Journal of Computational Physics 240 (2013) 100–120

Contents lists available at SciVerse ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

A two-dimensional Helmhotlz equation solution for the multiple cavity scattering problem

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ARTICLE INFO

Article history:
Received 10 October 2012
Received in revised form 21 December 2012
Accepted 24 December 2012
Available online 1 February 2013

Keywords:
Electromagnetic cavity
Helmholtz equation
Variational formulations
Existence and uniqueness
Finite element method

ABSTRACT

Here considered is the mathematical analysis and numerical computation of the electromagnetic wave scattering by multiple cavities embedded in an infinite ground plane. Above the ground plane the space is filled with a homogeneous medium, while the interiors of the cavities are filled with inhomogeneous media characterized by variable permittivities. By introducing a new transparent boundary condition on the cavity apertures, the multiple cavity scattering problem is reduced to a boundary value problem of the two-dimensional Helmholtz equation imposed in the separated interior domains of the cavities. The existence and uniqueness of the weak solution for the model problem is achieved via a variational approach. A block Gauss–Seidel iterative method is introduced to solve the coupled system of the multiple cavity scattering problem, where only a single cavity scattering problem is required to be solved at each iteration. Numerical examples demonstrate the efficiency and accuracy of the proposed method.

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1. Introduction

The phenomenon of electromagnetic scattering by cavity-backed apertures has received much attention by both the engineering and mathematical communities for its important applications. For instance, the radar cross section is a measure of the detectability of a target by a radar system. Deliberate control in the form of enhancement or reduction of the radar cross section of a target is of no less importance than many radar applications. The cavity radar cross section caused by jet engine inlet ducts or cavity-backed antennas can dominate the total radar cross section. A thorough understanding of the electromagnetic scattering characteristic of a target, particularly a cavity, is necessary for successful implementation of any desired control of its radar cross section, and is of high interest to the scientific and engineering community. Another example is that the cavity can be used to model cracks or holes in metallic surfaces such as aircraft wings. These cracks or holes would be invisible to a visual inspection but may be revealed by understanding the scattering characteristics of the cavity. As an inverse problem, the mathematical model can serve as a predictor of the scattering of electromagnetic waves by the cavity for use in non-destructive testing. Besides, this work is also motivated by the study of the optimal design problems of the cavity, where one wishes to design a cavity structure that reduces or enhances the radar cross section.

For the time-harmonic analysis of cavity-backed apertures with penetrable material filling the cavity interior, we mention, for example, works by Jin et al. [23,24,27], Wood and Wood [37], and references cited therein. Mathematical analysis

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1 The research was supported in part by NSF Grants DMS-0914595, DMS-1042958, and DMS-1151308.
2 The research was supported in part by AFOSR Grants F1ATA02059J001 and F1ATA01103J001.

0021-9991/$ - see front matter Published by Elsevier Inc.
http://dx.doi.org/10.1016/j.jcp.2012.12.022

of the cavity scattering problem including the scattering from overfilled cavities, where the cavity aperture is not planar and may protrude the ground plane, can be found in Ammari et al. [1–3], Bao et al. [9], Li et al. [26], Van and Wood [28–33], Wood [36]. Mode matching based analytical approaches are developed in Bao and Zhang [13], and Bao et al. [10] for solving the electromagnetic scattering problem involving large cavities. Much research has been devoted to solving the cavity scattering problem by various numerical methods, including finite element, finite difference, boundary element, and hybrid methods. See, for example, Bao and Sun [11], Du [15,16], Huang and Wood [19], Huang et al. [20], Wang et al. [34,35], Zhao et al. [39] and Zhang et al. [40]. To the best of our knowledge, all the known results in the open literature follow the model of a single cavity, which clearly limits the practical application of the model problem in industry and military. A major challenge in a multiple cavity model is how to fully capture the interactions among separate cavities. This paper aims to extend the single cavity model to the more general multiple cavity model, analyze and develop numerical methods for the associated scattering problem. Some preliminary numerical results are announced in Li and Wood [25].

Our approach calls for the development of a boundary condition over the cavity apertures based on Fourier transform. The boundary condition is nonlocal and transparent which connects the fields in all individual cavities. By using this boundary condition, we reduce the multiple cavity scattering problem into a boundary value problem of the two-dimensional Helmholtz equation imposed in the interiors of the cavities. The existence and uniqueness of the weak solution of the associated variational formulation for the model problem is achieved. A block Gauss–Seidel iterative method is introduced to solve the coupled system, where only a single cavity scattering problem is required to be solved at each iteration, rendering all efficient single cavity solvers applicable to the multiple cavity scattering problem. Two numerical examples are presented to show the efficiency and accuracy of the proposed method. Numerical methods for multiple obstacle scattering problems can be found in Grote and Kirsch [17], Huang et al. [18], Jiang and Zheng [22], and references therein. Recently, Bonnetier and Triki [14] and Babadjian et al. [8] studied the enhancement of electromagnetic fields by interacting subwavelength cavities. More general results could be found in Ammari et al. [6] on the existence of resonances by using layer potential techniques. We also refer to Ammari et al. [4,5,7] for related work on developing imaging functions to reconstruct perfectly conducting cracks and holes.

We organize the paper as follows. Section 2 presents a mathematical model for the single cavity scattering problem; the variational formulation is presented for an equivalent boundary value problem by using the transparent boundary condition; the uniqueness and existence of the solution are examined. Section 3 deals with two cavities. The major new ingredient is the introduction of a novel transparent boundary condition which reduces the double cavity problem to two coupled single cavity problems. The results are extended to the general multiple cavity scattering problem in Section 4. In Section 5, we examine issues related to the numerical implementation. Experiments on two example scenarios are performed and shown to be efficient and accurate. The paper is concluded in Section 6.

2. One cavity scattering

In this section, we study a mathematical model for a single cavity scattering problem, and discuss the existence and uniqueness of the solution based on its variational formulation. This section is intended to introduce the background for the cavity scattering problem and serve as a basis for the two cavity and the general multiple cavity scattering problems.

2.1. A model problem

We focus on a two-dimensional geometry by assuming that the medium and material are invariant in the z-direction. Throughout, the medium is assumed to be non-magnetic and has a constant magnetic permeability, i.e., $\mu = \mu_0$, where μ_0 is the magnetic permeability of vacuum. The electromagnetic property of the medium is characterized by the dielectric permittivity ε.

As shown in Fig. 1, an open cavity Ω, enclosed by the aperture Γ and the wall S, is placed on a perfectly conducting ground plane Γ^c. Above the flat surface $\{y = 0\} = \Gamma \cup \Gamma^c$, the medium is assumed to be homogeneous with a positive dielectric permittivity ε_0. The medium inside the cavity Ω is inhomogeneous with a variable relative dielectric permittivity $\varepsilon(x, y)$. Assume further that $\varepsilon(x, y) \in L^\infty(\Omega),\ \mathrm{Re}\,\varepsilon > 0,\ \mathrm{Im}\,\varepsilon \geqslant 0$.

Due to the uniformity in the z-axis, the scattering problem can be decomposed into two fundamental polarizations: transverse magnetic (TM) and transverse electric (TE). Its solution then can be expressed as a linear combination of the solutions to TM and TE problems. In this work, we focus on the TM polarization. The method can be extended to the TE polarization with obvious modifications.

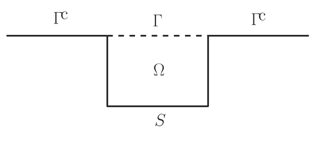

Fig. 1. The problem geometry for a single cavity scattering problem. An open cavity Ω, enclosed by the aperture Γ and the wall S, is placed on a perfectly conducting ground plane Γ^c.

For the TM polarization, the magnetic field is transverse to the invariant direction. The incident and the total electric fields are parallel to the invariant dimension. By the perfectly electrical conductor condition, the total field u vanishes on the cavity wall S and the ground plane Γ^c except over the aperture Γ. The time-harmonic Maxwell equations are reduced to the two-dimensional Helmholtz equation

$$\Delta u + \kappa^2 u = 0, \quad \text{in } \Omega \cup \mathbb{R}^2_+, \tag{2.1}$$

together with the homogeneous Dirichlet boundary condition

$$u = 0, \quad \text{on } \Gamma^c \cup S. \tag{2.2}$$

Here $\kappa^2 = \omega^2 \varepsilon \mu_0$, where ω is the angular frequency and κ is known as the wavenumber.

Let an incoming plane wave $u^{\mathrm{i}} = e^{\mathrm{i}\kappa_0 (x \sin\theta - y\cos\theta)}$ be incident on the cavity from above, where $\theta \in (-\pi/2, \pi/2)$ is the angle of incidence with respect to the positive y-axis, and $\kappa_0 = \omega\sqrt{\varepsilon_0 \mu_0}$ is the wavenumber of the free space.

