Numerical solution of an inverse diffraction grating problem from phaseless data

Gang Bao,[2,3] Peijun Li,[4] and Junliang Lv[1,2,*]

[1]School of Mathematics, Jilin University, Changchun 130012, China
[2]Department of Mathematics, Zhejiang University, Hangzhou 310027, China
[3]Department of Mathematics, Michigan State University, East Lansing, Michigan 48824, USA
[4]Department of Mathematics, Purdue University, West Lafayette, Indiana 47907, USA
*Corresponding author: junlianglv@gmail.com

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This paper is concerned with the numerical solution of an inverse diffraction grating problem, which is to reconstruct a periodic grating profile from measurements of the phaseless diffracted field at a constant height above the grating structure. An efficient continuation method is developed to recover the Fourier coefficients of the periodic grating profile. The continuation proceeds along the wavenumber and updates are obtained from the Landweber iteration at each step. Numerical results are presented to show that the method can effectively reconstruct the shape of the grating profile.

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1. INTRODUCTION

Diffractive optics is an important technology in science and engineering. One of its important applications is the design and fabrication of optic elements with periodic structures, often called diffraction gratings. The diffraction of time-harmonic electromagnetic waves by some periodic material can be reduced to a model problem of the two-dimensional Helmholtz equation. We refer to the monographs [1,2] for an introduction to the theory of electromagnetic diffraction by periodic structures. Numerical methods for forward problems can be found in [3–6].

A great deal of motivating applications are associated with the determination of the nature of a periodic structure from a measured diffracted field. For example, in the study of optimal design problems in diffractive optics, it is intended to design a grating structure that gives rise to some specified far-field patterns [7–10]. Uniqueness results and stability estimates for the inverse diffraction problem were obtained in [11–18]. Due to the nonlinearity and ill-posedness, it is challenging to develop efficient and stable numerical methods to solve the inverse diffraction problems. For an overview on inverse scattering problems in general (nonperiodic) structures, we refer to the book by Colton and Kress [19] and the references cited therein.

In this work we restrict our attention to the two-dimensional transverse electric (TE) case for perfectly conducting gratings. A number of numerical methods have been developed to solve these inverse problems. Within the range of validity of Rayleigh's hypothesis, Garcia and Nieto-Vesperinas [20] considered a near-field optics method. Ito and Reitich [21] proposed a high-order perturbation approach based on the methods of variation of boundaries. In [22], Arens and Kirsch applied the factorization method to scattering by a periodic surface. Iterative regularization methods were developed by Hettlich in [23] based on the shape derivatives with respect to the variations of the boundary. By reformulating the original inverse scattering as an optimization problem, Bruckner and Elschner [24] gave a two-step algorithm to reconstruct the grating profile. Elschner *et al.* [25] proposed an algorithm for the recovery of a two-dimensional periodic structure based on finite elements and optimization techniques. Recently, Bao *et al.* [26] presented an efficient continuation method to capture both the macro and micro structures of the grating profiles with multiple frequency data.

In applications, it is much harder to obtain data with accurate phase information than to just measure the intensity of the data. Therefore, it is often desirable to reconstruct the grating profile from the phaseless diffracted field at a constant height above the grating structure. However, little has been studied on such a problem both mathematically and numerically. Recently, Bao *et al.* [27] presented a continuation method to deal with the phaseless measurements for an inverse source problem. In this paper, we consider the problem of recovering a grating profile from the magnitude information of the diffracted field measured at a constant height above the grating structure. Following the idea in [27], we propose an efficient continuation method to solve the nonlinear inverse diffraction grating problem in a perfectly conducting structure. In order to recover both the macro information and the micro information of the grating profiles, we use the multiple frequency data in the algorithm, and the iterative steps are obtained by a continuation method with respect to the wavenumber. With the starting point given by the output from the previous step at a lower wavenumber, a new approximation to the grating surface filtered at a higher frequency is updated by a Landweber iteration. Our algorithm is based on the shape derivative with respect to variations of the boundary. Numerical results show that the continuation method cannot determine the location of the grating structure, but it can effectively reconstruct the grating shape from the phaseless data.

The rest of this paper is outlined as follows. In Section 2, we present the mathematical formulations of the forward and inverse diffraction problems. Section 3 is devoted to the continuation method for solving the inverse diffraction problem, along with details of the implementation. Numerical examples are presented in Section 4 to illustrate the performance of the method. The paper is concluded with some remarks in Section 5.

