J. Differential Equations 252 (2012) 3209–3225

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

# An inverse cavity problem for Maxwell's equations

Peijun Li [1]

*Department of Mathematics, Purdue University, West Lafayette, IN 47907, United States*

ARTICLE INFO

*Article history:*
Received 30 January 2011
Revised 27 October 2011
Available online 14 November 2011

*MSC:*
78A46
78M30

*Keywords:*
Maxwell's equations
Inverse problem
Cavity scattering
Domain derivative
Uniqueness
Local stability

ABSTRACT

Consider the scattering of a time-harmonic electromagnetic plane wave by an open cavity embedded in a perfect electrically conducting infinite ground plane, where the electromagnetic wave propagation is governed by the Maxwell equations. The upper half-space is filled with a lossless homogeneous medium above the flat ground surface; while the interior of the cavity is assumed to be filled with a lossy homogeneous medium accounting for the energy absorption. The inverse problem is to determine the cavity structure or the shape of the cavity from the tangential trace of the electric field measured on the aperture of the cavity. In this paper, results on a global uniqueness and a local stability are established for the inverse problem. A crucial step in the proof of the stability is to obtain the existence and characterization of the domain derivative of the electric field with respect to the shape of the cavity.

Published by Elsevier Inc.

## 1. Introduction

Consider a time-harmonic electromagnetic plane wave incident from the top on an arbitrarily shaped open cavity embedded in an infinite ground plane. The profile of the cavity wall is assumed to be sufficiently smooth, for example twice continuously differentiable. The ground plane and the cavity wall are perfect electric conductors, i.e., the tangential trace of the total electric field vanishes on these two surfaces. The upper half-space is filled with a homogeneous lossless medium with a fixed positive wavenumber; while the interior of the cavity is filled with another homogeneous lossy medium characterized by a fixed wavenumber with positive imaginary part accounting for the energy absorption. Given the structure or the shape of the cavity and a time-harmonic electromagnetic plane wave incident on the cavity, the direct scattering problem is to predict the field distributions away from

*E-mail address:* lipeijun@math.purdue.edu.

[1] The research was supported in part by the NSF grants DMS-0914595 and DMS-1042958.

0022-0396/$ – see front matter Published by Elsevier Inc.
doi:10.1016/j.jde.2011.10.023

the structure. We are interested in studying the inverse problem: what information can we extract about the structure or the shape of the cavity from the tangential trace of the electric field measured on the aperture of the cavity? The first result in this paper is a global uniqueness theorem for the inverse problem. The theorem indicates that any two cavity shapes are identical if they generate the same tangential trace of the electric fields on the aperture of the cavity. The proof is based on a combination of the Holmgren uniqueness and unique continuation. The second result is concerned with a local stability for the inverse problem: if $S_1$ and $S_2$ are "close" cavity walls to each other, then for any $\delta > 0$, the measurements of the two tangential trace of the electric fields being $\delta$-close implies that the two cavity walls are $O(\delta)$-close. A crucial step in the stability proof is to obtain the existence and characterization of the domain derivative of the electric field with respect to the shape of the cavity.

There are at least two major applications in terms of the direct and inverse cavity scattering problems: (1) The radar cross section is a measure of the detectability of a target by a radar system. Deliberate control in the form of enhancement or reduction of the radar cross section of a target is of no less importance than many radar applications. The cavity radar cross section caused by jet engine inlet ducts or cavity-backed antennas can dominate the total radar cross section. A thorough understanding of the electromagnetic scattering characteristic of a target, particularly a cavity, is necessary for successful implementation of any desired control of its radar cross section, and is of high interest to the scientific and engineering community; (2) The cavity can be used to model cracks or holes in metallic surfaces such as aircraft wings. These cracks or holes would be invisible to a visual inspection but may be revealed by understanding the scattering characteristics of the cavity. As an inverse problem, the mathematical model can serve as a predictor of the scattering of electromagnetic waves by the cavity for use in non-destructive testing. Besides, this work is also motivated by the study of the optimal design problems of the cavity, where one wishes to design a cavity structure that reduces or enhances the radar cross section.

The direct cavity scattering problem has been examined by numerous researchers from both numerical and mathematical viewpoints, such as Jin and Volakis [24], Liu and Jin [26], Van and Wood [31], Wood [32], Wood and Wood [33], Bao and Sun [9], Ammari et al. [3–5], and references cited therein. A good introduction to the problem of cavity scattering, along with some numerical methods, can be found in Jin [23]. One may consult Monk [28] and Nédélec [29] for recent accounts of finite element methods and integral equation methods for general direct electromagnetic scattering problems.

The inverse cavity scattering problem has recently received considerable attention. We refer to Liu [21], Feng and Ma [17], Bao et al. [8] for results on uniqueness and local stability of the two-dimensional Helmholtz equation. Although the scalar model problem has been extensively investigated, little is known for the electromagnetic case. This paper is a nontrivial extension of the work in [8] and considers the vector form of Maxwell's equations. The proofs are motivated by the techniques in Bao and Zhou [11] for the biperiodic diffraction grating problems, where a main ingredient was to estimate a quotient difference function due to the perturbation of the grating profile in the proof of local stability. Noticing that the quotient difference function is an approximation to the domain derivative, we directly investigate the domain derivative and provide the proof of the local stability. The results on uniqueness and stability of the closely related inverse grating problem may be found in Ammari [2], Bao [6], Bao and Friedman [7], and Bao et al. [10]. We refer to Elschner and Hu [15], Liu et al. [20], Liu et al. [22], and reference therein, for related work on the uniqueness of the inverse electromagnetic obstacle scattering problems, which is to determine a closed obstacle surface. A complete account of the general theory of inverse scattering problems may be found in Colton and Kress [13].

The outline of the paper is as follows. A model problem and governing equations of the electromagnetic cavity scattering problems are introduced in Section 2. In Section 3, the variational formulation for the direct problem is introduced, and some auxiliary results are also presented for the transparent boundary operator. Section 4 is devoted to the study of the inverse problem: a global uniqueness theorem is proved; the existence and characterization of the domain derivative are examined; as a consequence of the domain derivative, a local stability result is established. The paper is concluded with some general remarks and directions for future research in Section 5.

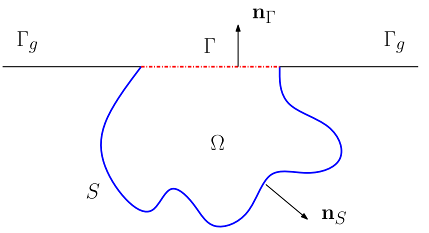


**Fig. 1.** The problem geometry. An open cavity with the wall $S$ is placed on a perfectly conducting ground plane $\Gamma_g$. The aperture $\Gamma$ covers the cavity and divides the physical domain into the interior $\Omega$, enclosed by $S$ and $\Gamma$, and the upper half-space above the flat surface $\Gamma_g \cup \Gamma$.

## 2. Maxwell's equations

We shall introduce a mathematical model problem and define some notation for the direct and inverse electromagnetic scattering problems by an open cavity. Let the structure or the shape of the cavity be described by the surface $S$, known as the cavity wall, which is assumed to be sufficiently smooth. The cavity is embedded in a perfect electrically conducting infinite ground plane $\Gamma_g$. Denote by $\Gamma$ the aperture of the cavity. So the cavity, represented by the finite domain $\Omega$, is enclosed by the wall $S$ and the aperture $\Gamma$, as seen in Fig. 1.

The electromagnetic wave propagation is governed by the time-harmonic Maxwell equations (time dependence $e^{-\mathrm{i}\omega t}$):

$$\nabla \times \mathbf{E} = \mathrm{i}\omega \mathbf{B}, \qquad \nabla \times \mathbf{H} = -\mathrm{i}\omega \mathbf{D} + \mathbf{J}, \tag{2.1}$$

where $\mathbf{E}$ is the electric field, $\mathbf{H}$ is the magnetic field, $\mathbf{B}$ is the magnetic flux density, $\mathbf{D}$ is the electric flux density, $\mathbf{J}$ is the electric current density, and $\omega$ is the angular frequency. The constitutive relations, describing the macroscopic properties of the medium, are taken as

$$\mathbf{B} = \mu \mathbf{H}, \qquad \mathbf{D} = \varepsilon \mathbf{E}, \quad \text{and} \quad \mathbf{J} = \sigma \mathbf{E}, \tag{2.2}$$

where the constitutive parameters $\mu$ and $\varepsilon$ denoted as the magnetic permeability and the electric permittivity, which are assumed to be positive constants everywhere in the physical domain, i.e., $\mu = \mu_0 > 0$ and $\varepsilon = \varepsilon_0 > 0$, and the constitutive parameter $\sigma$ is the conductivity of the medium, which is assumed to be a piecewise constant: $\sigma = 0$ above the flat surface $\Gamma_g \cup \Gamma$ and $\sigma = \sigma_0 > 0$ in the interior of the cavity accounting for the energy absorption. Substituting the constitutive relations (2.2) into (2.1) gives a coupled system for the electric and magnetic fields:

$$\nabla \times \mathbf{E} = \mathrm{i}\omega \mu_0 \mathbf{H}, \qquad \nabla \times \mathbf{H} = (-\mathrm{i}\omega \varepsilon_0 + \sigma)\mathbf{E}. \tag{2.3}$$

Eliminating the magnetic field from (2.3), we obtain a decoupled equation for the electric field:

$$\nabla \times (\nabla \times \mathbf{E}) - \kappa^2 \mathbf{E} = 0, \tag{2.4}$$

where $\kappa^2 = \omega^2 \mu_0 \varepsilon_0 + \mathrm{i}\omega \mu_0 \sigma$ and $\kappa$ is called the wavenumber. Due to the perfectly conducting material, the following homogeneous Dirichlet boundary condition is satisfied for the tangential trace of the electric field:

$$\mathbf{n} \times \mathbf{E} = 0 \quad \text{on } \Gamma_g \cup S, \tag{2.5}$$

where $\mathbf{n}$ is the unit outward normal vector.