Denote the reference field u^{ref} as the solution of the homogeneous Helmholtz equation in the upper half space:

$$\Delta u^{\mathrm{ref}} + \kappa_0^2 u^{\mathrm{ref}} = 0, \quad \text{in } \mathbb{R}^2_+, \tag{2.3}$$

together with the boundary condition

$$u^{\mathrm{ref}} = 0 \quad \text{on } \Gamma^c \cup \Gamma. \tag{2.4}$$

It can be shown from (2.3) and (2.4) that the reference field consists of the incident field u^{i} and the reflected field u^{r}:

$$u^{\mathrm{ref}} = u^{\mathrm{i}} + u^{\mathrm{r}},$$

where $u^{\mathrm{r}} = -e^{\mathrm{i}\kappa_0 (x\sin\theta + y\cos\theta)}$.

The total field u is composed of the reference field u^{ref} and the scattered field u^{s}:

$$u = u^{\mathrm{ref}} + u^{\mathrm{s}}.$$

It can be verified from (2.1) and (2.3) that the scattered field satisfies

$$\Delta u^{\mathrm{s}} + \kappa_0 u^{\mathrm{s}} = 0, \quad \text{in } \mathbb{R}^2_+. \tag{2.5}$$

In addition, the scattered field is required to satisfy the radiation condition

$$\lim_{\rho\to 0} \sqrt{\rho}\left(\frac{\partial u^{\mathrm{s}}}{\partial \rho} - \mathrm{i}\kappa_0 u^{\mathrm{s}}\right) = 0, \quad \rho = |(x, y)|. \tag{2.6}$$

To describe the boundary value problem and derive its variational formulation, we need to introduce some functional space notation. For $u \in L^2(\Gamma^c \cup \Gamma)$, which is identified with $L^2(\mathbb{R})$, we denote by $\hat{u}$ the Fourier transform of u defined as

$$\hat{u}(\xi) = \int_{\mathbb{R}} u(x) e^{\mathrm{i}x\xi} dx.$$

Using Fourier modes, the norm on the space $L^2(\mathbb{R})$ can be characterized by

$$\|u\|_{L^2(\mathbb{R})} = \left[\int_{\mathbb{R}} |u|^2 dx\right]^{1/2} = \left[\int_{\mathbb{R}} |\hat{u}|^2 d\xi\right]^{1/2}.$$

Denote the Sobolev space: $H^1(\Omega) = \{u : D^s u \in L^2(\Omega) \text{ for all } |s| \leqslant 1\}$. To describe the boundary operator and transparent boundary condition in the formulation of the boundary value problem, we define the trace functional space

$$H^s(\mathbb{R}) = \left\{u \in L^2(\mathbb{R}) : \int_{\mathbb{R}} (1+\xi^2)^s |\hat{u}|^2 d\xi < \infty\right\},$$

whose norm is defined by

$$\|u\|_{H^s(\mathbb{R})} = \left[\int_{\mathbb{R}} (1+\xi^2)^s |\hat{u}|^2 d\xi\right]^{1/2}.$$

It is clear that the dual space associated with $H^s(\mathbb{R})$ is the space $H^{-s}(\mathbb{R})$ with respect to the scalar product in $L^2(\mathbb{R})$ defined by

$$\langle u, v\rangle = \int_{\mathbb{R}} \hat{u}\,\bar{\hat{v}} d\xi. \tag{2.7}$$

To simply proofs, we shall employ positive constants C and C_i as generic constants whose precise values are not required and may be changed line by line but should be always clear from the context.

2.2. Transparent boundary condition

In this section, we shall derive a boundary operator, which maps the electric field to its normal derivative, and introduce a transparent boundary condition on the aperture of the cavity, under which the scattering problem may be reduced into a bounded domain, i.e., inside the cavity Ω, from the unbounded open domain $\Omega \cup \mathbb{R}^2_+$.

By taking the Fourier transform of (2.5) with respect to x, we have an ordinary differential equation with respect to y:

$$\frac{\partial \hat{u}^s(\xi, y)}{\partial y^2} + (\kappa_0^2 - \xi^2)\hat{u}^s(\xi, y) = 0, \quad y > 0. \tag{2.8}$$

Since the solution of (2.8) satisfies the radiation condition (2.6), we deduce that the solution of (2.8) has the analytical form

$$\hat{u}^s(\xi, y) = \hat{u}^s(\xi, 0)e^{\mathrm{i}\beta(\xi)y}, \tag{2.9}$$

where

$$\beta(\xi) = \begin{cases} (\kappa_0^2 - \xi^2)^{1/2} & \text{for } |\xi| < \kappa_0, \\ \mathrm{i}(\xi^2 - \kappa_0^2)^{1/2} & \text{for } |\xi| > \kappa_0. \end{cases}$$

Taking the inverse Fourier transform of (2.9), we find that

$$u^s(x, y) = \int_{\mathbb{R}} \hat{u}^s(\xi, 0)e^{\mathrm{i}\beta(\xi)y}e^{-\mathrm{i}\xi x}d\xi \quad \text{in } \mathbb{R}^2_+.$$

Taking the normal derivative on $\Gamma^c \cup \Gamma$, which is the partial derivative with respect to y on $\Gamma^c \cup \Gamma$, and evaluating at $y = 0$ yield

$$\partial_{\mathbf{n}} u^s(x, y)|_{y=0} = \int_{\mathbb{R}} \mathrm{i}\beta(\xi)\hat{u}^s(\xi, 0)e^{-\mathrm{i}\xi x}d\xi, \tag{2.10}$$

where $\mathbf{n}$ is the unit outward normal on $\Gamma^c \cup \Gamma$, i.e., $\mathbf{n} = (0, 1)^\top$.

For any given u on $\Gamma^c \cup \Gamma$, define the boundary operator T:

$$Tu = \int_{\mathbb{R}} \mathrm{i}\beta(\xi)\hat{u}(\xi, 0)e^{-\mathrm{i}\xi x}d\xi, \tag{2.11}$$

which leads to a transparent boundary condition for the scattered field on $\Gamma^c \cup \Gamma$:

$$\partial_{\mathbf{n}}(u - u^{\mathrm{ref}}) = T(u - u^{\mathrm{ref}}).$$

Equivalently it can be written as a transparent boundary condition for the total field

$$\partial_{\mathbf{n}} u = Tu + g \quad \text{on } \Gamma^c \cup \Gamma, \tag{2.12}$$

where

$$g(x) = \partial_{\mathbf{n}} u^{\mathrm{ref}} - Tu^{\mathrm{ref}} = -2\mathrm{i}\kappa_0 \cos\theta e^{\mathrm{i}\kappa_0 x \sin\theta}.$$

The following two lemmas are concerned with the continuity and analyticity of the boundary operator, and will play an important role in the proof of the uniqueness and existence of the solution for the cavity scattering problem.

Lemma 2.1. *The boundary operator $T : H^{1/2}(\mathbb{R}) \to H^{-1/2}(\mathbb{R})$ is continuous.*

Proof. For any $u, v \in H^{1/2}(\mathbb{R})$, it follows from the definitions (2.7) and (2.11) that

$$\langle Tu, v\rangle = \mathrm{i}\int_{\mathbb{R}} \beta\hat{u}\bar{\hat{v}}d\xi = \mathrm{i}\int_{\mathbb{R}} \beta(\xi)(1 + \xi^2)^{-1/2} \times (1 + \xi^2)^{1/4}\hat{u} \times (1 + \xi^2)^{1/4}\bar{\hat{v}}d\xi.$$

To prove the lemma, it is required to estimate

$$F(\xi) = \frac{|\beta(\xi)|}{(1 + \xi^2)^{1/2}}, \quad -\infty < \xi < \infty.$$

Explicitly, we have

$$F(\xi) = \begin{cases} [(\kappa_0^2 - \xi^2)/(1 + \xi^2)]^{1/2} & \text{for } |\xi| < \kappa_0, \\ [(\xi^2 - \kappa_0^2)/(1 + \xi^2)]^{1/2} & \text{for } |\xi| > \kappa_0. \end{cases}$$

It can be verified that the even function $F(\xi)$ decreases for $0 < \xi < \kappa_0$ and increases for $\kappa_0 < \xi < \infty$. Hence, a simple calculation yields

$$F(\xi) \leqslant \max\{F(0), F(\infty)\} = \max\{\kappa_0, 1\}.$$

Combining above estimates and using the Cauchy–Schwarz inequality yield

$$|\langle Tu, v\rangle| \leqslant C\|u\|_{H^{1/2}(\mathbb{R})}\|v\|_{H^{1/2}(\mathbb{R})},$$

where

$$C = \max\{\kappa_0, 1\}.$$

Thus we have

$$\|Tu\|_{H^{-1/2}(\mathbb{R})} \leqslant \sup_{v\in H^{1/2}(\mathbb{R})} \frac{|\langle Tu, v\rangle|}{\|v\|_{H^{1/2}(\mathbb{R})}} \leqslant C\|u\|_{H^{1/2}(\mathbb{R})}. \qquad \square$$