2. FORWARD AND INVERSE PROBLEMS

In this section, we outline the mathematical modeling of the diffraction grating. Throughout, we assume that the grating profile is a periodic function of the variable x with period 2π. Thus the problem can be restricted into a single period in x. Let the profile of the diffraction grating in one period be described by the curve

$$\Gamma_f = \{(x,y) \in \mathbb{R}^2 : y = f(x), 0 < x < 2\pi\},$$

where f is a periodic function of period 2π. Suppose that the domain

$$\Omega_f = \{(x,y) \in \mathbb{R}^2 : y > f(x), 0 < x < 2\pi\}$$

is filled with a homogeneous medium with a positive constant wavenumber κ. Let the plane wave

$$u^{\mathrm{inc}} = e^{i(\alpha x - \beta y)}$$

be incident on the grating surface Γ_f from the top, where $\alpha = \kappa \sin\theta$, $\beta = \kappa\cos\theta$, and $\theta \in (-\pi/2, \pi/2)$ is the angle of incidence. For $n \in \mathbb{Z}$, let $\alpha_n = \alpha + n$, and denote

$$\beta_n = \begin{cases} \sqrt{\kappa^2 - \alpha_n^2}, & \text{for } \kappa > |\alpha_n|, \\ i\sqrt{\alpha_n^2 - \kappa^2}, & \text{for } \kappa < |\alpha_n|. \end{cases}$$

We further exclude resonances by assuming that $\kappa \neq |\alpha_n|$ for all $n \in \mathbb{Z}$.

The forward diffracting problem in the TE mode by the perfectly conducting grating is to find the diffracted field u such that

$$\Delta u + \kappa^2 u = 0, \qquad \text{in } \Omega_f, \tag{1}$$

$$u + u^{\mathrm{inc}} = 0, \qquad \text{on } \Gamma_f. \tag{2}$$

Motivated by a uniqueness consideration, we seek the quasi-periodic solution; i.e., $u(x,y)e^{-i\alpha x}$ is a periodic function in x with period 2π. Moreover, the diffracted field u is required to satisfy an outgoing wave condition, which leads to the following Rayleigh expansion:

$$u = \sum_{n\in\mathbb{Z}} A_n e^{i\alpha_n x + i\beta_n y}, \qquad y > \max_{x\in(0,2\pi)} f(x), \tag{3}$$

with the Rayleigh coefficients $A_n \in \mathbb{C}$. Since β_n is real for at most a finite number of indices, we see that only a finite number of plane waves in the expansion (3) propagate into the far field, whereas the remaining terms represent the evanescent waves that exponentially decay with respect to y and $|n|$.

An inverse diffraction problem for the perfectly conducting grating can be formulated as follows: given the incident wave u^{inc}, the problem is to determine the profile $y = f(x)$ from the measurements of the diffracted field at a straight line segment:

$$\Gamma_0 = \{(x, y_0) \in \mathbb{R}^2 : x \in (0, 2\pi), \qquad y_0 > \max_{x\in(0,2\pi)} f(x)\},$$

i.e., the near-field data $u(x, y_0)$.

For a fixed incident field u^{inc}, we define a forward scattering operator $\mathcal{K} : C^2([0,2\pi]) \to L^2([0,2\pi])$ by

$$\mathcal{K}(f) = u(x, y_0). \tag{4}$$

It is shown by Kirsch [28] that the operator $\mathcal{K}$ is Fréchet differentiable and its Fréchet derivative can be characterized by a quasi-periodic solution u' of the Helmholtz equation.

Theorem 2.1. The operator $\mathcal{K}$ is Fréchet differentiable and its domain derivative

$$\mathcal{K}_f'(h) = u'(x, y_0)$$

at f in direction h is given by the quasi-periodic solution of the boundary value problem

$$\Delta u' + \kappa^2 u' = 0 \qquad \text{in } \Omega_f,$$

$$u' = -\frac{h}{\sqrt{1 + (f')^2}} \partial_{\mathbf{n}} u \qquad \text{on } \Gamma_f,$$

where $\mathbf{n}$ is the unit normal vector at Γ_f directed into Ω_f.

Our goal in this paper is to study an inverse problem of the profile reconstruction from the phaseless data: given the incident wave u^{inc}, the problem is to determine the profile $y = f(x)$ from the measurements of the phaseless diffracted field at Γ_0, i.e., the near-field data $|u(x, y_0)|^2$.

Define an operator $\mathcal{F}$ that maps the grating profile function $f(x)$ to the near-field phaseless data:

$$\mathcal{F}(f) = |u(x, y_0)|^2 = \mathcal{K}(f)\bar{\mathcal{K}}(f), \tag{5}$$

where the operator $\mathcal{K}$ is defined in Eq. (4) and the bar denotes the complex conjugate. The inverse problem can be formulated as follows: given the incident wave u^{inc}, the problem is to determine the periodic function $y = f(x)$ such that

$$\mathcal{F}(f) = |u(x, y_0)|^2. \tag{6}$$

Next we investigate some properties of the forward scattering operator $\mathcal{F}$.

Theorem 2.2. Assume $h, g \in C^2([0, 2\pi])$. The operator $\mathcal{F}(f)$ is Fréchet differentiable and its domain derivative can be represented by

$$\mathcal{F}_f'(h) = 2\,\mathrm{Re}[\bar{\mathcal{K}}(f)\mathcal{K}_f'(h)] = 2\,\mathrm{Re}[\bar{u}(x, y_0)u'(x, y_0)]$$

at f in direction h.