Denote $\kappa_0 = \omega\sqrt{\mu_0\varepsilon_0}$. Let $(\mathbf{E}^{\mathrm{inc}}, \mathbf{H}^{\mathrm{inc}})$ be the incoming plane wave that are incident upon the cavity from the above, where

$$\mathbf{E}^{\mathrm{inc}} = \mathbf{t}e^{\mathrm{i}\kappa_0\mathbf{q}\cdot\mathbf{x}}, \qquad \mathbf{H}^{\mathrm{inc}} = \mathbf{s}e^{\mathrm{i}\kappa_0\mathbf{q}\cdot\mathbf{x}}, \qquad \mathbf{s} = \frac{\mathbf{q}\times\mathbf{t}}{\omega_0\mu_0}, \qquad \mathbf{t}\cdot\mathbf{q} = 0.$$

Here $\mathbf{q} = (\alpha_1, \alpha_2, -\beta)^\top = (\sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2, -\cos\theta_1)^\top$, and $\theta_1, \theta_2$ are incident angles satisfying $0 \leqslant \theta_1 < \pi/2$, $0 \leqslant \theta_2 < 2\pi$. Let $\mathbf{q}^* = (\alpha_1, \alpha_2, \beta)^\top$ and then the total electric and magnetic fields can be decomposed as follows:

$$\mathbf{E} = \mathbf{E}^{\mathrm{inc}} - \mathbf{t}e^{\mathrm{i}\kappa_0\mathbf{q}^*\cdot\mathbf{x}} + \mathbf{E}^{\mathrm{s}} \quad \text{and} \quad \mathbf{H} = \mathbf{H}^{\mathrm{inc}} - z_0\mathbf{t}\times\mathbf{q}^* e^{\mathrm{i}\kappa_0\mathbf{q}^*\cdot\mathbf{x}} + \mathbf{H}^{\mathrm{s}}, \tag{2.6}$$

where $z_0 = \sqrt{\mu_0/\varepsilon_0}$, and $\mathbf{E}^{\mathrm{s}}, \mathbf{H}^{\mathrm{s}}$ are the scattered electric and magnetic fields, respectively. In addition, the scattered fields are required to satisfy the Silver–Müller radiation condition:

$$\lim_{|\mathbf{x}|\to\infty} \big(|\mathbf{x}|\mathbf{E}^{\mathrm{s}} - z_0\mathbf{H}^{\mathrm{s}} \times \mathbf{x}\big) = 0. \tag{2.7}$$

There are usually two types of problems posed for the above equations. Given the cavity domain $\Omega$ or the cavity wall $S$ and the incoming wave $(\mathbf{E}^{\mathrm{inc}}, \mathbf{H}^{\mathrm{inc}})$, the direct problem is to determine the electromagnetic field $(\mathbf{E}, \mathbf{H})$. On the contrary, the inverse problem is to determine the cavity wall $S$ from the tangential trace of the electric field, $\mathbf{n}_\Gamma \times \mathbf{E}$, measured on the aperture $\Gamma$.

To describe the boundary value problem and present its variational formulation, we introduce some Sobolev spaces. For $u \in L^2(\mathbb{R}^2)$, we denote by $\hat{u}$ the Fourier transform of $u$:

$$\hat{u}(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}^2} u(\rho)e^{-\mathrm{i}\rho\cdot\xi}\,d\rho,$$

where $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $\rho = (x_1, x_2) \in \mathbb{R}^2$. Using Fourier modes, the norm on the space $L^2(\mathbb{R}^2)$ can be characterized by

$$\|u\|_{L^2(\mathbb{R}^2)} = \left[\int_{\mathbb{R}^2} |u|^2\,d\rho\right]^{1/2} = \left[\int_{\mathbb{R}^2} |\hat{u}|^2\,d\xi\right]^{1/2}.$$

For $s \in \mathbb{R}$, define

$$H^s\big(\mathbb{R}^2\big) = \left\{u \in L^2\big(\mathbb{R}^2\big)\colon \int_{\mathbb{R}^2} \big(1 + |\xi|^2\big)^s |\hat{u}|^2\,d\xi < \infty\right\},$$

whose norm is characterized by

$$\|u\|_{H^s(\mathbb{R}^2)} = \left[\int_{\mathbb{R}^2} \big(1 + |\xi|^2\big)^s |\hat{u}|^2\,d\xi\right]^{1/2}.$$

It is clear that the spaces $H^s(\mathbb{R}^2)$ and $H^{-s}(\mathbb{R}^2)$ are mutually adjoint with respect to the scalar product in $L^2(\mathbb{R}^2)$.

For any vector field $\mathbf{u} = (u_1, u_2, u_3)^\top$, denote its tangential component on surface $\Gamma$ by

$$\mathbf{u}_\Gamma = -\mathbf{n}_\Gamma \times (\mathbf{n}_\Gamma \times \mathbf{u}) = \big(u_1(x_1, x_2, 0), u_2(x_1, x_2, 0), 0\big)^\top,$$

where $\mathbf{n}_\Gamma$ is the unit outward normal vector on $\Gamma$. For any smooth tangential vector $\mathbf{u} = (u_1, u_2, 0)^\top$ defined on $\Gamma$, denote by $\operatorname{div}_\Gamma \mathbf{u} = \partial_1 u_1 + \partial_2 u_2$ and $\operatorname{curl}_\Gamma \mathbf{u} = \partial_1 u_2 - \partial_2 u_1$ the surface divergence and the surface scalar curl of the field $\mathbf{u}$, respectively.

Introduce the trace functional spaces:

$$H^{1/2}(\Gamma) = \big\{u|_\Gamma\colon\ u \in H^{1/2}\big(\mathbb{R}^2\big)\big\},$$
$$H_0^{1/2}(\Gamma) = \big\{u|_\Gamma\colon\ u \in H^{1/2}\big(\mathbb{R}^2\big),\ \operatorname{supp}(u) \subset \overline{\Gamma}\big\}.$$

Denote by $H^{-1/2}(\Gamma)$ and $H_0^{-1/2}(\Gamma)$ the dual spaces of $H_0^{1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, respectively, i.e.,

$$H^{-1/2}(\Gamma) = \big(H_0^{1/2}(\Gamma)\big)' \quad \text{and} \quad H_0^{-1/2}(\Gamma) = \big(H^{1/2}(\Gamma)\big)'.$$

To study the transparent boundary operator, we introduce

$$H_{\mathrm{div}}^{-1/2}(\Gamma) = \big\{\mathbf{u} \in \big(H^{-1/2}(\Gamma)\big)^3\colon\ u_3 = 0,\ \operatorname{div}_\Gamma \mathbf{u} \in H^{-1/2}(\Gamma)\big\},$$
$$H_{\mathrm{curl}}^{-1/2}(\Gamma) = \big\{\mathbf{u} \in \big(H^{-1/2}(\Gamma)\big)^3\colon\ u_3 = 0,\ \operatorname{curl}_\Gamma \mathbf{u} \in H^{-1/2}(\Gamma)\big\}.$$

Denote by $\mathbf{n}_S$ the unit outward normal vector on $S$. We finally introduce

$$H_S(\mathrm{curl}, \Omega) = \big\{\mathbf{u} \in \big(L^2(\Omega)\big)^3,\ \nabla \times \mathbf{u} \in \big(L^2(\Omega)\big)^3,\ \mathbf{n}_S \times \mathbf{u} = 0 \text{ on } S\big\},$$

which is a Hilbert space for the norm:

$$\|\mathbf{u}\|_{H(\mathrm{curl},\Omega)} = \big(\|\mathbf{u}\|^2_{(L^2(\Omega))^3} + \|\nabla \times \mathbf{u}\|^2_{(L^2(\Omega))^3}\big)^{1/2}.$$

This space will be used as the solution space for the weak formulation of the direct scattering problem.

## 3. The direct problem

In this section, we present the transparent boundary condition and weak formulation of the direct problem. The following two lemmas play important roles in the proof of uniqueness and stability for the inverse problem. The first lemma is concerned with the existence and characterization of the pseudodifferential operator, and the second lemma concerns its injectivity. The detailed discussions and proofs may be found in Ammari et al. [5].