Lemma 2.2. *Let $u \in H^{1/2}(\mathbb{R})$. It holds that $Re\langle Tu, u\rangle \leqslant 0$ and $Im\langle Tu, u\rangle \geqslant 0$. Furthermore, if $\hat{u}$ is an analytical function with respect to ξ, $Re\langle Tu, u\rangle = 0$ or $Im\langle Tu, u\rangle = 0$ implies $u = 0$.*

Proof. By definitions (2.7) and (2.11), we find

$$\langle Tu, u\rangle = \mathrm{i}\int_{\mathbb{R}} \beta|\hat{u}|^2 d\xi.$$

Taking the real part gives

$$\mathrm{Re}\langle Tu, u\rangle = -\int_{|\xi|>\kappa_0} (\xi^2 - \kappa_0^2)^{1/2}|\hat{u}|^2 d\xi \leqslant 0$$

and taking the imaginary part yields

$$\mathrm{Im}\langle Tu, u\rangle = \int_{|\xi|<\kappa_0} (\kappa_0^2 - \xi^2)^{1/2}|\hat{u}|^2 d\xi \geqslant 0.$$

Furthermore, $\mathrm{Re}\langle Tu, u\rangle = 0$ implies $\hat{u} = 0$ for $|\xi| > \kappa_0$ and $\mathrm{Im}\langle Tu, u\rangle = 0$ implies $\hat{u} = 0$ for $|\xi| < \kappa_0$. If $\hat{u}$ is assumed to be an analytical function with respect to ξ, either $\mathrm{Re}\langle Tu, u\rangle = 0$ or $\mathrm{Im}\langle Tu, u\rangle = 0$ implies $\hat{u} = 0$ for $\xi \in \mathbb{R}$. Taking the inverse Fourier transform of $\hat{u} = 0$ yields $u = 0$, which completes the proof. $\square$

The transparent boundary condition is derived for $u \in H^{1/2}(\mathbb{R})$ and u is defined over $\Gamma^c \cup \Gamma$. To derive a transparent boundary condition for the total field only on the aperture Γ, we need to make the zero extension as follows: for any given u on Γ, define

$$\tilde{u}(x) = \begin{cases} u & \text{for } x \in \Gamma, \\ 0 & \text{for } x \in \Gamma^c. \end{cases}$$

The zero extension is consistent with the problem geometry where the cavity is placed on a perfectly conducting ground plane Γ^c, i.e., the total field u is required to be zero on Γ^c. Based on the extension and the transparent boundary condition (2.12), we have the transparent boundary condition for the total field on the aperture

$$\partial_{\mathbf{n}} u = T\tilde{u} + g \quad \text{on } \Gamma. \tag{2.13}$$

2.3. Well-posedness

We now present a variational formulation for the single cavity scattering problem and give a proof of the well-posedness of the boundary value problem.

Define a trace functional space

$$\widetilde{H}^{1/2}(\Gamma) = \{u : \tilde{u} \in H^{1/2}(\mathbb{R})\},$$

whose norm is defined as the $H^{1/2}(\mathbb{R})$-norm for its extension, i.e.,

$$\|u\|_{\widetilde{H}^{1/2}(\Gamma)} = \|\tilde{u}\|_{H^{1/2}(\mathbb{R})}. \tag{2.14}$$

Define a dual paring $\langle\cdot,\cdot\rangle_\Gamma$ by

$$\langle u, v\rangle_\Gamma = \int_\Gamma u\bar{v}.$$

Obviously, this dual paring for u and v is equivalent to the scalar product in $L^2(\mathbb{R})$ for their extensions, i.e.,

$$\langle u, v\rangle_\Gamma = \langle\tilde{u}, \tilde{v}\rangle.$$

Denote by $H^{-1/2}(\Gamma)$ the dual space of $\widetilde{H}^{1/2}(\Gamma)$, i.e., $H^{-1/2}(\Gamma) = (\widetilde{H}^{1/2}(\Gamma))'$. The norm on the space $H^{-1/2}(\Gamma)$ is characterized by

$$\|u\|_{H^{-1/2}(\Gamma)} = \sup_{v\in \widetilde{H}^{1/2}(\Gamma)} \frac{\langle u, v\rangle_\Gamma}{\|v\|_{\widetilde{H}^{1/2}(\Gamma)}} = \sup_{\tilde{v}\in H^{1/2}(\mathbb{R})} \frac{\langle \tilde{u}, \tilde{v}\rangle}{\|\tilde{v}\|_{H^{1/2}(\mathbb{R})}}. \tag{2.15}$$

Introduce the following space

$$H^1_S(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } S, \quad u|_\Gamma \in \widetilde{H}^{1/2}(\Gamma)\},$$

which is clearly a Hilbert space with the usual $H^1(\Omega)$-norm.

The following trace regularity in $\widetilde{H}^{1/2}(\Gamma)$ is useful in subsequent analysis.

Lemma 2.3. *For any $u \in H^1_S(\Omega)$, it holds the estimate*

$$\|u\|_{\widetilde{H}^{1/2}(\Gamma)} \leqslant C\|u\|_{H^1(\Omega)},$$

where C is a positive constant.

Proof. Choose a positive number b such that the domain

$$D = \{(x,y) \in \mathbb{R}^2 : -\infty < x < \infty, -b < y < 0\} = \mathbb{R} \times (-b, 0)$$

contains the cavity Ω, i.e., $\Omega \subset\subset D$.

Simple calculation yields

$$b|\zeta(0)|^2 = \int_{-b}^0 |\zeta(y)|^2 dy + \int_{-b}^0 \int_y^0 \frac{d}{dt}|\zeta(t)|^2 dt\, dy \leqslant \int_{-b}^0 |\zeta(y)|^2 dy + b\int_{-b}^0 2|\zeta(y)||\zeta'(y)|dy,$$

which implies by the Cauchy–Schwarz inequality that

$$(1+|\xi|^2)^{1/2}|\zeta(0)|^2 \leqslant C^2(1+|\xi|^2)\int_{-b}^0 |\zeta(y)|^2 dy + \int_{-b}^0 |\zeta'(y)|^2 dy, \tag{2.16}$$

where $C = (1+b^{-1})^{1/2}$.

Given u in $H^1_S(\Omega)$, consider the zero extension to the domain D:

$$\tilde{u}(x) = \begin{cases} u & \text{for } x \in \Omega, \\ 0 & \text{for } x \in D\setminus\Omega. \end{cases}$$

It follows from the definition (2.14) that

$$\|u\|^2_{\widetilde{H}^{1/2}(\Gamma)} = \|\tilde{u}\|^2_{H^{1/2}(\mathbb{R})} = \int_{\mathbb{R}} (1+\xi^2)^{1/2}|\hat{\tilde{u}}(\xi,0)|^2 d\xi.$$

Using (2.16) we obtain

$$(1+\xi^2)^{1/2}|\hat{\tilde{u}}(\xi,0)|^2 \leqslant C^2(1+\xi^2)\int_{-b}^0 |\hat{\tilde{u}}(\xi,y)|^2 dy + \int_{-b}^0 |\hat{\tilde{u}}'(\xi,y)|^2 dy \leqslant C^2\int_{-b}^0 \left[(1+\xi^2)|\hat{\tilde{u}}(\xi,y)|^2 + |\hat{\tilde{u}}'(\xi,y)|^2\right]dy. \tag{2.17}$$

Noting

$$\|\tilde{u}\|^2_{H^1(D)} = \int_{\mathbb{R}}\int_{-b}^0 \left[(1+\xi^2)|\hat{\tilde{u}}(\xi,y)|^2 + |\hat{\tilde{u}}'(\xi,y)|^2\right]dy\, d\xi. \tag{2.18}$$

Combining (2.14), (2.17) and (2.18) yields

$$\|u\|_{\widetilde{H}^{1/2}(\Gamma)} \leqslant C\|\tilde{u}\|_{H^1(D)} = C\|u\|_{H^1(\Omega)},$$

which completes the proof. □

Multiplying the complex conjugate of a test function $v \in H^1_S(\Omega)$ on both sides of (2.1), integrating over Ω, using the integration by parts and boundary conditions (2.2) and (2.13), we deduce the variational formulation for the single cavity scattering problem: find $u \in H^1_S(\Omega)$ such that

$$a_1(u, v) = \langle g, v\rangle_\Gamma \quad \text{for all } v \in H^1_S(\Omega), \tag{2.19}$$

where the sesquilinear form is

$$a_1(u, v) = \int_\Omega (\nabla u\cdot\nabla\bar{v} - \kappa^2 u\bar{v}) - \langle T\tilde{u}, v\rangle_\Gamma.$$

Noting the definition for the dual paring and using the scalar product in $L^2(\mathbb{R})$, we have an equivalent sesquilinear form

$$a_1(u,v) = \int_\Omega \left(\nabla u \cdot \nabla \bar{v} - \kappa^2 u \bar{v}\right) - \langle T\tilde{u}, \tilde{v}\rangle. \tag{2.20}$$