Proof. It follows from the well-known regularity results (cf. [29]) that the solutions $u_{f+h} = \mathcal{K}(f + h)$, $u_f = \mathcal{K}(f)$, and $u' = \mathcal{K}_f'(h)$ are all in $C^1([0, 2\pi])$ since f and h are both in $C^2([0, 2\pi])$.

By Theorem 2.1 (cf. [28]), we have

$$\mathcal{K}(f+h) = \mathcal{K}(f) + \mathcal{K}_f'(h) + \mathcal{R}_f(h), \tag{7}$$

where $\mathcal{R}_f(h) \in C^1([0,2\pi])$ with $\mathcal{R}_f(h) = o(\|h\|_{1,\infty})$.

Using Eqs. (5) and (7), we have from simple calculations that

$$\begin{aligned}\mathcal{F}(f+h) &= \mathcal{K}(f+h)\bar{\mathcal{K}}(f+h)\\ &= [\mathcal{K}(f) + \mathcal{K}_f'(h) + \mathcal{R}_f(h)][\bar{\mathcal{K}}(f) + \bar{\mathcal{K}}_f'(h) + \bar{\mathcal{R}}_f(h)]\\ &= \mathcal{F}(f) + 2\,\mathrm{Re}[\bar{\mathcal{K}}(f)\mathcal{K}_f'(h)] + |\mathcal{K}_f'(h)|^2 + |\mathcal{R}_f(h)|^2\\ &\quad + 2\,\mathrm{Re}[\mathcal{R}_f(h)(\bar{\mathcal{K}}(f) + \bar{\mathcal{K}}_f'(h))].\end{aligned}$$

Due to the Fréchet differentiability of $\mathcal{K}(f)$, the domain derivative $\mathcal{K}_f'$ is a bounded linear operator and it holds $\mathcal{K}_f'(h) = O(\|h\|_{1,\infty})$.

Combining the above estimates, we deduce that

$$\mathcal{K}(f+h) = \mathcal{K}(f) + 2\,\mathrm{Re}[\bar{\mathcal{K}}(f)\mathcal{K}_f'(h)] + o(\|h\|_{1,\infty}),$$

which proves the theorem.

To compute the updates in the Landweber iterations, we need to consider the adjoint operator of $\mathcal{F}_f'(h)$. Recall $\beta_n \neq 0$; the free-space quasi-periodic Green's function is

$$G(x,y;s,t) = \frac{i}{2\pi}\sum_{n\in\mathbb{Z}}\frac{1}{\beta_n}\exp(i\alpha_n(x-s) + i\beta_n|y-t|).$$

For any real function $g \in L^2([0,2\pi)]$, we define an auxiliary function

$$w_g^i(s,t) = \int_0^{2\pi} G(x,y_0;s,t)\bar{u}(x,y_0)g(x)\mathrm{d}x, \qquad t \neq y_0,$$

where $u(x,y_0)$ is the diffracted field on Γ_0.

Theorem 2.3. Given a real function $g \in L^2([0,2\pi])$, the adjoint operator of $\mathcal{F}_f'$ is given by

$$(\mathcal{F}_f')^*g = -2\,\mathrm{Re}[\partial_{\mathbf{n}}u(x,f(x))(\partial_{\mathbf{n}}w_g^i(x,f(x)) + \partial_{\mathbf{n}}w_g(x,f(x)))],$$

where u is the solution of the diffracted problem and w_g is the quasi-periodic solution of the following boundary value problem:

$$\Delta w_g + \kappa^2 w_g = 0 \qquad \text{in } \Omega_f,$$

$$w_g + w_g^i = 0 \qquad \text{on } \Gamma_f.$$

Proof. It follows from the integral representation theorem for the Helmholtz equation and the quasi-periodic Rayleigh expansion of u' that we have

$$u'(x,y_0) = \int_{\Gamma_f}[u'(s,t)\partial_{\mathbf{n}}G(x,y_0;s,t) - \partial_{\mathbf{n}}u'(s,t)G(x,y_0;s,t)]\mathrm{d}S.$$

By Theorem 2.2, we get

$$\begin{aligned}\langle\mathcal{F}_f'(h),g\rangle &= \int_0^{2\pi} 2\,\mathrm{Re}[\bar{u}(x,y_0)u'(x,y_0)]g(x)\mathrm{d}x\\ &= 2\,\mathrm{Re}\int_0^{2\pi}\bar{u}(x,y_0)g(x)\int_{\Gamma_f}[u'(s,t)\partial_{\mathbf{n}}G(x,y_0;s,t)\\ &\quad - \partial_{\mathbf{n}}u'(s,t)G(x,y_0;s,t)]\mathrm{d}S\mathrm{d}x\\ &= 2\,\mathrm{Re}\int_{\Gamma_f}u'(s,t)\int_0^{2\pi}\bar{u}(x,y_0)\partial_{\mathbf{n}}G(x,y_0;s,t)g(x)\mathrm{d}x\mathrm{d}S\\ &\quad - 2\,\mathrm{Re}\int_{\Gamma_f}\partial_{\mathbf{n}}u'(s,t)\int_0^{2\pi}\bar{u}(x,y_0)G(x,y_0;s,t)g(x)\mathrm{d}x\mathrm{d}S\\ &= 2\,\mathrm{Re}\int_{\Gamma_f}u'(s,t)\partial_{\mathbf{n}}w_g^i(s,t)\mathrm{d}S\\ &\quad - 2\,\mathrm{Re}\int_{\Gamma_f}\partial_{\mathbf{n}}u'(s,t)w_g^i(s,t)\mathrm{d}S\\ &= 2\,\mathrm{Re}\int_{\Gamma_f}u'(s,t)\partial_{\mathbf{n}}w_g^i(s,t)\mathrm{d}S\\ &\quad + 2\,\mathrm{Re}\int_{\Gamma_f}\partial_{\mathbf{n}}u'(s,t)w_g(s,t)\mathrm{d}S,\end{aligned}$$