**Lemma 3.1.** *There exists a linear continuous boundary operator $T : H_{\mathrm{curl}}^{-1/2}(\Gamma) \to H_{\mathrm{div}}^{-1/2}(\Gamma)$ such that it holds for any tangential vector $\mathbf{u} = (u_1, u_2, 0)^\top$ on $\Gamma$:*

$$T\mathbf{u} = (v_1, v_2, 0)^\top,$$

*where*

$$\hat{v}_1 = \frac{1}{\omega\mu}\left[\beta\hat{u}_1 + \frac{\xi_1}{\beta}(\xi_1\hat{u}_1 + \xi_2\hat{u}_2)\right], \qquad \hat{v}_2 = \frac{1}{\omega\mu}\left[\beta\hat{u}_2 + \frac{\xi_2}{\beta}(\xi_1\hat{u}_1 + \xi_2\hat{u}_2)\right],$$

*and*

$$\beta^2(\xi) = \kappa_0^2 - |\xi|^2.$$

**Lemma 3.2.** *For any* $\mathbf{u} \in H_{\mathrm{curl}}^{-1/2}(\Gamma)$, *if* $\mathrm{Re}\langle T\mathbf{u}, \mathbf{u}\rangle = 0$, *then* $\mathbf{u} = 0$.

Using the boundary operator, the following transparent boundary condition may be proposed on the cavity aperture $\Gamma$:

$$T\big(\mathbf{E}_\Gamma - \mathbf{E}_\Gamma^{\mathrm{inc}}\big) = \big(\mathbf{H} - \mathbf{H}^{\mathrm{inc}}\big) \times \mathbf{n}_\Gamma,$$

which map the tangential component of the scattered electric field to the tangential trace of the scattered magnetic field. Equivalently, it can be written as

$$(\nabla \times \mathbf{E}) \times \mathbf{n}_\Gamma = \mathrm{i}\omega\mu_0 T\mathbf{E}_\Gamma + \mathbf{f}, \tag{3.1}$$

where

$$\mathbf{f} = \mathrm{i}\omega\mu_0\big(\mathbf{H}^{\mathrm{inc}} \times \mathbf{n}_\Gamma - T\mathbf{E}_\Gamma^{\mathrm{inc}}\big).$$

Multiplying (2.4) by the complex conjugate of a test function $\mathbf{v} \in H_S(\mathrm{curl}, \Omega)$, integrating over $\Omega$, and using integration by parts and the transparent boundary condition (3.1), we arrive at the variational form for the direct problem: find $\mathbf{u} \in H_S(\mathrm{curl}, \Omega)$ such that

$$a(\mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v}\rangle \quad \text{for all } \mathbf{v} \in H_S(\mathrm{curl}, \Omega), \tag{3.2}$$

where the sesquilinear form

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \nabla \times \mathbf{u} \cdot \nabla \times \bar{\mathbf{v}} - \kappa^2 \int_\Omega \mathbf{u} \cdot \bar{\mathbf{v}} - \mathrm{i}\omega\mu_0 \int_\Gamma T\mathbf{u}_\Gamma \cdot \bar{\mathbf{v}}_\Gamma, \tag{3.3}$$

and the linear functional

$$\langle \mathbf{f}, \mathbf{v}\rangle = \int_\Gamma \mathbf{f} \cdot \bar{\mathbf{v}}_\Gamma. \tag{3.4}$$

For the direct problem, questions on existence and uniqueness are well understood, see for example [5]. It has been proved that the variational problem (3.2) admits a unique weak solution $\mathbf{u}$ in $H_S(\mathrm{curl}, \Omega)$.

## 4. The inverse problem

This section is concerned with the uniqueness and stability questions for the inverse problem. The uniqueness shows that the shape of the cavity is uniquely determined by the tangential trace of the electric field measured on the aperture. Based on the domain derivative, a local stability is established: if the two measurements of the tangential traces of the electric fields are “close” to each other, then the corresponding two cavities are also “close” to each other.

### 4.1. Global uniqueness

The following uniqueness result only requires a single incident field with one polarization, one frequency, and one incident direction. The proof is based on a combination of Holmgren's uniqueness and unique continuation theorems.

**Theorem 4.1.** *Let $\mathbf{u}_j$ be the solution of the variational problem* (3.2) *in $\Omega_j$ enclosed by $\Gamma$ and $S_j$ for $j = 1, 2$. If $\mathbf{n}_\Gamma \times \mathbf{u}_1 = \mathbf{n}_\Gamma \times \mathbf{u}_2$ on $\Gamma$, then $S_1 = S_2$.*

**Proof.** Assume that $S_1 \neq S_2$. Then $\Omega_1 \setminus (\Omega_1 \cap \Omega_2)$ or $\Omega_2 \setminus (\Omega_1 \cap \Omega_2)$ is a non-empty set. Without loss of generality, we assume that $D = \Omega_1 \setminus (\Omega_1 \cap \Omega_2) \neq \emptyset$. Denote $\partial D$ by $C_1 \cup C_2$ with $C_j \subset S_j$ for $j = 1, 2$.

Since $\mathbf{n}_\Gamma \times \mathbf{u}_1 - \mathbf{n}_\Gamma \times \mathbf{u}_2 = 0$ on $\Gamma$, it follows from the injectivity, Lemma 3.2, of the boundary operator $T$ and its definition (2.5) that $(\nabla \times \mathbf{u}_1) \times \mathbf{n}_\Gamma - (\nabla \times \mathbf{u}_2) \times \mathbf{n}_\Gamma = 0$ on $\Gamma$. An application of Holmgren's uniqueness theorem [1] yields $\mathbf{u}_1 - \mathbf{u}_2 = 0$ above $\Gamma_g \cup \Gamma$. By unique continuation [16], we get $\mathbf{u}_1 - \mathbf{u}_2 = 0$ in $\overline{\Omega_1 \cap \Omega_2}$ and especially $\mathbf{n} \times \mathbf{u}_1 - \mathbf{n} \times \mathbf{u}_2 = 0$ on $C_2$. It follows from $\mathbf{n} \times \mathbf{u}_2 = 0$ on $C_2$ that we have $\mathbf{n} \times \mathbf{u}_1 = 0$ on $C_2$ and the problem

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{u}_1) - \kappa^2 \mathbf{u}_1 = 0 \quad &\text{in } D, \\ \mathbf{n} \times \mathbf{u}_1 = 0 \quad &\text{on } \partial D. \end{aligned}$$

Recalling the expression of wavenumber, we have from the integration by parts that

$$\int_D |\nabla \times \mathbf{u}_1|^2 - \omega^2 \mu_0 \varepsilon_0 \int_D |\mathbf{u}_1|^2 - \mathrm{i}\omega\mu_0\sigma_0 \int_D |\mathbf{u}_1|^2 = 0,$$

which yields $\mathbf{u}_1 = 0$ in $D$. An application of the unique continuation again gives $\mathbf{u}_1 = 0$ in $\Omega_1$. But this contradicts the transparent boundary condition (3.1) since $\mathbf{f}$ is a nonzero function involving a downward incoming plane wave. $\square$

In general, the global uniqueness may not be possible when the conductivity $\sigma_0 = 0$ in the interior of the cavity. This is evident in the simplest case with a plane wave incident on a flat surface. In this case, the solution of the cavity scattering problem can be written explicitly:

$$\begin{aligned} \mathbf{E} &= \mathbf{E}^{\mathrm{inc}} - \mathbf{t} e^{\mathrm{i}\kappa_0 \mathbf{q}^* \cdot \mathbf{x}} = \mathbf{t} e^{\mathrm{i}\kappa_0 \mathbf{q} \cdot \mathbf{x}} - \mathbf{t} e^{\mathrm{i}\kappa_0 \mathbf{q}^* \cdot \mathbf{x}}, \\ \mathbf{H} &= \mathbf{H}^{\mathrm{inc}} - z_0 \mathbf{t} \times \mathbf{q}^* e^{\mathrm{i}\kappa_0 \mathbf{q}^* \cdot \mathbf{x}} = \mathbf{s} e^{\mathrm{i}\kappa_0 \mathbf{q} \cdot \mathbf{x}} - z_0 \mathbf{t} \times \mathbf{q}^* e^{\mathrm{i}\kappa_0 \mathbf{q}^* \cdot \mathbf{x}}, \end{aligned}$$

i.e., the total fields consist of the plane incident fields and the plane reflected fields; no scattered fields are present. The nonuniqueness is obvious since the fields will remain the same when the flat surface is moved up or down in certain multiples of the wavelength.

### 4.2. Domain derivative

The calculation of domain derivative, or more generally of the Fréchet derivative of the wave field with respect to the perturbation to the boundary of the medium, is an essential step for inverse scattering problems, which plays an important role in the stability analysis. The domain derivatives for related inverse obstacle scattering problems have been discussed by a number of researchers, e.g., Haddar and Kress [18], Hettlich [19], Kirsch [25], Meyer et al. [27], and Potthast [30]. Recently, Cagnol and Eller [12] have considered the shape sensitivity for Maxwell's equation in the time domain, where the shape derivative is studied by formulating the Maxwell equation into a hyperbolic system. We shall investigate the domain derivative for the time-harmonic Maxwell equation from the variational approach in the context of the cavity scattering problem.

Introduce a cavity domain $\Omega_h$ bounded by $S_h$ and $\Gamma$, where

$$S_h = \big\{\mathbf{x} + h\mathbf{p}(\mathbf{x}):\ \mathbf{x} \in S\big\}.$$

Here the cavity wall $S$ is assumed to be in $C^2$, the constant $h > 0$, and the function $\mathbf{p} = (p_1(\mathbf{x}), p_2(\mathbf{x}), p_3(\mathbf{x}))^\top \in C^2(S, \mathbb{R}^3)$ satisfying $\mathbf{p}(\mathbf{x}) = 0$ on the boundary of the aperture $\Gamma$. Obviously, if $h$ is small enough then $S_h \in C^2$ is a small perturbation of $S$.

