phi_j]$ *be the solution of (4.9) in* Ω_j *with* $\partial\Omega_j = \Gamma \cup S_j$ *for* $j = 1, 2$. *If* $u_1 = u_2$ *on* Γ, *then* $S_1 = S_2$.

Proof. Following essentially the same arguments for TE case, we consider

$$\nabla \cdot (\kappa^{-2} \nabla u_1) + u_1 = 0 \quad \text{in}\, \Omega_1 \setminus (\Omega_1 \cap \Omega_2), \tag{4.10}$$

$$\partial_{\mathbf{n}} u_1 = 0 \quad \text{on}\, C_1 \cup C_2. \tag{4.11}$$

Recalling the expression of wavenumber, we have from (4.10)–(4.11) that

$$\int_{\Omega_1 \setminus (\Omega_1 \cap \Omega_2)} \kappa^{-2} |\nabla u_1|^2 - \int_{\Omega_1 \setminus (\Omega_1 \cap \Omega_2)} |u_1|^2 = 0,$$

which yields $\nabla u_1 = 0$ in $\Omega_1 \setminus (\Omega_1 \cap \Omega_2)$. The proof is complete by noticing $u_1 = 0$ from (4.10). $\square$

The above uniqueness theorem again relies on the fact that the wavenumber has nonzero imaginary part. When the medium is not lossy, the uniqueness result is not obvious and needs further investigation since there is no monotonicity property for the eigenvalues as in the TE case.

Introduce the domain Ω_h bounded by S_h and Γ, where

$$S_h = \{\mathbf{x} + h\mathbf{p}(\mathbf{x}) : \mathbf{x} \in S\},$$

where the cavity wall S is assumed to be in C^2, the constant $h > 0$, and the function $\mathbf{p} \in C^2(S, \mathbb{R}^2)$ satisfying $\mathbf{p}(\mathbf{x}) = 0$ at the two end points of S.

Define a nonlinear map

$$M_{\mathrm{TM}} : S_h \to \mathbf{u}_h|_{\Gamma}.$$

The domain derivative of the operator M_{TM} on the boundary S along the direction $\mathbf{p}$ is define by

$$M'_{\mathrm{TM}}(S, \mathbf{p}) := \lim_{h \to 0} \frac{\mathbf{u}_h|_{\Gamma} - \mathbf{u}|_{\Gamma}}{h}.$$

The weak formulation for $\mathbf{u}_h$ is

$$a^h_{\rm TM}(\mathbf{u}_h,\mathbf{v}_h)=\langle g,v_h\rangle+\langle f,\psi_h\rangle\quad\text{for all}\,\mathbf{v}_h\in\mathscr{V}_{\rm TM},\tag{4.12}$$

where $a^h_{\rm TM}$ is the bilinear form defined in (4.8) over Ω_h.

Using the change of variable and identities (3.24)–(3.25), we may arrive

$$\begin{aligned}a_{\rm TM}\left(\frac{\breve{\mathbf{u}}_h-\mathbf{u}}{h},\mathbf{v}\right)=&\int_\Omega\kappa^{-2}\nabla\breve{u}_h[J_{\mathbf{p}}+J_{\mathbf{p}}^\top-\nabla\cdot\mathbf{p}I+O(h)]\nabla\overline{v}\\&+\int_\Omega[\nabla\cdot\mathbf{p}+O(h)]\breve{u}_h\overline{v}.\end{aligned}\tag{4.13}$$

Based on this variational form, we have the following result for the domain derivative.

Theorem 4.3. *Let $[u,\phi]$ be the solution of (4.9) in Ω and $\mathbf{n}$ the outward normal to S. Then the domain derivative can be expressed as $M'_{\rm TM}(S,\mathbf{p})=[u'|_\Gamma,\phi']$, where $=[u',\phi']\in\mathscr{V}_{\rm TM}$ is the weak solution of the following boundary value problem*

$$\nabla\cdot(\kappa^{-2}\nabla u')+u'=0\quad in\,\Omega,\tag{4.14}$$

$$u'=(\frac{1}{2}I+K_{\rm TM})u'-V_{\rm TM}\phi'\quad on\,\Gamma,\tag{4.15}$$

$$\phi'=-D_{\rm TM}u'+(\frac{1}{2}I-K^*_{\rm TM})\phi'\quad on\,\Gamma,\tag{4.16}$$

$$\kappa^{-2}\phi'=\nabla_S\cdot[\kappa^{-2}(\mathbf{p}\cdot\mathbf{n})\nabla_S u]+(\mathbf{p}\cdot\mathbf{n})u\quad on\,S,\tag{4.17}$$

where $\phi'=\partial_{\mathbf{n}}u'$ on Γ.