where the boundary condition for w_g is used in the last equality. Because u' and w_g are quasi-periodic solutions of the Helmholtz equation, we may apply the Green's theorem and the Rayleigh expansions of u' and w_g to obtain

$$\int_{\Gamma_f}(u'\partial_{\mathbf{n}}w_g - \partial_{\mathbf{n}}u'w_g)\mathrm{d}S = 0.$$

It follows from the boundary condition of u' that

$$\begin{aligned}\langle h,(\mathcal{F}_f')^*g\rangle &= \langle\mathcal{F}_f'h,g\rangle\\ &= 2\,\mathrm{Re}\int_{\Gamma_f}u'(s,t)(\partial_{\mathbf{n}}w_g^i(s,t) + \partial_{\mathbf{n}}w_g(s,t))\mathrm{d}S\\ &= -2\int_0^{2\pi}h(x)\mathrm{Re}[\partial_{\mathbf{n}}u(x,f(x))\\ &\quad \times(\partial_{\mathbf{n}}w_g^i(x,f(x)) + \partial_{\mathbf{n}}w_g(x,f(x)))]\mathrm{d}x,\end{aligned}$$

which completes the proof.

3. RECONSTRUCTION ALGORITHM

In this section, we present a continuation algorithm to reconstruct the grating profile. Implicitly, let $\mathcal{K}_\kappa$ and $\mathcal{F}_\kappa$ denote the operators $\mathcal{K}$ and $\mathcal{F}$ at the wavenumber κ, respectively. By Theorem 2.2, the Fréchet derivative of $\mathcal{F}_\kappa(f)$ is

$$\mathcal{F}_{\kappa,f}' = 2\,\mathrm{Re}\,\bar{\mathcal{K}}_\kappa(f)\mathcal{K}_{\kappa,f}', \tag{8}$$

where $\mathcal{K}_{\kappa,f}'$ is the Fréchet derivative of the operator $\mathcal{K}_\kappa$. Here the subscript f is introduced to denote that the Fréchet derivatives are linear operators depending on f at the fixed κ.

Next we describe how the continuation method proceeds along with the wavenumber to reconstruct the grating profile. Assume that at $\kappa = \kappa_m$, the reconstructed grating profile function is $f_m(x)$. At a higher wavenumber $\kappa = \kappa_{m+1}$, the forward map $\mathcal{F}_{\kappa_{m+1}}(f)$ can be approximated at the previously reconstructed grating profile function f_m:

$$\mathcal{F}_{\kappa_{m+1}}(f) \approx \mathcal{F}_{\kappa_{m+1}}(f_m) + \mathcal{F}'_{\kappa_{m+1}f_m}(f - f_m). \tag{9}$$

Note that $\mathcal{F}_{\kappa_{m+1}}(f)$ is the measured intensity data $|u(x, y_0)|^2$ of the diffracted field at the wavenumber κ_{m+1}. We denote the residual by $\mathcal{R}_{m+1} = \mathcal{F}_{\kappa_{m+1}}(f) - \mathcal{F}_{\kappa_{m+1}}(f_m)$. Let $\delta f = f - f_m$; then the formula can be rewritten as the following linearized equation:

$$\mathcal{F}'_{\kappa_{m+1}f_m}\delta f = \mathcal{R}_{m+1}. \tag{10}$$

An application of the Landweber regularization method to solve the linearized equation yields

$$\delta f = \tau(\mathcal{F}'_{\kappa_{m+1}f_m})^* \mathcal{R}_{m+1},$$

where $(\mathcal{F}'_{\kappa_{m+1}f_m})^*$ is the adjoint operator of $\mathcal{F}'_{\kappa_{m+1}f_m}$ and τ is a relaxation parameter. Therefore, the reconstruction at $\kappa = \kappa_{m+1}$ is updated by setting

$$f_{m+1} = f_m + \delta f. \tag{11}$$

The above description is for general function $f(x)$. Since the grating profile $f(x)$ is a periodic function of period 2π, we can choose trigonometric polynomials as a finite-dimensional basis, i.e.,

$$f(x) \approx c_0 + \sum_{m=1}^{M}[c_{2m-1}\cos(mx) + c_{2m}\sin(mx)]. \tag{12}$$

Clearly, it is desirable to determine all the Fourier coefficients $c_0, c_1, c_2, \ldots c_{2M-1}, c_{2M}$ in order to reconstruct an approximated grating profile. For the convenience of the description of the algorithm, we decompose the sum in Eq. (12) into low-frequency and high-frequency parts:

$$f(x) \approx c_0 + \sum_{m=1}^{N}[c_{2m-1}\cos(mx) + c_{2m}\sin(mx)] + \sum_{m=N+1}^{M}[c_{2m-1}\cos(mx) + c_{2m}\sin(mx)], \tag{13}$$

where $N < M$ is some nonnegative integer.