Theorem 2.1. *The variational problem (2.19) has at most one solution.*

Proof. It suffices to show that $u = 0$ in Ω if $g = 0$. If u satisfies the homogeneous variational problem (2.19), then we have

$$\int_\Omega (|\nabla u|^2 - \kappa^2 |u|^2) - \langle T\tilde{u}, \tilde{u}\rangle = 0.$$

Taking the imaginary part of the above identity yields

$$\int_\Omega \mathrm{Im}\kappa^2 |u|^2 + \mathrm{Im}\langle T\tilde{u}, \tilde{u}\rangle = 0.$$

Recall $\kappa^2 = \omega^2 \varepsilon \mu_0, \mathrm{Im}\,\varepsilon \geqslant 0$, and $\mathrm{Im}\langle T\tilde{u}, \tilde{u}\rangle \geqslant 0$ from Lemma 2.2, we get

$$\mathrm{Im}\langle T\tilde{u}, \tilde{u}\rangle = 0.$$

Since $\tilde{u}$ has a compact support on the x-axis, $\hat{\tilde{u}}$ is analytical with respect to ξ. It follows from Lemma 2.2 again that $\tilde{u} = 0$. The transparent boundary condition (2.12) yields that $\partial_{\mathbf{n}}\tilde{u} = 0$ on $\Gamma^c \cup \Gamma$. An application of Holmgren uniqueness theorem yields $u = 0$ in $\mathbb{R}^2_+$. A unique continuation result in [21] concludes that $u = 0$ in Ω. $\square$

Theorem 2.2. *The variational problem (2.19) has a unique weak solution in $H^1_S(\Omega)$ and the solution satisfies the estimate*

$$\|u\|_{H^1(\Omega)} \leqslant C\|g\|_{H^{-1/2}(\Gamma)},$$

where C is a positive constant.

Proof. Decompose the sesquilinear form (2.20) into $a_1 = a_{11} - a_{12}$, where

$$a_{11}(u,v) = \int_\Omega \nabla u \cdot \nabla \bar{v} - \langle T\tilde{u}, \tilde{v}\rangle$$

and

$$a_{12}(u,v) = \int_\Omega \kappa^2 u \bar{v}.$$

We conclude from Lemma 2.2 and Poincaré inequality that a_1 is coercive from

$$\mathrm{Re}\, a_{11}(u,u) = \int_\Omega |\nabla u|^2 - \mathrm{Re}\langle T\tilde{u}, \tilde{u}\rangle \geqslant \int_\Omega |\nabla u|^2 \geqslant C\|u\|^2_{H^1(\Omega)} \quad \text{for all } u \in H^1_S(\Omega).$$

Next we prove the compactness of a_{12}. Define an operator $K_1 : L^2(\Omega) \to H^1(\Omega)$ by

$$a_{11}(K_1 u, v) = a_{12}(u,v) \quad \text{for all } v \in H^1_S(\Omega),$$

which explicitly gives

$$\int_\Omega \nabla K_1 u \cdot \nabla \bar{v} - \langle TK_1\tilde{u}, \tilde{v}\rangle = \int_\Omega \kappa^2 u \bar{v} \quad \text{for all } v \in H^1_S(\Omega).$$

Using the coercivity of a_{11} and the Lax–Milgram Lemma, it follows that

$$\|K_1 u\|_{H^1(\Omega)} \leqslant C\|u\|_{L^2(\Omega)}. \tag{2.21}$$

Thus K_1 is bounded from $L^2(\Omega)$ to $H^1(\Omega)$ and $H^1(\Omega)$ is compactly imbedded into $L^2(\Omega)$. Hence $K_1 : L^2(\Omega) \to L^2(\Omega)$ is a compact operator.

Define a function $w \in L^2(\Omega)$ by requiring $w \in H^1_S(\Omega)$ and satisfying

$$a_{11}(w,v) = \langle g, v\rangle_\Gamma \quad \text{for all } v \in H^1_S(\Omega).$$

It follows from Lemma 2.3 and the Lax–Milgram Lemma that

$$\|w\|_{H^1(\Omega)} \leqslant C\|g\|_{H^{-1/2}(\Gamma)}. \tag{2.22}$$

Using the operator K_1, we can see that the variational problem (2.19) is equivalent to find $u \in L^2$ such that

$$(I - K_1)u = w. \tag{2.23}$$

It follows from the uniqueness result of Theorem 2.1 and the Fredholm alternative that the operator $I - K_1$ has a bounded inverse. We then have the estimate that

$$\|u\|_{L^2(\Omega)} \leqslant C\|w\|_{L^2(\Omega)}. \tag{2.24}$$

Combining (2.21)–(2.24), we deduce that

$$\|u\|_{H^1(\Omega)} \leqslant \|K_1 u\|_{H^1(\Omega)} + \|w\|_{H^1(\Omega)} \leqslant C\|u\|_{L^2(\Omega)} + \|w\|_{H^1(\Omega)} \leqslant C\|w\|_{H^1(\Omega)} \leqslant C\|g\|_{H^{-1/2}(\Gamma)},$$

which completes the proof. $\square$

3. Two cavity scattering

To address the general multiple cavity scattering problem, we begin with the discussion on the two cavity scattering problem. The two cavity scattering problem shares the same features with the general multiple cavity scattering problem, but it is easier to present the major ideas in the proof of the well-posedness of the solution for the multiple cavity scattering problem.

3.1. A model problem

As shown in Fig. 2, two open cavity Ω_1 and Ω_2, enclosed by the aperture Γ_1 and Γ_2 and the walls S_1 and S_2, are placed on a perfectly conducting ground plane Γ^c. Above the flat surface $\{y = 0\} = \Gamma_1 \cup \Gamma_2 \cup \Gamma^c$, the medium is assumed to be homogeneous with a positive dielectric permittivity ε_0. The medium inside the cavity Ω_1 and Ω_2 is inhomogeneous with a variable dielectric permittivity $\varepsilon_1(x,y)$ and $\varepsilon_2(x,y)$, respectively. Assume further that $\varepsilon_j(x,y) \in L^\infty(\Omega),\ \mathrm{Re}\,\varepsilon_j > 0,\ \mathrm{Im}\,\varepsilon_j \geqslant 0$ for $j = 1, 2$.

We consider the TM polarization, where the time-harmonic Maxwell equations are reduced to the two-dimensional Helmholtz equation

$$\Delta u + \kappa^2 u = 0, \quad \text{in } \Omega_1 \cup \Omega_2 \cup \mathbb{R}^2_+, \tag{3.1}$$

together with the homogeneous Dirichlet boundary condition

$$u = 0, \quad \text{on } \Gamma^c \cup S_1 \cup S_2. \tag{3.2}$$

Let the plane wave u^i be incident on the cavities from above. Due to the interaction between the incident wave and the ground plane and the cavities, the total field u is assumed to be consisted of the incident field u^i, the reflected field u^r, and the scattered field u^s. The scattered field is required to satisfy the radiation condition (2.6).

To reduce the scattering problem from the open domain $\Omega_1 \cup \Omega_2 \cup \mathbb{R}^2_+$ into the bounded domains Ω_1 and Ω_2, we need to derive transparent boundary conditions on the aperture Γ_1 and Γ_2. Rewrite (3.1) and (3.2) into two single cavity scattering problem:

$$\Delta u_1 + \kappa_1^2 u_1 = 0 \quad \text{in } \Omega_1, \tag{3.3}$$

$$u_1 = 0 \quad \text{on } S_1 \tag{3.4}$$

and

$$\Delta u_2 + \kappa_2^2 u_2 = 0 \quad \text{in } \Omega_2, \tag{3.5}$$

$$u_2 = 0 \quad \text{on } S_2, \tag{3.6}$$

where $\kappa_j^2 = \omega^2 \varepsilon_j \mu_0,\ j = 1, 2$. Clearly, if u is the solution of (3.1) and (3.2), and u_1 and u_2 are solutions of (3.3) and (3.4) and (3.5) and (3.6), respectively, then we have

$$u_1 = u|_{\Omega_1} \quad \text{and} \quad u_2 = u|_{\Omega_2}.$$

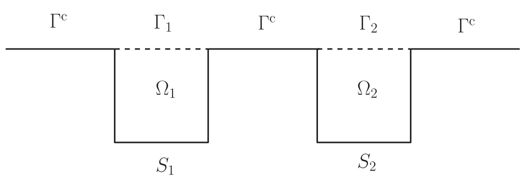

Fig. 2. The problem geometry for the two cavity scattering problem. Two open cavities Ω_1 and Ω_2, enclosed by the apertures Γ_1 and Γ_2 and the walls S_1 and S_2, are placed on a perfectly conducting ground plane Γ^c.