Proof. Taking $h\to0$ in (4.13) yields

$$a_{\rm TM}\left(\lim_{h\to0}\frac{\breve{\mathbf{u}}_h-\mathbf{u}}{h},\mathbf{v}\right)=\int_\Omega\kappa^{-2}\nabla u(J_{\mathbf{p}}+J_{\mathbf{p}}^\top-\nabla\cdot\mathbf{p}I)\nabla\overline{v}+(\nabla\cdot\mathbf{p})u\overline{v}.\tag{4.18}$$

Therefore $(\breve{\mathbf{u}}_h-\mathbf{u})/h$ is convergent in $\mathscr{V}_{\rm TM}$ as $h\to0$. Denote by $\mathbf{w}$ this limit and rewrite (4.18) as

$$a_{\rm TM}(\mathbf{w},\mathbf{v})=l(\mathbf{p})(u,v).\tag{4.19}$$

Next we compute $l(\mathbf{p})(u,v)$. For test function $v\in H^1(\Omega)\cap H^2(\Omega)$, we obtain from $\mathbf{p}=0$ on Γ that

$$\begin{aligned}l(\mathbf{p})(u,v)=&\int_\Omega(\nabla\cdot\mathbf{p})u\overline{v}-(\mathbf{p}\cdot\nabla u)\nabla\cdot(\kappa^{-2}\nabla\overline{v})-(\mathbf{p}\cdot\nabla\overline{v})\nabla\cdot(\kappa^{-2}\nabla u)\\&+\int_S\kappa^{-2}[(\mathbf{p}\cdot\nabla u)\nabla\overline{v}+(\mathbf{p}\cdot\nabla\overline{v})\nabla u-(\nabla u\cdot\nabla\overline{v})\mathbf{p}]\cdot\mathbf{n}\\&+\int_\Omega(\nabla u\cdot\nabla\overline{v})\mathbf{p}\cdot\nabla\kappa^{-2}\end{aligned}\tag{4.20}$$

Here we applied the divergence theorem and the identity

$$\kappa^{-2}\nabla u(J_{\mathbf{p}} + J_{\mathbf{p}}^\top - \nabla\cdot\mathbf{p}I)\nabla\overline{v} = \nabla\cdot[\kappa^{-2}((\mathbf{p}\cdot\nabla u)\nabla\overline{v} + (\mathbf{p}\cdot\nabla\overline{v})\nabla u - (\nabla u\cdot\nabla\overline{v})\mathbf{p})]$$
$$+(\nabla u\cdot\nabla\overline{v})\mathbf{p}\cdot\nabla\kappa^{-2} - (\mathbf{p}\cdot\nabla u)\nabla\cdot(\kappa^{-2}\nabla\overline{v}) - (\mathbf{p}\cdot\nabla\overline{v})\nabla\cdot(\kappa^{-2}\nabla u),$$

By applying $\nabla\cdot(\kappa^{-2}\nabla u) + u = 0$ in Ω, $\partial_{\mathbf{n}}u = 0$ on S, and Green's formula for the test function v, we have

$$l(\mathbf{p})(u,v) = \int_\Omega (\nabla\cdot\mathbf{p})u\overline{v} + (\mathbf{p}\cdot\nabla\overline{v})u + \kappa^{-2}\nabla\overline{v}\cdot\nabla(\mathbf{p}\cdot\nabla u)$$
$$+\int_\Omega (\nabla u\cdot\nabla\overline{v})\mathbf{p}\cdot\nabla\kappa^{-2} - \int_S \kappa^{-2}(\nabla u\cdot\nabla\overline{v})\mathbf{p}\cdot\mathbf{n}.$$

Applying

$$(\nabla\cdot\mathbf{p})u\overline{v} + (\mathbf{p}\cdot\nabla\overline{v})u = \nabla\cdot(u\overline{v}\mathbf{p}) - (\mathbf{p}\cdot\nabla u)\overline{v}$$

and the divergence theorem yields

$$l(\mathbf{p})(u,v) = \int_\Omega \kappa^{-2}\nabla(\mathbf{p}\cdot\nabla u)\cdot\nabla\overline{v} - (\mathbf{p}\cdot\nabla u)\overline{v} + \int_\Omega (\nabla u\cdot\nabla\overline{v})\mathbf{p}\cdot\nabla\kappa^{-2}$$
$$+\int_S u\overline{v}\mathbf{p}\cdot\mathbf{n} - \int_S \kappa^{-2}(\nabla u\cdot\nabla\overline{v})\mathbf{p}\cdot\mathbf{n},$$

which leads to

$$l(\mathbf{p})(u,v) = \int_\Omega \nabla(\kappa^{-2}(\mathbf{p}\cdot\nabla u)\cdot\nabla\overline{v} - (\mathbf{p}\cdot\nabla u)\overline{v} - \int_S \kappa^{-2}(\nabla u\cdot\nabla\overline{v})\mathbf{p}\cdot\mathbf{n} + \int_S u\overline{v}\mathbf{p}\cdot\mathbf{n}.$$