According to a standard continuity argument for elliptic boundary value problems, there exists a unique solution $\mathbf{u}^h$ to the variational problem (3.2) corresponding to the domain $\Omega_h$ for any small enough $h$. Define a nonlinear map

$$M : S_h \to \mathbf{n}_\Gamma \times \mathbf{u}^h|_\Gamma.$$

The domain derivative of the operator scattering $M$ on the boundary $S$ along the direction $\mathbf{p}$ is defined by

$$M'(S, \mathbf{p}) := \lim_{h\to 0} \frac{\mathbf{n}_\Gamma \times \mathbf{u}^h|_\Gamma - \mathbf{n}_\Gamma \times \mathbf{u}|_\Gamma}{h}.$$

Obviously, the weak formulation for $\mathbf{u}^h$ is

$$a_h\big(\mathbf{u}^h, \mathbf{v}\big) = \langle \mathbf{f}, \mathbf{v}\rangle \quad \text{for all } \mathbf{v} \in H_{S_h}(\mathrm{curl}, \Omega_h), \tag{4.1}$$

where $a_h$ is the sesquilinear form defined in (3.3) over the perturbed domain $\Omega_h$.

Motivated by the technique adopted in Kirsch [25], we make change of variables to convert the integral in $\Omega_h$ into $\Omega$. For $\mathbf{p} \in C^2(S, \mathbb{R}^3)$, we extend the definition of function $\mathbf{p}(\mathbf{x})$ to $\overline{\Omega}$ satisfying: $\mathbf{p}(\mathbf{x}) \in C^2(\Omega, \mathbb{R}^3) \cap C(\overline{\Omega})$; $\mathbf{p}(\mathbf{x}) = 0$ on $\Gamma$; $\xi^h(\mathbf{x}) = (\xi_1^h(\mathbf{x}), \xi_2^h(\mathbf{x}), \xi_3^h(\mathbf{x}))^\top := \mathbf{x} + h\mathbf{p}(\mathbf{x})$ maps $\Omega \to \Omega_h$. In this way, $\xi^h$ is a diffeomorphism from $\Omega$ to $\Omega_h$ for small enough $h$. Denote by $\eta^h(\mathbf{y}) = (\eta_1^h(\mathbf{y}), \eta_2^h(\mathbf{y}), \eta_3^h(\mathbf{y}))^\top : \Omega_h \to \Omega$ the inverse map of $\xi^h$.

For $\mathbf{y} \in \Omega_h$, make the change of variable $\mathbf{y} = \xi^h(\mathbf{x})$: $\mathbf{x} \in \Omega$ and define $\breve{\mathbf{u}}^h(\mathbf{x}) := \mathbf{u}^h \circ \xi^h$. The volume integrals over $\Omega_h$ in the sesquilinear form (3.3) can be converted into integrals over the unperturbed domain $\Omega$:

$$\begin{aligned}
&\int_{\Omega_h} \nabla \times \mathbf{u}^h \cdot \nabla \times \bar{\mathbf{v}} - \kappa^2 \int_{\Omega_h} \mathbf{u}^h \cdot \bar{\mathbf{v}} \\
&\quad = \int_{\Omega} \big[\big(\nabla\eta^h \otimes \nabla\breve{\mathbf{u}}^h\big) \cdot \big(\nabla\eta^h \otimes \nabla\bar{\breve{\mathbf{v}}}\big) - \kappa^2 \breve{\mathbf{u}}^h \cdot \bar{\breve{\mathbf{v}}}\big] \det(J_{\xi^h}),
\end{aligned} \tag{4.2}$$

where $\breve{\mathbf{v}} = \mathbf{v} \circ \xi^h$ and $J_{\xi^h}$ is the Jacobian matrix of the transform $\xi^h$. For convenience, given two vectors $\mathbf{w} = (w_1, w_2, w_3)^\top$ and $\mathbf{v} = (v_1, v_2, v_3)^\top$, the symbol $\otimes$ is defined as follows:

$$\nabla\mathbf{w} \otimes \nabla\mathbf{v} = \begin{bmatrix} \partial_2\mathbf{w} \cdot \nabla v_3 - \partial_3\mathbf{w} \cdot \nabla v_2 \\ \partial_3\mathbf{w} \cdot \nabla v_1 - \partial_1\mathbf{w} \cdot \nabla v_3 \\ \partial_1\mathbf{w} \cdot \nabla v_2 - \partial_2\mathbf{w} \cdot \nabla v_1 \end{bmatrix},$$

where $\partial_j\mathbf{w} = (\partial_j w_1, \partial_j w_2, \partial_j w_3)^\top$.

For an arbitrary test function $\mathbf{v}$ in domain $\Omega_h$, the function $\breve{\mathbf{v}}$ is a test function for domain $\Omega$ according to the transform. Recalling (4.2), we can rewrite the bilinear form $a_h$ in (4.1) as

$$a_h\big(\breve{\mathbf{u}}^h, \mathbf{v}\big) = \int\limits_\Omega \big[\big(\nabla\eta^h \otimes \nabla\breve{\mathbf{u}}^h\big)\cdot\big(\nabla\eta^h \otimes \nabla\bar{\mathbf{v}}\big) - \kappa^2\breve{\mathbf{u}}^h\cdot\bar{\mathbf{v}}\big]\det(J_{\xi^h}) - \mathrm{i}\omega\mu_0\int\limits_\Gamma T\breve{\mathbf{u}}^h_\Gamma\cdot\bar{\mathbf{v}}_\Gamma, \tag{4.3}$$

which leads to an equivalent variational formulation to (4.1):

$$a_h(\breve{\mathbf{u}}_h, \mathbf{v}) = \langle\mathbf{f}, \mathbf{v}\rangle \quad \text{for all } \mathbf{v}\in H_S(\mathrm{curl}, \Omega). \tag{4.4}$$

It follows from (4.3) and (4.4) that

$$\begin{aligned} a(\breve{\mathbf{u}}_h - \mathbf{u}, \mathbf{v}) &= a(\breve{\mathbf{u}}_h, \mathbf{v}) - \langle\mathbf{f}, \mathbf{v}\rangle = a(\breve{\mathbf{u}}_h, \mathbf{v}) - a_h(\breve{\mathbf{u}}_h, \mathbf{v}) \\ &= \int\limits_\Omega \big[\nabla\times\breve{\mathbf{u}}^h\cdot\nabla\times\bar{\mathbf{v}} - \big(\nabla\eta^h\otimes\nabla\breve{\mathbf{u}}^h\big)\cdot\big(\nabla\eta^h\otimes\nabla\bar{\mathbf{v}}\big)\det(J_{\xi^h})\big] \\ &\quad - \kappa^2\int\limits_\Omega \big[1-\det(J_{\xi^h})\big]\breve{\mathbf{u}}_h\cdot\bar{\mathbf{v}}. \end{aligned} \tag{4.5}$$

Using the definition of Jacobian matrix, we have

$$\det(J_{\xi^h}) = 1 + h\nabla\cdot\mathbf{p} + O\big(h^2\big).$$

Let $J_{\eta^h}$ be the Jacobian matrix of the transform $\eta^h$. Denote by $J_\mathbf{p} = [\partial_i p_j]$ the $3\times 3$ Jacobian matrix of $\mathbf{p}$. Simple calculation yields

$$J_{\eta^h} = J_{\xi^h}^{-1}\circ\eta^h = I - hJ_\mathbf{p} + O\big(h^2\big),$$

which gives after equating two sides

$$\partial_i\eta^h_j = \delta_{ij} - h\partial_i p_j + O\big(h^2\big).$$

Here $\delta_{ij}$ is the Kronecker delta symbol. Recalling the definition for the symbol $\otimes$ and using the above explicit expression for the partial derivatives of $\eta^h$, we can deduce that

$$\nabla\eta^h\otimes\nabla\breve{\mathbf{u}}^h = \nabla\times\breve{\mathbf{u}}^h - h\nabla\mathbf{p}\otimes\nabla\breve{\mathbf{u}}^h + O\big(h^2\big)$$

and

$$\nabla\eta^h\otimes\nabla\bar{\mathbf{v}} = \nabla\times\bar{\mathbf{v}} - h\nabla\mathbf{p}\otimes\nabla\bar{\mathbf{v}} + O\big(h^2\big).$$

Upon plugging in above identities, we have

$$\begin{aligned} &\big(\nabla\eta^h\otimes\nabla\breve{\mathbf{u}}^h\big)\cdot\big(\nabla\eta^h\otimes\nabla\bar{\mathbf{v}}\big)\det(J_{\xi^h}) \\ &\quad = (1+h\nabla\cdot\mathbf{p})\nabla\times\breve{\mathbf{u}}^h\cdot\nabla\times\bar{\mathbf{v}} \\ &\qquad - h\big[(\nabla\mathbf{p}\otimes\nabla\bar{\mathbf{v}})\cdot\nabla\times\breve{\mathbf{u}}^h + \big(\nabla\mathbf{p}\otimes\nabla\breve{\mathbf{u}}^h\big)\cdot\nabla\times\bar{\mathbf{v}}\big] + O\big(h^2\big). \end{aligned} \tag{4.6}$$

Substituting (4.6) into (4.5) and dividing by $h$ yield

$$a\left(\frac{\breve{\mathbf{u}}_h - \mathbf{u}}{h}, \mathbf{v}\right) = \int_\Omega \left[(\nabla \mathbf{p} \otimes \nabla \bar{\mathbf{v}}) \cdot \nabla \times \breve{\mathbf{u}}^h + \left(\nabla \mathbf{p} \otimes \nabla \breve{\mathbf{u}}^h\right) \cdot \nabla \times \bar{\mathbf{v}}\right] - \int_\Omega (\nabla \cdot \mathbf{p})\left(\nabla \times \breve{\mathbf{u}}^h \cdot \nabla \times \bar{\mathbf{v}} - \kappa^2 \breve{\mathbf{u}}^h \cdot \bar{\mathbf{v}}\right) + O(h). \tag{4.7}$$

Based on this variational form, we have the following result for the domain derivative.