Specifically, the reconstruction algorithm from intensity with multifrequency data is summarized as the following three steps:

Step 1. Initialization. Choose a nonnegative integer N according to specific problems. Given an initial value for the wavenumber κ_0, we choose an initial guess $c_0 = f_0$ with $c_j = 0, j = 1, 2, \ldots, 2N + 2k_0$, where k_0 is taken to be the largest integer that is smaller or equal to the wavenumber κ_0.

Step 2. Update the reconstructed grating profile function at wavenumber κ by linearization. Denote $\mathbf{C}_k = [c_0, c_1, \ldots,$

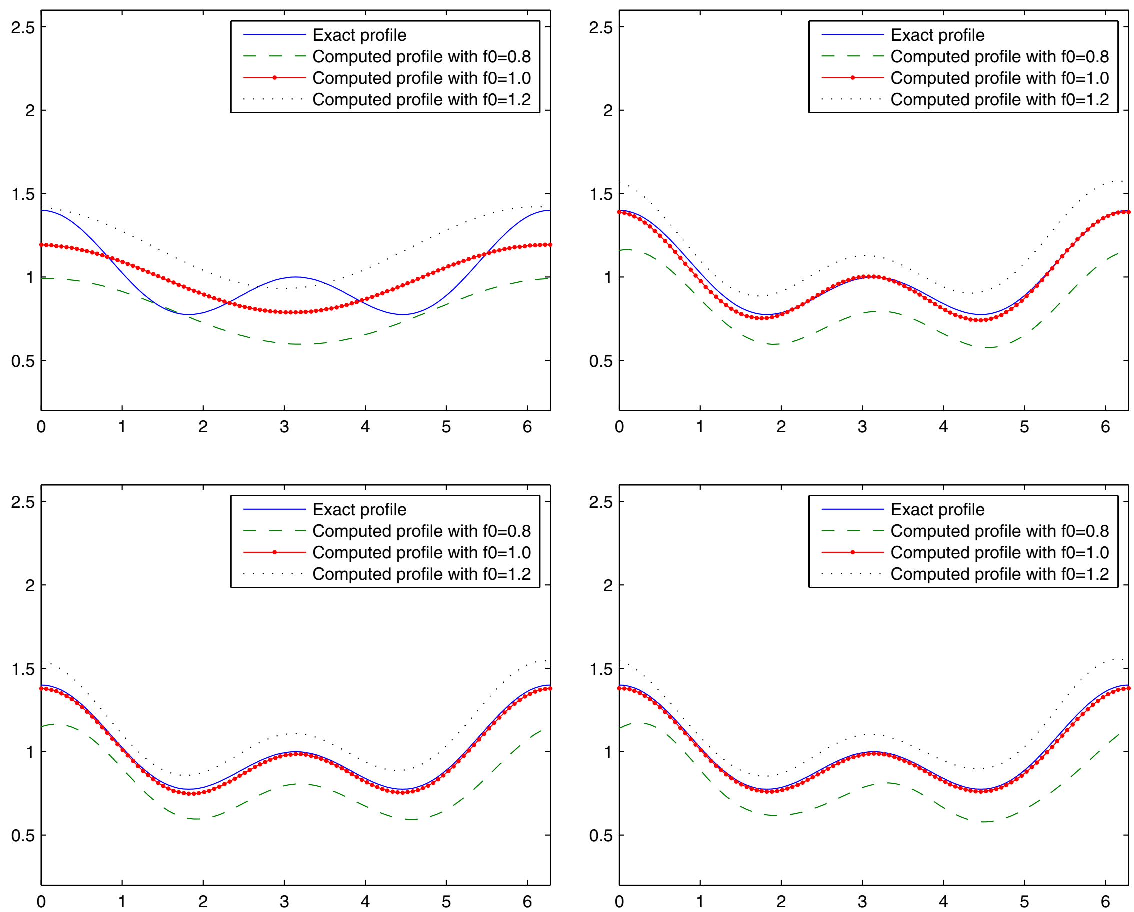

Fig. 1. (Color online) Evolution of the reconstructions in Example 1. Left column from top to bottom: reconstructions at $\kappa = 1$, reconstructions at $\kappa = 3$. Right column from top to bottom: reconstructions at $\kappa = 2$, reconstructions at $\kappa = 4$.