It follows from Eq. (4.19) and the definition of the bilinear form a_{TM} that $\mathbf{u}' := \mathbf{w} - [\mathbf{p}\cdot\nabla u, 0]$ satisfies

$$a_{\mathrm{TM}}(\mathbf{u}',\mathbf{v}) = -\int_S \kappa^{-2}(\nabla u\cdot\nabla\overline{v})\mathbf{p}\cdot\mathbf{n} + \int_S uv\mathbf{p}\cdot\mathbf{n} \quad \text{for all}\,\mathbf{v}\in\mathscr{V}_{\mathrm{TM}}. \tag{4.21}$$

On the other hand, we have on Γ that

$$\mathbf{w} = \lim_{h\to 0}\frac{\breve{\mathbf{u}}_h - \mathbf{u}}{h} = \lim_{h\to 0}\frac{\mathbf{u}_h - \mathbf{u}}{h} = M'_{\mathrm{TM}}(S,\mathbf{p}). \tag{4.22}$$

Finally we consider the right-hand side of (4.21). The spatial gradient and surface gradient has the relation

$$\nabla u = \nabla_S u + (\partial_{\mathbf{n}}u)\mathbf{n} \quad \text{and} \quad \nabla v = \nabla_S v + (\partial_{\mathbf{n}}v)\mathbf{n}$$

Since $\partial_{\mathbf{n}}u = 0$ on S, we have from the surface divergence theorem that

$$a_{\mathrm{TM}}(\mathbf{u}',\mathbf{v}) = \int_S [\nabla_S\cdot(\kappa^{-2}[(\mathbf{p}\cdot\mathbf{n})\nabla_S u)] + (\mathbf{p}\cdot\mathbf{n})u]\overline{v} \quad \text{for all}\,\mathbf{v}\in\mathscr{V}_{\mathrm{TM}},$$

which completes the proof. $\square$

Now we consider the local stability. Introduce domains Ω_h bounded by S_h and Γ, where

$$S_h : \mathbf{x} + hp(\mathbf{x})\mathbf{n},$$

where $p \in C^2(S, \mathbb{R})$.

Theorem 4.4. *If $p \in C^2(S, \mathbb{R})$ and $h > 0$ is sufficiently small, then*

$$d(\Omega_h, \Omega) \le C \parallel u_h - u \parallel_{H^{1/2}(\Gamma)} \tag{4.23}$$

where C is a constant independent of h.

Proof. We prove it by contradiction. Suppose now that the assertion is not true, we have for any given $p \in C^2(S, \mathbb{R})$ that

$$\left\| \frac{u_h - u}{h} \right\|_{H^{1/2}(\Gamma)} \to \parallel u' \parallel_{H^{1/2}(\Gamma)} = 0 \quad \text{as}\, h \to 0, \tag{4.24}$$

which yields $u' = 0$ on Γ. It follows from (4.15) and the injectivity of the single-layer potential operator V_{TM} that $\phi' = 0$ on Γ. We infer by unique continuation that $u' = 0$ in Ω and thus $\phi' = 0$ on S. We have by (4.17)

$$\nabla_S \cdot (\kappa^{-2} p \nabla_S u) + pu = 0 \quad \text{on}\, S.$$

Multiplying the complex conjugate of u and integrating over S yield

$$\int_S p(\kappa^{-2}|\nabla_S u|^2 - |u|^2) = 0.$$

Since p is arbitrary, we have from the definition of the wavenumber that

$$\kappa^{-2}|\nabla_S u|^2 - |u|^2 = |\nabla_S u|^2 - (\omega^2 \mu_0 \varepsilon + i\omega\mu_0\sigma)|u|^2 = 0 \quad \text{on}\, S.$$

It follows that $u = 0$ on S. Recalling that also $\partial_{\mathbf{n}} u = 0$ on S, we infer by the Holmgren uniqueness result and unique continuation that $u = 0$ in Ω, which is a contradiction to (4.4) since f is a nonzero function. $\square$

5. Concluding remarks

A method of symmetric coupling of the finite element and boundary integral methods is developed for solving the electromagnetic scattering of a cavity embedded in perfect electrically conductor ground plane for both the transverse electrical and magnetic polarization cases. The method is to enclose the inhomogeneous sample with a fictitious surface to separate the finite element region from the exterior region where the boundary integral equation applies. The field inside the surface is formulated using the finite element method, whereas those exterior to the surface can be expressed in terms of surface integrals. The interior and exterior are finally coupled via the field continuity conditions, leading to a complete system for the solution of interior and surface fields. Variational formulations for the direct scattering problems are presented, existence and uniqueness of weak solutions are studied, and the domain derivatives of the field with respect to the cavity shape are derived. Computationally, the variational approaches give an efficient method to solve the direct scattering problem for regular and overfilled cavities. We have established uniqueness and local stability results in terms of the inverse problem, which indicates that the cavity is uniquely determined by boundary measurement of the total field and that for small h, if the boundary measurements are $O(h)$ close to the wave field either in the $\widetilde{H}^{1/2}$ norm for the TE case or in the $H^{1/2}$ norm for the TM case, then Ω_h is $O(h)$ close to Ω in the Hausdorff distance.

We are currently extending the method and the techniques developed in this paper to more complicated three-dimensional Maxwell's equations. The results will be reported elsewhere.

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