Due to the homogeneous medium in the upper half space $\mathbb{R}^2_+$ and the radiation condition (2.6), the scattered field u^s still satisfies the same ordinary differential equation (2.9) after taking the Fourier transform with respect to x, and thus the total field u satisfies the transparent boundary condition (2.12), which is now written as

$$\partial_{\mathbf{n}} u = Tu + g \quad \text{on } \Gamma^c \cup \Gamma_1 \cup \Gamma_2. \tag{3.7}$$

For $u_j(x,0),\ j = 1,2$, define their extensions to the whole x-axis by

$$\tilde{u}_j(x,0) = \begin{cases} u_j(x,0) & \text{for } x \in \Gamma_j, \\ 0 & \text{for } x \in \mathbb{R} \setminus \Gamma_j. \end{cases}$$

For the total field $u(x,0)$, define its extension to the whole x-axis by

$$\tilde{u}(x,0) = \begin{cases} u_1(x,0) & \text{for } x \in \Gamma_1, \\ u_2(x,0) & \text{for } x \in \Gamma_2, \\ 0 & \text{for } x \in \Gamma^c. \end{cases}$$

It follows from the definitions of these extensions that we have

$$\tilde{u} = \tilde{u}_1 + \tilde{u}_2 \quad \text{on } \Gamma^c \cup \Gamma_1 \cup \Gamma_2.$$

The transparent boundary condition (3.7) can be written as

$$\partial_{\mathbf{n}} \tilde{u} = T\tilde{u} + g \quad \text{on } \Gamma^c \cup \Gamma_1 \cup \Gamma_2, \tag{3.8}$$

which leads to the transparent boundary conditions for u_1 and u_2:

$$\partial_{\mathbf{n}} u_1 = T\tilde{u}_1 + T\tilde{u}_2 + g \quad \text{on } \Gamma_1 \tag{3.9}$$

and

$$\partial_{\mathbf{n}} u_2 = T\tilde{u}_2 + T\tilde{u}_1 + g \quad \text{on } \Gamma_2. \tag{3.10}$$

As we can see from (3.9) and (3.10), the boundary conditions for u_1 and u_2 are coupled with each other, which is the major difference between the single cavity scattering problem and the multiple cavity scattering problem.

The following lemma is analogous to Lemma 2.2 and plays an important role in the analysis of the uniqueness and existence for the solution of the two cavity scattering problem.

Lemma 3.1. *Let $u, v \in H^{1/2}(\mathbb{R})$. It holds that*

$$\mathrm{Re}(\langle Tu, u\rangle + \langle Tv, v\rangle + \langle Tu, v\rangle + \langle Tv, u\rangle) \leqslant 0$$

and

$$\mathrm{Im}(\langle Tu, u\rangle + \langle Tv, v\rangle + \langle Tu, v\rangle + \langle Tv, u\rangle) \geqslant 0.$$

Furthermore, if $\hat{u}$ and $\hat{v}$ are analytical functions with respect to ξ, either

$$\mathrm{Re}(\langle Tu, u\rangle + \langle Tv, v\rangle + \langle Tu, v\rangle + \langle Tv, u\rangle) = 0$$

or

$$\mathrm{Im}(\langle Tu, u\rangle + \langle Tv, v\rangle + \langle Tu, v\rangle + \langle Tv, u\rangle) = 0$$

implies

$$u + v = 0.$$

Proof. By definitions (2.7) and (2.11), we have

$$\langle Tu, u\rangle + \langle Tv, v\rangle + \langle Tu, v\rangle + \langle Tv, u\rangle = \mathrm{i}\int_{\mathbb{R}} \beta\left[|\hat{u}|^2 + |\hat{v}|^2 + 2\mathrm{Re}(\hat{u}\bar{\hat{v}})\right] d\xi = \mathrm{i}\int_{\mathbb{R}} \beta|\hat{u} + \hat{v}|^2 d\xi.$$

Taking the real part of above identity gives

$$\mathrm{Re}(\langle Tu, u\rangle + \langle Tv, v\rangle + \langle Tu, v\rangle + \langle Tv, u\rangle) = -\int_{|\xi|>\kappa_0} (\xi^2 - \kappa_0^2)^{1/2} |\hat{u} + \hat{v}|^2 d\xi \leqslant 0$$

and taking the imaginary part of above identity yields

$$\mathrm{Im}(\langle Tu, u\rangle + \langle Tv, v\rangle + \langle Tu, v\rangle + \langle Tv, u\rangle) = \int_{|\xi|<\kappa_0} (\kappa_0^2 - \xi^2)^{1/2} |\hat{u} + \hat{v}|^2 d\xi \geqslant 0.$$

Furthermore,

$$\mathrm{Re}(\langle Tu, u\rangle + \langle Tv, v\rangle + \langle Tu, v\rangle + \langle Tv, u\rangle) = 0 \tag{3.11}$$

implies

$$\hat{u} + \hat{v} = 0 \quad \text{for } |\xi| > \kappa_0$$

and

$$\mathrm{Im}(\langle Tu, u\rangle + \langle Tv, v\rangle + \langle Tu, v\rangle + \langle Tv, u\rangle) = 0 \tag{3.12}$$

implies

$$\hat{u} + \hat{v} = 0 \quad \text{for} |\xi| < \kappa_0.$$

If $\hat{u}$ and $\hat{v}$ are assumed to be analytical functions with respect to ξ, then either (3.11) or (3.12) implies that $\hat{u} + \hat{v} = 0$ for all $\xi \in \mathbb{R}$. Taking the inverse Fourier transform of $\hat{u} + \hat{v} = 0$ yields $u + v = 0$, which completes the proof. □

3.2. Well-posedness

We now present a variational formulation for the two cavity scattering problem and give a proof of the well-posedness of solution for the boundary value problem.

Denote $\Omega = \Omega_1 \cup \Omega_2, \Gamma = \Gamma_1 \cup \Gamma_2$, and $S = S_1 \cup S_2$. Let

$$u = \begin{cases} u_1 & \text{in } \Omega_1, \\ u_2 & \text{in } \Omega_2. \end{cases}$$

Define a trace functional space

$$\widetilde{\mathbb{H}}^{1/2}(\Gamma) = \widetilde{H}^{1/2}(\Gamma_1) \times \widetilde{H}^{1/2}(\Gamma_2),$$

whose norm is characterized by

$$\|u\|^2_{\widetilde{\mathbb{H}}^{1/2}(\Gamma)} = \|u_1\|^2_{\widetilde{H}^{1/2}(\Gamma_1)} + \|u_2\|^2_{\widetilde{H}^{1/2}(\Gamma_2)}.$$

Denote $\mathbb{H}^{-1/2}(\Gamma) = H^{-1/2}(\Gamma_1) \times H^{-1/2}(\Gamma_2)$, which is the dual space of $\widetilde{\mathbb{H}}^{1/2}(\Gamma)$. The norm on the space $\mathbb{H}^{-1/2}(\Gamma)$ is characterized by

$$\|u\|^2_{\mathbb{H}^{-1/2}(\Gamma)} = \|u_1\|^2_{H^{-1/2}(\Gamma_1)} + \|u_2\|^2_{H^{-1/2}(\Gamma_2)}.$$

Introduce the following space

$$\mathbb{H}^1_S(\Omega) = H^1_{S_1}(\Omega_1) \times H^1_{S_2}(\Omega_2),$$

which is a Hilbert space with norm characterized by

$$\|u\|^2_{\mathbb{H}^1(\Omega)} = \|u_1\|^2_{H^1(\Omega_1)} + \|u_2\|^2_{H^1(\Omega_2)}.$$

Multiplying the complex conjugate of test function $v_1 \in H^1_{S_1}(\Omega_1)$ on both sides of (3.3), integrating over Ω_1, and using the integration by parts and the boundary conditions (3.4) and (3.9), we have

$$\int_{\Omega_1} (\nabla u_1 \cdot \nabla \bar{v}_1 - \kappa_1^2 u_1 \bar{v}_1) - \langle T\tilde{u}_1, \tilde{v}_1\rangle - \langle T\tilde{u}_2, \tilde{v}_1\rangle = \langle g, v_1\rangle_{\Gamma_1}. \tag{3.13}$$

Similarly, regarding (3.5), (3.6), and (3.10), we have

$$\int_{\Omega_2} (\nabla u_2 \cdot \nabla \bar{v}_2 - \kappa_2^2 u_2 \bar{v}_2) - \langle T\tilde{u}_1, \tilde{v}_2\rangle - \langle T\tilde{u}_2, \tilde{v}_2\rangle = \langle g, v_2\rangle_{\Gamma_2} \quad \text{for all } v_2 \in H^1_{S_2}(\Omega_2). \tag{3.14}$$

Adding (3.13) and (3.14), we deduce the variational formulation for the two cavity scattering problem: find $u \in \mathbb{H}^1_S(\Omega)$, where $u_1 = u|_{\Omega_1} \in H_{S_1}(\Omega_1)$ and $u_2 = u|_{\Omega_2} \in H_{S_2}(\Omega_2)$, such that for all $v \in \mathbb{H}^1_S(\Omega)$, where $v_1 = v|_{\Omega_1} \in H_{S_1}(\Omega_1)$ and $v_2 = v|_{\Omega_2} \in H_{S_2}(\Omega_2)$, it holds

$$a_2(u, v) = \langle g, v_1\rangle_{\Gamma_1} + \langle g, v_2\rangle_{\Gamma_2}, \tag{3.15}$$

with the sesquilinear form

$$a_2(u, v) = \int_{\Omega_1} (\nabla u_1 \cdot \nabla \bar{v}_1 - \kappa_1^2 u_1 \bar{v}_1) + \int_{\Omega_2} (\nabla u_2 \cdot \nabla \bar{v}_2 - \kappa_2^2 u_2 \bar{v}_2) - \langle T\tilde{u}_1, \tilde{v}_1\rangle - \langle T\tilde{u}_2, \tilde{v}_1\rangle - \langle T\tilde{u}_1, \tilde{v}_2\rangle - \langle T\tilde{u}_2, \tilde{v}_2\rangle. \tag{3.16}$$