$c_{2(N+k)-1}, c_{2(N+k)}]^\top$, where k is taken to be the largest integer that is smaller than or equal to the wavenumber κ. Note that some entries in $\mathbf{C}_k$ have been obtained at the previous wavenumber. Set initial vector $\mathbf{C}_k^{(0)} = \mathbf{C}_k$. For $n = 0, 1, \ldots$, solve the forward problem

$$\Delta u_n + \kappa^2 u_n = 0 \qquad \text{in } \Omega_{f_k^{(n)}},$$

$$u_n + e^{i(\alpha x - \beta f_k^{(n)}(x))} = 0 \qquad \text{on } \Gamma_{f_k^{(n)}},$$

by the integral equation method [23]. We get a solution $u_n(x, y)$ and its normal derivative $\partial_{\mathbf{n}} u_n$ on $\Gamma_{f_k^{(n)}}$. Here $f_k^{(n)}(x) = \Phi_k \mathbf{C}_k^{(n)}$, where the row vector

$$\Phi_k = [1, \cos x, \sin x, \ldots, \cos((N+k)x), \sin((N+k)x)].$$

Thus we have

$$\mathcal{K}_\kappa(f_k^{(n)}) = u_n(x, y_0),$$

$$\mathcal{F}_\kappa(f_k^{(n)}) = |u_n(x, y_0)|^2.$$

The residual is

$$\mathcal{R}^{(n)} = \mathcal{F}_\kappa(f) - \mathcal{F}_\kappa(f_k^{(n)}),$$

where $\mathcal{F}_\kappa(f)$ is the measured data on Γ_0 at the wavenumber κ. We consider the Landweber iteration

$$f_k^{(n+1)} = f_k^{(n)} + \tau_\kappa (\mathcal{F}'_{\kappa, f_k^{(n)}})^* \mathcal{R}^{(n)}, \tag{14}$$

where τ_κ is a relaxation parameter. Since $f_k'(x)$ and $f_k''(x)$ are required in the integral equation method, we solve the equation

$$\Phi_k \mathbf{C}_k^{(n)} = f_k^{(n)}$$

to obtain $\mathbf{C}_k^{(n)}$. A stopping rule for the iteration (14) is set as follows: the loop stops when

$$\|\mathbf{C}_k^{(n+1)} - \mathbf{C}_k^{(n)}\| > \|\mathbf{C}_k^{(n)} - \mathbf{C}_k^{(n-1)}\|. \tag{15}$$

The resulting solution represents the Fourier coefficients of $f(x)$ corresponding to the frequencies not exceeding $N + k$.

Step 3. March along the wavenumber. Increase the wavenumber κ to a new value $\tilde{\kappa} > \kappa$ and seek a new approximation to the profile function $f(x)$ by the Fourier series

$$f_{\tilde{k}}(x) = \tilde{c}_0 + \sum_{m=1}^{N+\tilde{k}} [\tilde{c}_{2m-1} \cos(mx) + \tilde{c}_{2m} \sin(mx)]$$