**Theorem 4.2.** *Let* $\mathbf{u}$ *be the solution of the variational problem* (3.2) *in* $\Omega$. *Given* $\mathbf{p} \in C^2(S, \mathbb{R}^3)$, *the domain derivative of the scattering operator is* $M'(S, \mathbf{p}) = \mathbf{n}_\Gamma \times \mathbf{u}'|_\Gamma$, *where* $\mathbf{u}' \in H(\mathrm{curl}, \Omega)$ *is the weak solution of the boundary value problem*:

$$\nabla \times \left(\nabla \times \mathbf{u}'\right) - \kappa^2 \mathbf{u}' = 0 \quad \textit{in } \Omega, \tag{4.8}$$

$$\left(\nabla \times \mathbf{u}'\right) \times \mathbf{n}_\Gamma = \mathrm{i}\omega\mu T \mathbf{u}'_\Gamma \quad \textit{on } \Gamma, \tag{4.9}$$

$$\mathbf{n}_S \times \mathbf{u}' = \left[(\mathbf{n}_S \cdot \mathbf{p})\partial_S \mathbf{u}_S + (\mathbf{n}_S \cdot \mathbf{u})\nabla_S(\mathbf{n}_S \cdot \mathbf{p})\right] \times \mathbf{n}_S \quad \textit{on } S. \tag{4.10}$$

*Here* $\nabla_S$ *and* $\partial_S$ *denote the surface gradient and the normal derivative on* $S$, *respectively, and* $\mathbf{u}_S$ *is the tangential component of* $\mathbf{u}$ *on* $S$, *i.e.*, $\mathbf{u}_S = -\mathbf{n}_S \times (\mathbf{n}_S \times \mathbf{u})$.

**Proof.** The proof consists of two steps: the first step is to show that the domain derivative $\mathbf{u}'$ satisfies the Maxwell equation (4.8) and the transparent boundary condition (4.9); the second step is to verify that the tangential trace of the domain derivative $\mathbf{n}_S \times \mathbf{u}'$ satisfies the nonhomogeneous boundary condition (4.10).

First we prove (4.8) and (4.9). Given $\mathbf{p}$, it follows from the well-posedness of the variational problem that $\breve{\mathbf{u}}_h \to \mathbf{u}$ in $H_S(\mathrm{curl}, \Omega)$ as $h \to 0$. Taking the limit in (4.7) gives

$$a\left(\lim_{h\to 0} \frac{\breve{\mathbf{u}}^h - \mathbf{u}}{h}, \mathbf{v}\right) = \int_\Omega \left[(\nabla \mathbf{p} \otimes \nabla \bar{\mathbf{v}}) \cdot \nabla \times \mathbf{u} + (\nabla \mathbf{p} \otimes \nabla \mathbf{u}) \cdot \nabla \times \bar{\mathbf{v}}\right] - \int_\Omega (\nabla \cdot \mathbf{p})\left(\nabla \times \mathbf{u} \cdot \nabla \times \bar{\mathbf{v}} - \kappa^2 \mathbf{u} \cdot \bar{\mathbf{v}}\right). \tag{4.11}$$

Therefore $(\breve{\mathbf{u}}^h - \mathbf{u})/h$ is convergent in $H_S(\mathrm{curl}, \Omega)$ as $h \to 0$. Denote by $\dot{\mathbf{u}}$ this limit, which is known as the material derivative, and rewrite (4.11) as

$$a(\dot{\mathbf{u}}, \mathbf{v}) = b_1(\mathbf{p})(\mathbf{u}, \mathbf{v}) + b_2(\mathbf{p})(\mathbf{u}, \mathbf{v}) + b_3(\mathbf{p})(\mathbf{u}, \mathbf{v}) + b_4(\mathbf{p})(\mathbf{u}, \mathbf{v}), \tag{4.12}$$

where

$$b_1(\mathbf{p})(\mathbf{u}, \mathbf{v}) = \int_\Omega (\nabla \mathbf{p} \otimes \nabla \bar{\mathbf{v}}) \cdot \nabla \times \mathbf{u},$$

$$b_2(\mathbf{p})(\mathbf{u}, \mathbf{v}) = \int_\Omega (\nabla \mathbf{p} \otimes \nabla \mathbf{u}) \cdot \nabla \times \bar{\mathbf{v}},$$

$$b_3(\mathbf{p})(\mathbf{u},\mathbf{v}) = -\int_\Omega (\nabla\cdot\mathbf{p})\nabla\times\mathbf{u}\cdot\nabla\times\bar{\mathbf{v}},$$

$$b_4(\mathbf{p})(\mathbf{u},\mathbf{v}) = \kappa^2\int_\Omega (\nabla\cdot\mathbf{p})\mathbf{u}\cdot\bar{\mathbf{v}}.$$

The goal is to prove

$$b_1+b_2+b_3+b_4 = \int_\Omega \nabla\times(\mathbf{p}\cdot\nabla\mathbf{u})\cdot\nabla\times\bar{\mathbf{v}} - \kappa^2\int_\Omega (\mathbf{p}\cdot\nabla\mathbf{u})\cdot\bar{\mathbf{v}}. \tag{4.13}$$

In the following, we simplify the integrals of $b_j$ to deduce the desired identity (4.13).

Following the identity

$$(\nabla\mathbf{p}\otimes\nabla\mathbf{u})\cdot\nabla\times\bar{\mathbf{v}} = \nabla\times(\mathbf{p}\cdot\nabla\mathbf{u})\cdot\nabla\times\bar{\mathbf{v}} - (\mathbf{p}\cdot\nabla)(\nabla\times\mathbf{u})\cdot\nabla\times\bar{\mathbf{v}},$$

and the integration by parts, we have

$$\begin{aligned} b_2 &= \int_\Omega \nabla\times(\mathbf{p}\cdot\nabla\mathbf{u})\cdot\nabla\times\bar{\mathbf{v}} - \int_\Omega (\mathbf{p}\cdot\nabla)(\nabla\times\mathbf{u})\cdot\nabla\times\bar{\mathbf{v}} \\ &= \int_\Omega \nabla\times(\mathbf{p}\cdot\nabla\mathbf{u})\cdot\nabla\times\bar{\mathbf{v}} - \int_\Omega \big[\nabla\times(\mathbf{p}\cdot\nabla)(\nabla\times\mathbf{u})\big]\cdot\bar{\mathbf{v}} - \int_S \bar{\mathbf{v}}\times\big[(\mathbf{p}\cdot\nabla)(\nabla\times\mathbf{u})\big]\cdot\mathbf{n}_S. \end{aligned}$$

It can be verified that

$$\begin{aligned} \nabla\times(\mathbf{p}\cdot\nabla)(\nabla\times\mathbf{u}) &= \nabla\mathbf{p}\otimes\nabla(\nabla\times\mathbf{u}) + (\mathbf{p}\cdot\nabla)\nabla\times(\nabla\times\mathbf{u}) \\ &= \nabla\mathbf{p}\otimes\nabla(\nabla\times\mathbf{u}) + \kappa^2(\mathbf{p}\cdot\nabla)\mathbf{u}, \end{aligned}$$

where we have used $\nabla\times(\nabla\times\mathbf{u}) - \kappa^2\mathbf{u} = 0$. Hence

$$\begin{aligned} b_2 = {}& \int_\Omega \nabla\times(\mathbf{p}\cdot\nabla\mathbf{u})\cdot\nabla\times\bar{\mathbf{v}} - \kappa^2\int_\Omega (\mathbf{p}\cdot\nabla)\mathbf{u}\cdot\bar{\mathbf{v}} - \int_\Omega \big[\nabla\mathbf{p}\otimes\nabla(\nabla\times\mathbf{u})\big]\cdot\bar{\mathbf{v}} \\ & - \int_S \bar{\mathbf{v}}\times\big[(\mathbf{p}\cdot\nabla)(\nabla\times\mathbf{u})\big]\cdot\mathbf{n}_S. \end{aligned} \tag{4.14}$$

Using the definition of the symbol $\otimes$ and rearranging some terms, we get

$$\begin{aligned} \int_\Omega \big[\nabla\mathbf{p}\otimes\nabla(\nabla\times\mathbf{u})\big]\cdot\bar{\mathbf{v}} = {}& \int_\Omega (\bar{v}_1\partial_2\mathbf{p} - \bar{v}_2\partial_1\mathbf{p})\cdot\nabla(\partial_1 u_2 - \partial_2 u_1) \\ & + \int_\Omega (\bar{v}_3\partial_1\mathbf{p} - \bar{v}_1\partial_3\mathbf{p})\cdot\nabla(\partial_3 u_1 - \partial_1 u_3) \\ & + \int_\Omega (\bar{v}_2\partial_3\mathbf{p} - \bar{v}_3\partial_2\mathbf{p})\cdot\nabla(\partial_2 u_3 - \partial_3 u_2). \end{aligned} \tag{4.15}$$

An application of the triple product gives

$$\int_S \bar{\mathbf{v}} \times \big[(\mathbf{p}\cdot\nabla)(\nabla\times\mathbf{u})\big]\cdot\mathbf{n}_S = \int_S (\mathbf{n}_S\times\bar{\mathbf{v}})\cdot\big[(\mathbf{p}\cdot\nabla)(\nabla\times\mathbf{u})\big] = 0, \tag{4.16}$$

since $\mathbf{v}\in H_S(\mathrm{curl},\Omega)$.