Theorem 3.1. *The variational problem (3.15) has at most one solution.*

Proof. It suffices to show that $u_1 = 0$ in Ω_1 and $u_2 = 0$ in Ω_2 if $g = 0$. If u_1 and u_2 satisfy the homogeneous variational problem (3.15) in Ω_1 and Ω_2, then we have

$$\int_{\Omega_1}(|\nabla u_1|^2 - \kappa_1^2|u_1|^2) + \int_{\Omega_2}(|\nabla u_2|^2 - \kappa_2^2|u_2|^2) - \langle T\tilde{u}_1, \tilde{u}_1\rangle - \langle T\tilde{u}_2, \tilde{u}_1\rangle - \langle T\tilde{u}_1, \tilde{u}_2\rangle - \langle T\tilde{u}_2, \tilde{u}_2\rangle = 0.$$

Recall $\mathrm{Im}\varepsilon_1 \geqslant 0$ and $\mathrm{Im}\varepsilon_2 \geqslant 0$. Taking the imaginary part of the above identity yields

$$\mathrm{Im}(\langle T\tilde{u}_1, \tilde{u}_1\rangle + \langle T\tilde{u}_2, \tilde{u}_1\rangle + \langle T\tilde{u}_1, \tilde{u}_2\rangle + \langle T\tilde{u}_2, \tilde{u}_2\rangle) = 0.$$

Since $\tilde{u}_1$ and $\tilde{u}_2$ have compact supports on the x-axis, $\hat{\tilde{u}}_1$ and $\hat{\tilde{u}}_2$ are analytical with respect to ξ. Hence we have from Lemma 3.1 that

$$\tilde{u}_1 + \tilde{u}_2 = \tilde{u} = 0.$$

The transparent boundary condition (3.8) yields that $\partial_{\mathbf{n}}\tilde{u} = 0$ on $\Gamma^c \cup \Gamma_1 \cup \Gamma_2$. An application of Holmgren uniqueness theorem yields $u = 0$ in $\mathbb{R}^2_+$. A unique continuation result in [21] concludes that $u_1 = 0$ in Ω_1 and $u_2 = 0$ in Ω_2. □

Theorem 3.2. *The variational problem (3.15) has a unique weak solution u in $\mathbb{H}^1_S(\Omega)$ and the solution satisfies the estimate*

$$\|u\|_{\mathbb{H}^1(\Omega)} \leqslant C\|g\|_{\mathbb{H}^{-1/2}(\Gamma)},$$

where C is a positive constant.

Proof. The proof is analogous to that for Theorem 2.2. We sketch it here. Decompose the sesquilinear form (3.16) into $a_2 = a_{21} - a_{22}$, where

$$a_2(u, v) = \int_{\Omega_1}\nabla u_1 \cdot \nabla \bar{v}_1 + \int_{\Omega_2}\nabla u_2 \cdot \nabla \bar{v}_2 - \langle T\tilde{u}_1, \tilde{v}_1\rangle - \langle T\tilde{u}_2, \tilde{v}_1\rangle - \langle T\tilde{u}_1, \tilde{v}_2\rangle - \langle T\tilde{u}_2, \tilde{v}_2\rangle$$

and

$$a_2(u, v) = \int_{\Omega_1}\kappa_1^2 u_1 \bar{v}_1 + \int_{\Omega_2}\kappa_2^2 u_2 \bar{v}_2.$$

We conclude from Lemma 3.1 and Poincaré inequality that a_{21} is coercive from

$$\begin{aligned}\mathrm{Re}\, a_{21}(u, u) &= \int_{\Omega_1}|\nabla u_1|^2 + \int_{\Omega_2}|\nabla u_2|^2 - \mathrm{Re}(\langle T\tilde{u}_1, \tilde{u}_1\rangle + \langle T\tilde{u}_2, \tilde{u}_1\rangle + \langle T\tilde{u}_1, \tilde{u}_2\rangle + \langle T\tilde{u}_2, \tilde{u}_2\rangle) \geqslant \int_{\Omega_1}|\nabla u_1|^2 + \int_{\Omega_2}|\nabla u_2|^2 \\ &\geqslant C_1\|u_1\|^2_{H^1(\Omega_1)} + C_2\|u_2\|^2_{H^1(\Omega_2)} \geqslant C\|u\|^2_{\mathbb{H}^1(\Omega)} \quad \text{for all } u \in \mathbb{H}^1_S(\Omega).\end{aligned}$$

Next we prove the compactness of a_{22}. Define an operator $K_2 : L^2(\Omega) \to H^1(\Omega)$ by

$$a_{21}(K_2 u, v) = a_{22}(u, v) \quad \text{for all } v \in \mathbb{H}^1_S(\Omega).$$

The continuity of the sesquilinear form a_{22} follows from the Cauchy–Schwarz inequality

$$|a_{22}(u, v)| \leqslant C_1\|u_1\|_{L^2(\Omega_1)}\|v_1\|_{L^2(\Omega_1)} + C_2\|u_2\|_{L^2(\Omega_2)}\|v_2\|_{L^2(\Omega_2)} \leqslant C\|u\|_{\mathbb{L}^2(\Omega)}\|v\|_{\mathbb{L}^2(\Omega)},$$

where $\mathbb{L}^2(\Omega) = L^2(\Omega_1) \times L^2(\Omega_2)$. Using the Lax–Milgram Lemma and the continuity of a_{22}, we obtain

$$\|K_2 u\|_{\mathbb{H}^1(\Omega)} \leqslant C\|u\|_{\mathbb{L}^2(\Omega)}. \tag{3.17}$$

Thus K_2 is bounded from $\mathbb{L}^2(\Omega)$ to $\mathbb{H}^1(\Omega)$ and $\mathbb{H}^1(\Omega)$ is compactly imbedded into $\mathbb{L}^2(\Omega)$. Hence $K_2 : \mathbb{L}^2(\Omega) \to \mathbb{L}^2(\Omega)$ is a compact operator.

Define a function $w \in \mathbb{L}^2(\Omega)$ by requiring $w \in \mathbb{H}^1_S(\Omega)$ and satisfying

$$a_{21}(w, v) = \langle g, v_1\rangle_{\Gamma_1} + \langle g, v_2\rangle_{\Gamma_2} \quad \text{for all } v \in \mathbb{H}^1_S(\Omega).$$

It follows from Lemma 2.3 and the Lax–Milgram Lemma that

$$\|w\|_{\mathbb{H}^1(\Omega)} \leqslant C\|g\|_{\mathbb{H}^{-1/2}(\Gamma)}. \tag{3.18}$$

Using the operator K_2, we can see that the variational problem (3.15) is equivalent to find $u \in \mathbb{L}^2(\Omega)$ such that

$$(I - K_2)u = w. \tag{3.19}$$

It follows from the uniqueness result of Theorem 3.1 and the Fredholm alternative that the operator $I - K_2$ has a bounded inverse. We then have the estimate that

$$\|u\|_{\mathbb{L}^2(\Omega)} \leqslant C\|w\|_{\mathbb{L}^2(\Omega)}. \tag{3.20}$$

From (3.17)–(3.20), we deduce that

$$\|u\|_{\mathbb{H}^1(\Omega)} \leqslant \|K_2 u\|_{\mathbb{H}^1(\Omega)} + \|w\|_{\mathbb{H}^1(\Omega)} \leqslant C\|u\|_{\mathbb{L}^2(\Omega)} + \|w\|_{\mathbb{H}^1(\Omega)} \leqslant C\|w\|_{\mathbb{H}^1(\Omega)} \leqslant C\|g\|_{\mathbb{H}^{-1/2}(\Gamma)},$$

which completes the proof. □

4. Multiple cavity scattering

Now we generalize the model problem and techniques for the two cavity scattering to the case of multiple cavity scattering. The proofs are analogous to those for the two cavity scattering problem. For completeness, we shall briefly discuss the model problem and the solution for the multiple scattering problem.