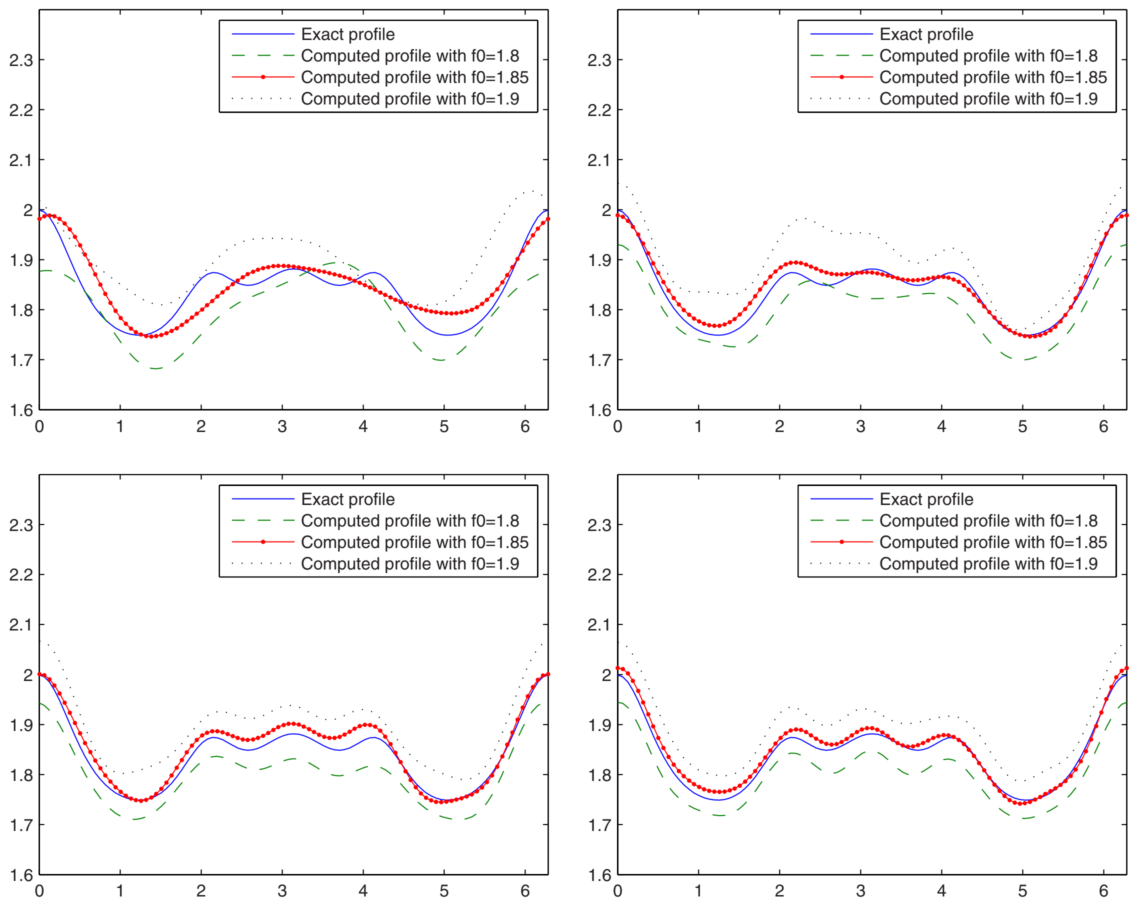

Fig. 2. (Color online) Evolution of the reconstructions in Example 2. Left column from top to bottom: reconstructions at $\kappa = 1$, reconstructions at $\kappa = 2$. Right column from top to bottom: reconstructions at $\kappa = 3$, reconstructions at $\kappa = 4$.

and determine the coefficients $\tilde{c}_j, j = 0, 1, \ldots, 2(N + \tilde{k})$, where $\tilde{k}$ is again the largest integer smaller than or equal to the wavenumber $\tilde{\kappa}$. Specifically, we repeat Step 2 with the previous approximation to $f(x)$ as our starting point:

$$\tilde{c}_j = \begin{cases} c_j, & \text{for } j \leq 2(N+k), \\ 0, & \text{for } j > 2(N+k), \end{cases}$$

where the coefficients c_j come from Step 2. The resulting solution in this step represents the Fourier coefficients of $f(x)$ corresponding to the frequencies not exceeding $N + \tilde{k}$. Repeat Step 3 until a prescribed frequency is reached.

Next, we discuss some practical implementation issues. The above algorithm requires the numerical solutions of two forward diffraction grating problems: solve one forward problem for u_n and $\partial_{\mathbf{n}} u_n$ with the obtained profile function $f_k^{(n)}$; solve another forward problem for w_g to obtain the updates. Since u_n and w_g are quasi-periodic solutions of the same Helmholtz equation, they can be obtained by the integral equation method [23,30]. In our algorithm, the Landweber iteration should be stopped after finite steps according to the stopping criterion described in Step 2. The choice of N relies on the *a priori* information about the grating profile. If f has finitely many Fourier modes, a small N can avoid unnecessary computation. If f contains infinitely many Fourier modes, each iteration can obtain more information with a larger N at lower wavenumbers.

4. NUMERICAL EXPERIMENTS

In this section, some results of numerical experiments are presented to illustrate the performance of our method. We consider three gratings, and all of the near-field measurements $|u(x, y_0)|^2$ are simulated by solving the direct problem with added noise, i.e.,

$$|u(x, y_0)|^2 := |u(x, y_0)|^2(1 + \sigma\,\mathrm{rand}), \tag{16}$$

where "rand" represents normally distributed random numbers in $[-1, 1]$ and σ is the noise level. In each example, we choose three different initial guesses and draw the reconstructions from them in the same figure. The maximum number of the Landweber iterations and the noise level σ are taken as 30 and 0.05, respectively.

Example 1. The first grating profile is given by

$$f(x) = 1.0 + 0.2\ \cos\ x + 0.2\ \cos\ 2x,$$

which is illuminated by an incident plane wave with incident direction $\theta = -\pi/3$. Intensity data at 100 equidistant points on