Using the identity

$$\nabla\times(\nabla\cdot\mathbf{p}\nabla\times\mathbf{u}) = \nabla(\nabla\cdot\mathbf{p})\times(\nabla\times\mathbf{u}) + (\nabla\cdot\mathbf{p})\nabla\times(\nabla\times\mathbf{u})$$

and the integration by parts, we obtain

$$\begin{aligned}
b_3 &= -\int_\Omega \big[\nabla\times(\nabla\cdot\mathbf{p}\nabla\times\mathbf{u})\big]\cdot\bar{\mathbf{v}} - \int_S (\nabla\cdot\mathbf{p})\big[\bar{\mathbf{v}}\times(\nabla\times\mathbf{u})\big]\cdot\mathbf{n}_S \\
&= -\int_\Omega \big[\nabla(\nabla\cdot\mathbf{p})\times(\nabla\times\mathbf{u})\big]\cdot\bar{\mathbf{v}} - \int_\Omega (\nabla\cdot\mathbf{p})\big[\nabla\times(\nabla\times\mathbf{u})\big]\cdot\bar{\mathbf{v}} \\
&\quad - \int_S (\nabla\cdot\mathbf{p})\big[\bar{\mathbf{v}}\times(\nabla\times\mathbf{u})\big]\cdot\mathbf{n}_S \\
&= -\int_\Omega \big[\bar{\mathbf{v}}\times\nabla(\nabla\cdot\mathbf{p})\big]\cdot(\nabla\times\mathbf{u}) - \int_\Omega (\nabla\cdot\mathbf{p})\big[\nabla\times(\nabla\times\mathbf{u})\big]\cdot\bar{\mathbf{v}} \\
&\quad - \int_S (\nabla\cdot\mathbf{p})\big[\bar{\mathbf{v}}\times(\nabla\times\mathbf{u})\big]\cdot\mathbf{n}_S.
\end{aligned}$$

Noticing $\nabla\times(\nabla\times\mathbf{u}) - \kappa^2\mathbf{u} = 0$ in $\Omega$ and $\mathbf{n}_S\times\mathbf{v} = 0$ on $S$, we have

$$\begin{aligned}
b_3 + b_4 &= -\int_\Omega \big[\bar{\mathbf{v}}\times\nabla(\nabla\cdot\mathbf{p})\big]\cdot(\nabla\times\mathbf{u}) - \int_\Omega (\nabla\cdot\mathbf{p})\big[\nabla\times(\nabla\times\mathbf{u}) - \kappa^2\mathbf{u}\big]\cdot\bar{\mathbf{v}} \\
&\quad - \int_S (\nabla\cdot\mathbf{p})\big[\bar{\mathbf{v}}\times(\nabla\times\mathbf{u})\big]\cdot\mathbf{n}_S \\
&= -\int_\Omega \big[\bar{\mathbf{v}}\times\nabla(\nabla\cdot\mathbf{p})\big]\cdot(\nabla\times\mathbf{u}) - \int_S (\nabla\cdot\mathbf{p})(\mathbf{n}_S\times\bar{\mathbf{v}})\cdot(\nabla\times\mathbf{u}) \\
&= -\int_\Omega \big[\bar{\mathbf{v}}\times\nabla(\nabla\cdot\mathbf{p})\big]\cdot(\nabla\times\mathbf{u}).
\end{aligned}$$

A simple calculation yields

$$\bar{\mathbf{v}}\times\nabla(\nabla\cdot\mathbf{p}) = \begin{bmatrix} \bar{v}_2\partial_3(\nabla\cdot\mathbf{p}) - \bar{v}_3\partial_2(\nabla\cdot\mathbf{p}) \\ \bar{v}_3\partial_1(\nabla\cdot\mathbf{p}) - \bar{v}_1\partial_3(\nabla\cdot\mathbf{p}) \\ \bar{v}_1\partial_2(\nabla\cdot\mathbf{p}) - \bar{v}_2\partial_1(\nabla\cdot\mathbf{p}) \end{bmatrix}.$$

Hence

$$\nabla\mathbf{p}\otimes\nabla\bar{\mathbf{v}}-\bar{\mathbf{v}}\times\nabla(\nabla\cdot\mathbf{p})=\begin{bmatrix}\nabla\cdot(\bar{v}_3\partial_2\mathbf{p}-\bar{v}_2\partial_3\mathbf{p})\\\nabla\cdot(\bar{v}_1\partial_3\mathbf{p}-\bar{v}_3\partial_1\mathbf{p})\\\nabla\cdot(\bar{v}_2\partial_1\mathbf{p}-\bar{v}_1\partial_2\mathbf{p})\end{bmatrix},$$

which gives after the use of integration by parts

$$\begin{aligned}b_1+b_3+b_4&=\int_\Omega\big[\nabla\mathbf{p}\otimes\nabla\bar{\mathbf{v}}-\bar{\mathbf{v}}\times\nabla(\nabla\cdot\mathbf{p})\big]\cdot(\nabla\times\mathbf{u})\\&=\int_\Omega(\partial_2u_3-\partial_3u_2)\nabla\cdot(\bar{v}_3\partial_2\mathbf{p}-\bar{v}_2\partial_3\mathbf{p})+\int_\Omega(\partial_3u_1-\partial_1u_3)\nabla\cdot(\bar{v}_1\partial_3\mathbf{p}-\bar{v}_3\partial_1\mathbf{p})\\&\quad+\int_\Omega(\partial_1u_2-\partial_2u_1)\nabla\cdot(\bar{v}_2\partial_1\mathbf{p}-\bar{v}_1\partial_2\mathbf{p})\\&=-\int_\Omega(\bar{v}_3\partial_2\mathbf{p}-\bar{v}_2\partial_3\mathbf{p})\cdot\nabla(\partial_2u_3-\partial_3u_2)-\int_\Omega(\bar{v}_1\partial_3\mathbf{p}-\bar{v}_3\partial_1\mathbf{p})\cdot\nabla(\partial_3u_1-\partial_1u_3)\\&\quad-\int_\Omega(\bar{v}_2\partial_1\mathbf{p}-\bar{v}_1\partial_2\mathbf{p})\cdot\nabla(\partial_1u_2-\partial_2u_1).\end{aligned}\tag{4.17}$$

Combining (4.14)–(4.17) yields (4.13).

Noticing $\mathbf{p}=0$ on $\Gamma$, we have from (4.12) and (4.13) that

$$a(\dot{\mathbf{u}},\mathbf{v})=\int_\Omega\nabla\times(\mathbf{p}\cdot\nabla\mathbf{u})\cdot\nabla\times\bar{\mathbf{v}}-\kappa^2\int_\Omega(\mathbf{p}\cdot\nabla\mathbf{u})\cdot\bar{\mathbf{v}}=a\big((\mathbf{p}\cdot\nabla)\mathbf{u},\mathbf{v}\big).\tag{4.18}$$

Denote the domain derivative $\mathbf{u}'=\dot{\mathbf{u}}-(\mathbf{p}\cdot\nabla)\mathbf{u}$. It follows from (4.18) that

$$a\big(\mathbf{u}',\mathbf{v}\big)=0\quad\text{for all }\mathbf{v}\in H_S(\mathrm{curl},\Omega),$$

which yields

$$\int_\Omega\nabla\times\mathbf{u}'\cdot\nabla\times\bar{\mathbf{v}}-\kappa^2\int_\Omega\mathbf{u}'\cdot\bar{\mathbf{v}}-\mathrm{i}\omega\mu_0\int_\Gamma T\mathbf{u}'_\Gamma\cdot\bar{\mathbf{v}}_\Gamma=0\quad\text{for all }\mathbf{v}\in H_S(\mathrm{curl},\Omega).$$

Following the integration by parts, it holds that for all $\mathbf{v}\in H_S(\mathrm{curl},\Omega)$

$$\int_\Omega\big[\nabla\times\big(\nabla\times\mathbf{u}'\big)-\kappa^2\mathbf{u}'\big]\cdot\bar{\mathbf{v}}+\int_\Gamma\big[\big(\nabla\times\mathbf{u}'\big)\times\mathbf{n}_\Gamma-\mathrm{i}\omega\mu_0T\mathbf{u}'_\Gamma\big]\cdot\bar{\mathbf{v}}=0,$$

which completes the first part of the proof.