4.1. A model problem

As shown in Fig. 3, we consider a situation with n cavities, where the multiple open cavity $\Omega_1, \ldots, \Omega_n$, enclosed by the aperture $\Gamma_1, \ldots, \Gamma_n$ and the walls $S_1, \ldots, S_n$, are placed on a perfectly conducting ground plane Γ^c. Above the flat surface $\{y = 0\} = \Gamma_1 \cup \cdots \cup \Gamma_n \cup \Gamma^c$, the medium is assumed to be homogeneous with a positive dielectric permittivity ε_0. The medium inside the cavity Ω_j is inhomogeneous with a variable dielectric permittivity $\varepsilon_j(x,y)$. Assume further that $\varepsilon_j(x,y) \in L^\infty(\Omega),\ \mathrm{Re}\varepsilon_j > 0,\ \mathrm{Im}\varepsilon_j \geqslant 0$ for $j = 1, \ldots, n$.

We consider the same model of the two-dimensional Helmholtz equation for the total field:

$$\Delta u + \kappa^2 u = 0, \quad \text{in } \Omega_1 \cup \cdots \cup \Omega_n \cup \mathbb{R}^2_+, \tag{4.1}$$

together with the perfectly electric conductor condition

$$u = 0, \quad \text{on } \Gamma^c \cup \Gamma_1 \cup \cdots \cup \Gamma_n. \tag{4.2}$$

The total field u is assumed to consist of the incident field u^{i}, the reflected field u^{r}, and the scattered field u^{s}, where the scattered field is required to satisfy the radiation condition (2.6).

To derive the transparent boundary condition on the aperture $\Gamma_j,\ j = 1, \ldots, n$, we reformulate the multiple cavity scattering problem (4.1) and (4.2) into n single cavity scattering problems which are coupled through the boundary condition.

$$\Delta u_j + \kappa_j^2 u_j = 0 \quad \text{in } \Omega_j, \tag{4.3}$$

$$u_j = 0 \quad \text{on } \Gamma_j, \tag{4.4}$$

where $\kappa_j^2 = \omega^2 \varepsilon_j \mu_0,\ j = 1, \ldots, n$.

For $u_j(x,0),\ j = 1, \ldots, n$, define its zero extension to the whole x-axis,

$$\tilde{u}_j(x,0) = \begin{cases} u_j(x,0) & \text{for } x \in \Gamma_j, \\ 0 & \text{for } x \in \mathbb{R} \setminus \Gamma_j. \end{cases}$$

For the total field $u(x,0)$, define its extension to the whole x-axis by

$$\tilde{u}(x,0) = \begin{cases} u_j(x,0) & \text{for } x \in \Gamma_j, \\ 0 & \text{for } x \in \Gamma^c. \end{cases}$$

It follows from the definition of the extension that we have

$$\tilde{u} = \sum_{j=1}^{n} \tilde{u}_j \quad \text{on } \Gamma^c \cup \Gamma_1 \cup \cdots \cup \Gamma_n.$$

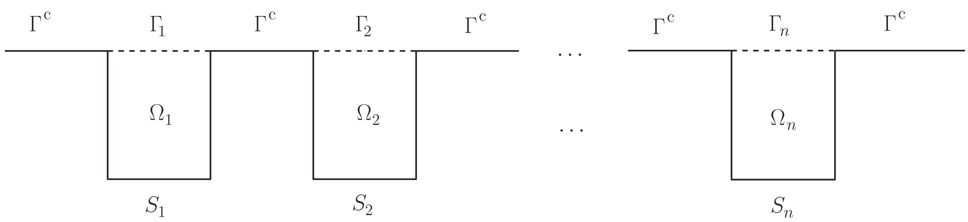

Fig. 3. The problem geometry for the multiple cavity scattering problem. Multiple open cavities $\Omega_1, \ldots, \Omega_n$, enclosed by the apertures $\Gamma_1, \ldots, \Gamma_n$ and the walls $S_1, \ldots, S_n$, are placed on a perfectly conducting ground plane Γ^c.

The transparent boundary condition can be written as

$$\partial_{\mathbf{n}}\tilde{u} = T\tilde{u} + g \quad \text{on } \Gamma^c \cup \Gamma_1 \cup \cdots \cup \Gamma_n, \tag{4.5}$$

which leads to the transparent boundary condition for u_j:

$$\partial_{\mathbf{n}} u_j = T\tilde{u}_j + \sum_{\substack{i=1 \\ i\neq j}}^{n} T\tilde{u}_i + g \quad \text{on } \Gamma_j. \tag{4.6}$$

Lemma 4.1. *Let $u_j \in H^{1/2}(\mathbb{R}),\ j = 1, \ldots, n$. It holds that*

$$\mathrm{Re}\sum_{i=1}^{n}\sum_{j=1}^{n}\langle Tu_j, u_i\rangle \leqslant 0$$

and

$$\mathrm{Im}\sum_{i=1}^{n}\sum_{j=1}^{n}\langle Tu_j, u_i\rangle \geqslant 0.$$

Furthermore, if $\hat{u}_j,\ j = 1, \ldots, n$ are analytical functions with respect to ξ, either

$$\mathrm{Re}\sum_{i=1}^{n}\sum_{j=1}^{n}\langle Tu_j, u_i\rangle = 0$$

or

$$\mathrm{Im}\sum_{i=1}^{n}\sum_{j=1}^{n}\langle Tu_j, u_i\rangle = 0$$

implies

$$\sum_{j=1}^{n} u_j = 0.$$

Proof. By definitions (2.7) and (2.11), we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\langle Tu_j, u_i\rangle = \mathrm{i}\int \beta(\xi)\left|\sum_{j=1}^{n}\hat{u}_j\right|^2 d\xi.$$

Taking the real part gives

$$\mathrm{Re}\sum_{i=1}^{n}\sum_{j=1}^{n}\langle Tu_j, u_i\rangle = -\int_{|\xi|>\kappa_0}(\xi^2 - \kappa_0^2)^{1/2}\left|\sum_{j=1}^{n}\hat{u}_j\right|^2 d\xi \leqslant 0$$

and taking the imaginary part yields

$$\mathrm{Im}\sum_{i=1}^{n}\sum_{j=1}^{n}\langle Tu_j, u_i\rangle = \int_{|\xi|<\kappa_0}(\kappa_0^2 - \xi^2)^{1/2}\left|\sum_{j=1}^{n}\hat{u}_j\right|^2 d\xi \geqslant 0.$$

Furthermore,

$$\mathrm{Re}\sum_{i=1}^{n}\sum_{j=1}^{n}\langle Tu_j, u_i\rangle = 0 \tag{4.7}$$

implies

$$\sum_{j=1}^{n}\hat{u}_j = 0 \quad \text{for } |\xi| > \kappa_0,$$

and

$$\mathrm{Im}\sum_{i=1}^{n}\sum_{j=1}^{n}\langle Tu_j, u_i\rangle = 0 \tag{4.8}$$

implies

$$\sum_{j=1}^{n} \hat{u}_j = 0 \quad \text{for } |\xi| < \kappa_0.$$

If $\hat{u}_j,\ j = 1, \ldots, n$, are assumed to be analytical functions with respect to ξ, then either (4.7) or (4.8) implies that

$$\sum_{j=1}^{n} \hat{u}_j = 0 \quad \text{for all } \xi \in \mathbb{R}. \tag{4.9}$$

The proof is completed by taking the inverse Fourier transform of (4.9). □

4.2. Well-posedness

We now present a variational formulation for the multiple cavity scattering problem and sketch the proof for the well-posedness of the boundary value problem.

Denote $\Omega = \Omega_1 \cup \cdots \cup \Omega_n,\ \Gamma = \Gamma_1 \cup \cdots \cup \Gamma_n$, and $S = S_1 \cup \cdots \cup S_n$. For simplicity, we shall use the same notation as those adopted in Section 3 for the two cavity scattering problem. The following notation for the general multiple cavity scattering problem are actually consistent with those for the two cavity scattering problem when letting $n = 2$.

Define a trace functional space

$$\widetilde{\mathbb{H}}^{1/2}(\Gamma) = \widetilde{H}^{1/2}(\Gamma_1) \times \cdots \times \widetilde{H}^{1/2}(\Gamma_n).$$

Its norm is characterized by

$$\|u\|^2_{\widetilde{\mathbb{H}}^{1/2}(\Gamma)} = \sum_{j=1}^{n} \|u_j\|^2_{\widetilde{H}^{1/2}(\Gamma_j)}.$$

Denote $\mathbb{H}^{-1/2}(\Gamma) = H^{-1/2}(\Gamma_1) \times \cdots \times H^{-1/2}(\Gamma_n)$, which is the dual space of $\widetilde{\mathbb{H}}^{1/2}(\Gamma)$. The norm on the space $\mathbb{H}^{-1/2}(\Gamma)$ is characterized by

$$\|u\|^2_{\mathbb{H}^{-1/2}(\Gamma)} = \sum_{j=1}^{n} \|u_j\|^2_{H^{-1/2}(\Gamma_j)}.$$