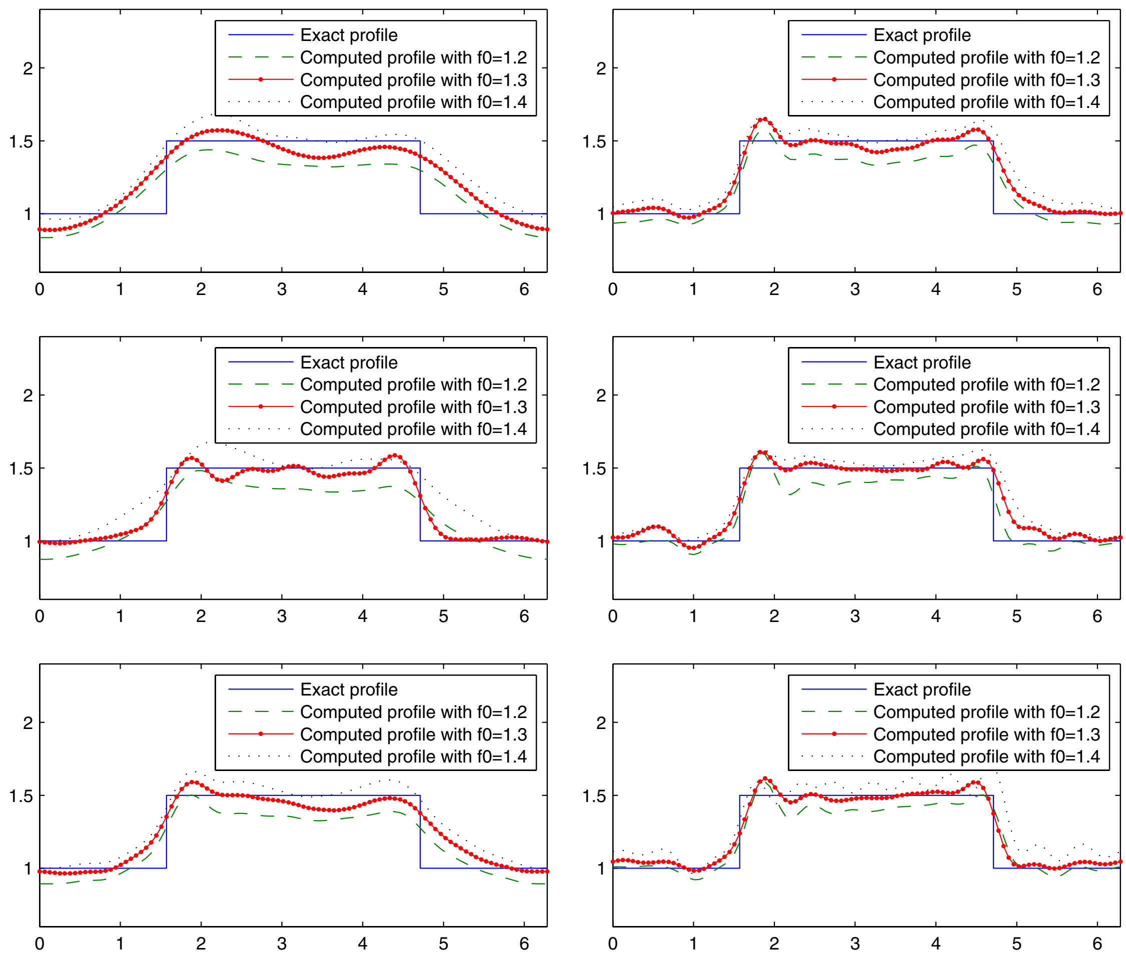

Fig. 3. (Color online) Evolution of the reconstructions in Example 3. Left column from top to bottom: reconstructions at $\kappa = 1$, reconstructions at $\kappa = 2$, and reconstructions at $\kappa = 3$. Right column from top to bottom: reconstructions at $\kappa = 4$, reconstructions at $\kappa = 5$, and reconstructions at $\kappa = 6$.

the line $y_0 = 1.7$ were computed by the integral equation method and then adding distributed random noise. Since the profile function only contains a few Fourier modes, we choose $N = 0$. Figure 1 shows the reconstructions by the Landweber iteration method with $\tau_\kappa = 0.05/\kappa$. Here the initial guesses are chosen as $f_0 = 0.8$, 1.0, and 1.2, respectively.

Example 2. The second grating profile is given by

$$f(x) = 1.7 + 0.06e^{\cos(2x)} + 0.05e^{\cos(3x)},$$

which is not in the finite-dimensional subspace of trigonometric polynomials where we seek reconstructions. The diffractive field is measured at $y_0 = 2.2$, and the initial guess is $f_0 = 1.8$, 1.85, and 1.9, respectively. The incident angle $\theta = \pi/4$, $N = 4$, and $\tau_\kappa = 0.05/\kappa$. The numerical results are shown in Fig. 2.

Example 3. The third grating profile is a binary grating

$$f(x) = \begin{cases} 1.5, & \text{for } x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \\ 1.0, & \text{otherwise.} \end{cases}$$

The intensity of the diffractive field is measured at the line $y_0 = 1.8$, and relaxation parameter $\tau_\kappa = 0.02/\kappa$. The initial guess is taken as $f_0 = 1.2$, 1.3, and 1.4, respectively. The incident angle $\theta = \pi/4$. Since the profile function contains infinitely many Fourier modes, we choose $N = 8$ in our experiments. The graphs of this tested profile and the reconstructed profiles with different wavenumbers are shown in Fig. 3.

From these examples, we observe that the proposed algorithm can effectively recover the shape of the grating profiles, but cannot accurately locate the position of the grating profiles unless a good initial guess of c_0 is available. The reason may be insufficient knowledge of the diffracted field, e.g., phase information of the data is missing.

5. CONCLUSION

In this paper, an efficient continuation method is proposed for reconstructing the diffraction grating profile from the intensity of the diffraction waves measured at a constant height above the grating structure. The induced nonlinear problem is linearized by a continuation method and then a Landweber iteration is applied at each step. At each iteration, two forward diffraction grating problems need to be solved. Although the numerical results are sensitive to initial guesses, the algorithm can effectively determine the shape of the grating profiles.

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