Next is to prove the boundary condition (4.10). The proof is similar to what is used in [12] for the derivation of the boundary condition of the shape derivative. Observe that

$$\mathbf{p}\cdot\nabla=\mathbf{p}_S\cdot\nabla_S+(\mathbf{n}_S\cdot\mathbf{p})\partial_S,$$

where $\nabla_S$ denotes the surface gradient on $S$, $\mathbf{p}_S$ is the tangential component of the direction vector $\mathbf{p}$, i.e., $\mathbf{p}_S = -\mathbf{n}_S \times (\mathbf{n}_S \times \mathbf{p}) = \mathbf{p} - (\mathbf{n}_S \cdot \mathbf{p})\mathbf{n}_S$, and $\partial_S$ is the normal derivative on $S$. Since $\mathbf{n}_S \times \mathbf{u} = 0$ along $S$, it follows

$$0 = (\mathbf{p}_S \cdot \nabla_S)(\mathbf{n}_S \times \mathbf{u}) = (\mathbf{p}_S \cdot \nabla_S)\mathbf{n}_S \times \mathbf{u} + \mathbf{n}_S \times (\mathbf{p}_S \cdot \nabla_S)\mathbf{u},$$

which gives

$$\mathbf{n}_S \times (\mathbf{p} \cdot \nabla)\mathbf{u} = \mathbf{n}_S \times (\mathbf{n}_S \cdot \mathbf{p})\partial_S \mathbf{u} - (\mathbf{p}_S \cdot \nabla_S)\mathbf{n}_S \times \mathbf{u}.$$

Noticing

$$\begin{aligned} 0 = \lim_{h\to 0} \frac{\mathbf{n}_{S_h} \times \mathbf{u}^h - \mathbf{n}_S \times \mathbf{u}}{h} &= \lim_{h\to 0} \frac{\mathbf{n}_{S_h} \times \mathbf{u}^h - \mathbf{n}_S \times \mathbf{u}^h}{h} + \lim_{h\to 0} \frac{\mathbf{n}_S \times \mathbf{u}^h - \mathbf{n}_S \times \mathbf{u}}{h} \\ &= \dot{\mathbf{n}}_S \times \mathbf{u} + \mathbf{n}_S \times \dot{\mathbf{u}}, \end{aligned}$$

we have

$$\mathbf{n}_S \times \dot{\mathbf{u}} = -\dot{\mathbf{n}}_S \times \mathbf{u}.$$

Hence the domain derivative satisfies the boundary condition

$$\begin{aligned} \mathbf{n}_S \times \mathbf{u}' &= \mathbf{n}_S \times \dot{\mathbf{u}} - \mathbf{n}_S \times (\mathbf{p} \cdot \nabla)\mathbf{u} = -\dot{\mathbf{n}}_S \times \mathbf{u} - \mathbf{n}_S \times (\mathbf{p} \cdot \nabla)\mathbf{u} \\ &= -\mathbf{n}_S \times (\mathbf{n}_S \cdot \mathbf{p})\partial_S \mathbf{u} + \big[(\mathbf{p}_S \cdot \nabla_S)\mathbf{n}_S - \dot{\mathbf{n}}_S\big] \times \mathbf{u}. \end{aligned} \tag{4.19}$$

It can be verified that

$$\nabla_S(\mathbf{n}_S \cdot \mathbf{p}) = \big(J_{\mathbf{p}}^\top \mathbf{n}_S\big)_S + \big(J_S^\top \mathbf{p}\big)_S,$$

where $J_{\mathbf{p}}$ and $J_S$ are the Jacobian matrices of the vector $\mathbf{p}$ and the unit outward normal $\mathbf{n}_S$, respectively. We obtain

$$\begin{aligned} (\mathbf{p}_S \cdot \nabla_S)\mathbf{n}_S - \dot{\mathbf{n}}_S &= (\mathbf{p}_S \cdot \nabla_S)\mathbf{n}_S + \big(J_{\mathbf{p}}^\top \mathbf{n}_S\big)_S = (J_S \mathbf{p})_S + \big(J_{\mathbf{p}}^\top \mathbf{n}_S\big)_S \\ &= \nabla_S(\mathbf{n}_S \cdot \mathbf{p}) + (J_S \mathbf{p})_S - \big(J_S^\top \mathbf{p}\big)_S. \end{aligned}$$

Since $S$ is assumed to be of class $C^2$, it is shown in [14] that the distance function $\phi(\mathbf{x}) = \mathrm{dist}(\mathbf{x}, S)$ is of class $C^2$ in the set $\{\mathbf{x} \in \overline{\Omega}\colon\ \phi(\mathbf{x}) < c\}$, where $c$ is some positive number, and furthermore $\mathbf{n}_S = -\nabla\phi$ for $\mathbf{x} \in S$. Hence $J_S = J_S^\top$, which yields

$$(\mathbf{p}_S \cdot \nabla_S)\mathbf{n}_S - \dot{\mathbf{n}}_S = \nabla_S(\mathbf{n}_S \cdot \mathbf{p}).$$

The boundary condition (4.19) reduces to

$$\mathbf{n}_S \times \mathbf{u}' = -\mathbf{n}_S \times (\mathbf{n}_S \cdot \mathbf{p})\partial_S \mathbf{u} + \nabla_S(\mathbf{n}_S \cdot \mathbf{p}) \times \mathbf{u}.$$

It is clear that

$$
\begin{aligned}
-\mathbf{n}_S \times (\mathbf{n}_S \cdot \mathbf{p})\partial_S \mathbf{u} &= (\mathbf{n}_S \cdot \mathbf{p})\partial_S \big[\mathbf{u}_S + (\mathbf{n}_S \cdot \mathbf{u})\mathbf{n}_S\big] \times \mathbf{n}_S \\
&= (\mathbf{n}_S \cdot \mathbf{p})\partial_S \mathbf{u}_S \times \mathbf{n}_S.
\end{aligned}
$$

Finally, since $\mathbf{n}_S \times \mathbf{u} = 0$ we have also $\mathbf{u}_S = 0$ and hence

$$
\begin{aligned}
\nabla_S(\mathbf{n}_S \cdot \mathbf{p}) \times \mathbf{u} &= \nabla_S(\mathbf{n}_S \cdot \mathbf{p}) \times \big[\mathbf{u}_S + (\mathbf{n}_S \cdot \mathbf{u})\mathbf{n}_S\big] \\
&= \nabla_S(\mathbf{n}_S \cdot \mathbf{p}) \times (\mathbf{n}_S \cdot \mathbf{u})\mathbf{n}_S = (\mathbf{n}_S \cdot \mathbf{u})\nabla_S(\mathbf{n}_S \cdot \mathbf{p}) \times \mathbf{n}_S.
\end{aligned}
$$

Combining the last two formulas yields the boundary condition for the tangential trace of the domain derivative:

$$
\mathbf{n}_S \times \mathbf{u}' = \big[(\mathbf{n}_S \cdot \mathbf{p})\partial_S \mathbf{u}_S + (\mathbf{n}_S \cdot \mathbf{u})\nabla_S(\mathbf{n}_S \cdot \mathbf{p})\big] \times \mathbf{n}_S,
$$

which completes the second part of the proof, and hence the proof of the whole theorem. □

Besides the application for the proof of the stability, the characterization of the domain derivative, i.e., the boundary value problem (4.8)–(4.10), can be utilized for an implementation of Newton-based methods for solving shape optimization problems involving Maxwell's equations.

### 4.3. *Local stability*

In applications, it is impossible to make exact measurements. Stability is crucial in the practical reconstruction of cavity walls since it contains necessary information to determine to what extent the data can be trusted.

For any two domains $D_1$ and $D_2$ in $\mathbb{R}^2$, define $d(D_1, D_2)$ the Hausdorff distance between them by

$$
\operatorname{dist}(D_1, D_2) = \max\big\{\rho(D_1, D_2), \rho(D_2, D_1)\big\}
$$

where

$$
\rho(D_m, D_n) = \sup_{\mathbf{x}\in D_m} \inf_{\mathbf{y}\in D_n} |\mathbf{x} - \mathbf{y}|.
$$

Introduce domains $\Omega_h$ bounded by $S_h$ and $\Gamma$, where

$$
S_h: \quad \mathbf{x} + hp(\mathbf{x})\mathbf{n}_S,
$$

where $p \in C^2(S, \mathbb{R})$. It is easily seen that the Hausdorff distance between $\Omega$ and $\Omega_h$ is of the order $h$, i.e., $\operatorname{dist}(\Omega_h, \Omega) = O(h)$.

We have the following local stability result.

**Theorem 4.3.** *If $p \in C^2(S, \mathbb{R})$ and $h > 0$ is sufficiently small, then*

$$
\operatorname{dist}(\Omega_h, \Omega) \leqslant C\big\|\mathbf{n}_\Gamma \times \mathbf{u}^h - \mathbf{n}_\Gamma \times \mathbf{u}\big\|_{H_0^{1/2}(\Gamma)} \tag{4.20}
$$

*where $C$ is a positive constant independent of $h$.*

**Proof.** We prove it by contradiction. Suppose now that the assertion is not true, for any given $p \in C^2(S, \mathbb{R})$, there exists a subsequence from $\{\mathbf{u}^h\}$, which is still denoted as $\{\mathbf{u}^h\}$ for simplicity, such that

$$\left\| \frac{\mathbf{n}_\Gamma \times \mathbf{u}^h - \mathbf{n}_\Gamma \times \mathbf{u}}{h} \right\|_{H_0^{1/2}(\Gamma)} \to \left\| \mathbf{n}_\Gamma \times \mathbf{u}' \right\|_{H_0^{1/2}(\Gamma)} = 0 \quad \text{as } h \to 0, \tag{4.21}$$

which yields $\mathbf{n}_\Gamma \times \mathbf{u}' = 0$ on $\Gamma$. Based on Theorem 4.2, it follows from the boundary condition of $\mathbf{n}_\Gamma \times \mathbf{u}'$ on $\Gamma$ in (4.9) and the injectivity of the boundary operator $T$ that $(\nabla \times \mathbf{u}') \times \mathbf{n}_\Gamma = 0$ on $\Gamma$. An application of Holmgren's uniqueness theorem yields that $\mathbf{u}' = 0$ in $\mathbb{R}^3_+$. We infer by unique continuation that $\mathbf{u}' = 0$ in $\Omega$. The boundary condition of $\mathbf{u}'$ in (4.10) gives

$$\mathbf{n}_S \times \mathbf{u}' = \left[ p \partial_S \mathbf{u}_S + (\mathbf{n}_S \cdot \mathbf{u}) \nabla_S p \right] \times \mathbf{n}_S = 0 \quad \text{on } S.$$

Since $p$ is arbitrary, we have $\partial_S \mathbf{u}_S = 0$ and $\mathbf{n}_S \cdot \mathbf{u} = 0$, which yields $\partial_S \mathbf{u} = 0$. Since both the tangential trace $\mathbf{n}_S \times \mathbf{u} = 0$ and the normal component $\mathbf{n}_S \cdot \mathbf{u} = 0$, then $\mathbf{u} = 0$ on $S$. Therefore $\mathbf{u} = 0$ and $\partial_S \mathbf{u} = 0$ on $S$. We infer by unique continuation once again that $\mathbf{u} = 0$ in $\Omega$, which is a contradiction to the transparent boundary condition (3.1). □

The result indicates that for small $h$, if the boundary measurements are $O(h)$ close to the wave field in the $H_0^{1/2}(\Gamma)$ norm, then $\Omega_h$ is $O(h)$ close to $\Omega$ in the Hausdorff distance.

## 5. Concluding remarks

In this paper, we have presented two results for the inverse electromagnetic cavity scattering problem: a global uniqueness and a local stability. The uniqueness shows that the cavity shape is uniquely determined by the tangential trace of the electric field measured on the cavity aperture corresponding to a single incoming wave; the stability shows that the Hausdorff distance of two cavities is bounded above by the distance of corresponding tangential trace of the electric fields if they are close enough. To prove the stability, a crucial step is to establish the domain derivative of the electric field with respect to the change of the cavity shape. We have deduced that the domain derivative satisfies a boundary value problem of the Maxwell equation, which is similar to the model problem of the direct scattering.

Throughout, the medium is assumed to be lossy inside the cavity. A challenging problem is to investigate the case of lossless medium in the whole physical domain. Although the global uniqueness may not be possible, it is desirable to extract the information: to what extend the uniqueness may become available? Another interesting project is concerned with the optimal design problem, which is to design the cavity structure and material to reduce or enhance the radar cross section. The results will be reported elsewhere.

## References

[1] T. Abbound, J.-C. Nédélec, Electromagnetic waves in an inhomogeneous medium, J. Math. Anal. Appl. 164 (1992) 40–58.
[2] H. Ammari, Théorèmes d'unicité pour un problèm inverse dans une structure bipériodique, C. R. Acad. Sci. Paris Sér. I 320 (1995) 815–820.
[3] H. Ammari, G. Bao, A. Wood, An integral equation method for the electromagnetic scattering from cavities, Math. Methods Appl. Sci. 23 (2000) 1057–1072.
[4] H. Ammari, G. Bao, A. Wood, Analysis of the electromagnetic scattering from a cavity, Jpn. J. Ind. Appl. Math. 19 (2001) 301–308.
[5] H. Ammari, G. Bao, A. Wood, A cavity problem for Maxwell's equation, Methods Appl. Anal. 9 (2002) 249–260.
[6] G. Bao, A uniqueness theorem for an inverse problem in periodic diffractive optics, Inverse Problems 10 (1994) 335–340.
[7] G. Bao, A. Friedman, Inverse problems for scattering by periodic structures, Arch. Ration. Mech. Anal. 132 (1995) 49–72.
[8] G. Bao, J. Gao, P. Li, Analysis of direct and inverse cavity scattering problems, Numer. Math. Theory Methods Appl. 4 (2011) 419–442.
[9] G. Bao, W. Sun, A fast algorithm for the electromagnetic scattering from a large cavity, SIAM J. Sci. Comput. 27 (2005) 553–574.

[10] G. Bao, H. Zhang, J. Zou, Unique determination of periodic polyhedral structures by scattered electromagnetic fields, Trans. Amer. Math. Soc. 363 (2011) 4527–4551.
[11] G. Bao, Z. Zhou, An inverse problem for scattering by a doubly periodic structure, Trans. Amer. Math. Soc. 350 (1998) 4089–4103.
[12] J. Cagnol, M. Eller, Boundary regularity for Maxwell's equations with application to shape optimization, J. Differential Equations 250 (2011) 1114–1136.
[13] D. Colton, R. Kress, Inverse Acoustic and Electromagnetic Scattering Theory, 2nd ed., Appl. Math. Sci., vol. 93, Springer-Verlag, Berlin, 1998.
[14] M. Delfour, J.-P. Zolésio, Oriented distance function and its evolution equation for initial sets with thin boundary, SIAM J. Control Optim. 42 (2004) 2286–2304.
[15] J. Elschner, G. Hu, Global uniqueness in determining polygonal periodic structures with a minimal number of incident plane waves, Inverse Problems 26 (2010) 115002.
[16] M. Eller, V. Isakov, G. Nakamura, D. Tataru, Uniqueness and stability in the Cauchy problem for Maxwell's and elasticity systems, in: D. Gioranescu, J.-L. Lions (Eds.), Nonlinear Partial Differential Equations and Their Applications, vol. 14, North-Holland, Elsevier Science, 2002, pp. 329–351.
[17] L. Feng, F. Ma, Uniqueness and local stability for the inverse scattering problem of determining the cavity, Sci. China Ser. A 48 (2005) 1113–1123.
[18] H. Haddar, R. Kress, On the Fréchet derivative for obstacle scattering with an impedance boundary condition, SIAM J. Appl. Math. 65 (2004) 194–208.
[19] F. Hettlich, Fréchet derivative in inverse obstacle scattering, Inverse Problems 11 (1995) 371–382.
[20] H. Liu, M. Yamamoto, J. Zou, Reflection principle for Maxwell's equations and its application to inverse electromagnetic scattering problem, Inverse Problems 23 (2007) 2357–2366.
[21] J.-J. Liu, On uniqueness and linearization of an inverse electromagnetic scattering problem, Appl. Math. Comput. 171 (2005) 406–419.
[22] X. Liu, B. Zhang, J. Yang, The inverse electromagnetic scattering problem in a piecewise homogeneous medium, Inverse Problems 26 (2010) 125001.
[23] J.M. Jin, The Finite Element Method in Electromagnetics, Wiley & Sons, New York, 2002.
[24] J.M. Jin, J.L. Volakis, A hybrid finite element method for scattering and radiation by micro strip patch antennas and arrays residing in a cavity, IEEE Trans. Antennas and Propagation 39 (1991) 1598–1604.
[25] A. Kirsch, The domain derivative and two applications in inverse scattering theory, Inverse Problems 9 (1993) 301–310.
[26] J. Liu, J.M. Jin, A special higher order finite-element method for scattering by deep cavities, IEEE Trans. Antennas and Propagation 48 (2000) 694–703.
[27] P.S. Meyer, M. Capistran, Y. Chen, On the naturally induced sources for obstacle scattering, Commun. Comput. Phys. 1 (2006) 974–983.
[28] P. Monk, Finite Element Methods for Maxwell's Equations, Oxford University Press, New York, 2003.
[29] J.-C. Nédélec, Acoustic and Electromagnetic Equations: Integral Representations for Harmonic Problems, Springer, New York, 2000.
[30] R. Potthast, Domain derivatives in electromagnetic scattering, Math. Methods Appl. Sci. 19 (1996) 1157–1175.
[31] T. Van, A. Wood, Finite element analysis for 2-D cavity problem, IEEE Trans. Antennas and Propagation 51 (2003) 1–8.
[32] A. Wood, Analysis of electromagnetic scattering from an overfilled cavity in the ground plane, J. Comput. Phys. 215 (2006) 630–641.
[33] W. Wood, A. Wood, Development and numerical solution of integral equations for electromagnetic scattering from a trough in a ground plane, IEEE Trans. Antennas and Propagation 47 (1999) 1318–1322.