Introduce the following space

$$\mathbb{H}^1_S(\Omega) = H^1_{S_1}(\Omega_1) \times \cdots \times H^1_{S_n}(\Omega_n),$$

which is a Hilbert space with norm characterized by

$$\|u\|^2_{\mathbb{H}^1(\Omega)} = \sum_{j=1}^{n} \|u_j\|^2_{H^1(\Omega_j)}.$$

Multiplying the complex conjugate of test function $v_j \in H^1_{S_j}(\Omega_j)$ on both sides of (4.3), integrating over Ω_j, and using the integration by parts and boundary conditions (4.4) and (4.6), we obtain

$$\int_{\Omega_j} (\nabla u_j \cdot \nabla \bar{v}_j - \kappa_j^2 u_j \bar{v}_j) - \sum_{i=1}^{n} \langle T\tilde{u}_i, \tilde{v}_j \rangle = \langle g, v_j \rangle_{\Gamma_j}. \tag{4.10}$$

Taking summation of (4.10) for $j = 1, \ldots, n$, we deduce the variational formulation for the multiple cavity scattering problem: find $u \in \mathbb{H}^1_S(\Omega)$ with $u_j = u|_{\Omega_j}$ such that

$$a_3(u, v) = \sum_{j=1}^{n} \langle g, v_j \rangle_{\Gamma_j} \quad \text{for all } v \in \mathbb{H}^1_S(\Omega), \tag{4.11}$$

where the sesquilinear form

$$a_3(u, v) = \sum_{j=1}^{n} \int_{\Omega_j} (\nabla u_j \cdot \nabla \bar{v}_j - \kappa_j^2 u_j \bar{v}_j) - \sum_{j=1}^{n} \sum_{i=1}^{n} \langle T\tilde{u}_i, \tilde{v}_j \rangle. \tag{4.12}$$

Theorem 4.1. *The variational problem (4.11) has at most one solution.*

Proof. It suffices to show that $u_j = 0$ in Ω_j for $j = 1, \ldots, n$ if $g = 0$. If u_j satisfy the homogeneous variational problem in Ω_j, then we have

$$\sum_{j=1}^{n} \int_{\Omega_j} (|\nabla u_j|^2 - \kappa_j^2 |u_j|^2) - \sum_{j=1}^{n}\sum_{i=1}^{n} \langle T\tilde{u}_i, \tilde{u}_j\rangle = 0.$$

Noting $\mathrm{Im}\varepsilon_j \geqslant 0$ and taking the imaginary part yields

$$\mathrm{Im}\sum_{i=j}^{n}\sum_{i=1}^{n} \langle T\tilde{u}_i, \tilde{u}_j\rangle = 0.$$

Since $\tilde{u}_j$ has a compact support on the x-axis, $\hat{\tilde{u}}_j$ is analytical with respect to ξ. Hence we have from Lemma 4.1 that

$$\tilde{u} = \sum_{j=1}^{n} \tilde{u}_j = 0.$$

By the definition of the extensions $\tilde{u}_j$ we obtain

$$\tilde{u}_j = 0 \quad \text{on } \Gamma^c \cup \Gamma_1 \cup \cdots \cup \Gamma_n.$$

The transparent boundary condition (4.5) yields that $\partial_{\mathbf{n}}\tilde{u} = 0$ on $\Gamma^c \cup \Gamma_1 \cup \cdots \cup \Gamma_n$. An application of Holmgren uniqueness theorem yields $u = 0$ in $\mathbb{R}^2_+$. A unique continuation result in [21] concludes that $u_j = 0$ in Ω_j for $j = 1, \ldots, n$. $\square$

We have the following well-posedness result for the general multiple cavity scattering problem. The proof is similar in nature as that of the two cavity model problem and is omitted here for brevity.

Theorem 4.2. *The variational problem (4.11) has a unique weak solution u in $\mathbb{H}^1_S(\Omega)$ and the solution satisfies the estimate*

$$\|u\|_{\mathbb{H}^1(\Omega)} \leqslant C\|g\|_{\mathbb{H}^{-1/2}(\Gamma)},$$

where C is a positive constant.

5. Numerical experiments

In this section, we discuss the computational aspects for solving the multiple scattering problems, including the finite element approximation, a block Gauss–Seidel iteration method for the discrete weak formulation, and an alternative transparent boundary condition.

5.1. Finite element formulation

Let $\mathcal{M}_j$ be a regular conforming triangulation of Ω_j with M_j small triangular element and $V_j \subset H^1_{S_j}$ be the conforming linear finite element space over $\mathcal{M}_j$. Denote $\mathbb{V} = V_1 \times \ldots \times V_n$. The finite element approximation to the multiple cavity scattering problem (4.11) is: find $u^h \in V$ with $u^h_j \in V_j$ such that

$$a_3(u^h, v^h) = \sum_{j=1}^{n} \langle g, v^h_j\rangle_{\Gamma_j} \quad \text{for all } v^h \in \mathbb{V}, \tag{5.1}$$

where the sesquilinear form

$$a_3(u^h, v^h) = \sum_{j=1}^{n} \int_{\Omega_j} (\nabla u^h_j \cdot \nabla \bar{v}^h_j - \kappa_j^2 u^h_j \bar{v}^h_j) - \sum_{j=1}^{n}\sum_{i=1}^{n} \langle T\tilde{u}^h_i, \tilde{v}^h_j\rangle. \tag{5.2}$$

For any $1 \leqslant j \leqslant n$, we denote by $\mathcal{V}_j$ the set of N_j vertices of $\mathcal{M}_j$ which are not on the cavity wall S_j, and let $\varphi_j(\mathbf{r}) \in V_j$ be the nodal basis function belonging to vertex $\mathbf{r} \in \mathcal{V}_j$. Using the basis functions, the solution of (5.1) is represented as

$$u^h_j = \sum_{\mathbf{r}\in\mathcal{V}_j} u_j(\mathbf{r})\varphi_j(\mathbf{r}).$$

The discrete problem (5.1) is equivalent to the following system of algebraic equations

$$\begin{bmatrix} A_1 - B_{1,1} & -B_{2,1} & \cdots & -B_{n,1} \\ -B_{1,2} & A_2 - B_{2,2} & \cdots & -B_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ -B_{1,n} & -B_{2,n} & \cdots & A_n - B_{n,n} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_n \end{bmatrix} = \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \vdots \\ \mathbf{g}_n \end{bmatrix}, \tag{5.3}$$

where each $\mathbf{u}_j$ is an unknown vector whose entries are $\mathbf{u}_j(\mathbf{r}) = u_j^h(\mathbf{r})$ for all $\mathbf{r} \in \mathcal{V}_j$, A_j is the stiffness matrix for the discrete problem of u_j^h and its entries are defined by

$$A_j(\mathbf{r}, \mathbf{r}') = \int_{\Omega_j} \left[\nabla \varphi_j(\mathbf{r}) \cdot \nabla \varphi_j(\mathbf{r}') - \kappa_j^2 \varphi_j(\mathbf{r}) \varphi_j(\mathbf{r}') \right] \quad \text{for all } \mathbf{r}, \mathbf{r}' \in \mathcal{V}_j,$$

the entries of B_{ij} are defined by

$$B_{j,i}(\mathbf{r}, \mathbf{r}') = \langle T\tilde{\varphi}_i(\mathbf{r}), \tilde{\varphi}_j(\mathbf{r}') \rangle \quad \text{for all } \mathbf{r}, \mathbf{r}' \in \mathcal{V}_j \cap \Gamma_j$$

and the entries of each vector $\mathbf{g}_j$ are given by

$$\mathbf{g}_j(\mathbf{r}) = \langle g, \varphi_j(\mathbf{r}) \rangle_{\Gamma_j} \quad \text{for all } \mathbf{r} \in \mathcal{V}_j \cap \Gamma_j.$$

The linear system (5.3) requires to solve a large and coupled equation, particularly for multiple cavities. We present an efficient block Gauss–Seidel method, which may be written as follows: given $(\mathbf{u}_1^{(0)}, \cdots, \mathbf{u}_n^{(0)})^\top$, define $(\mathbf{u}_1^{(k)}, \ldots, \mathbf{u}_n^{(k)})^\top, k \geqslant 1$ by the solution of the following system of equations

$$(A_j - B_{j,j})\mathbf{u}_j^{(k)} = \mathbf{g}_j + \sum_{i=1}^{j-1} B_{j,i}\mathbf{u}_i^{(k)} + \sum_{i=j+1}^{n} B_{j,i}\mathbf{u}_i^{(k-1)}, \quad 1 \leqslant j \leqslant n. \tag{5.4}$$

The block Gauss–Seidel iteration (5.4) is equivalent to apply the finite element method for solving the following problem: Let $(u_1^{(0)}, \ldots, u_n^{(0)})^\top = (0, \ldots, 0)^\top$, define $(u_1^{(k)}, \ldots, u_n^{(k)})^\top$ for $k \geqslant 1$ by the solutions of the following decoupled equations for $j = 1, \ldots, n$:

$$\Delta u_j^{(k)} + \kappa_j^2 u_j^{(k)} = 0 \quad \text{in } \Omega_j, \tag{5.5}$$

$$u_j^{(k)} = 0 \quad \text{on } S_j, \tag{5.6}$$

$$\partial_{\mathbf{n}} u_j^{(k)} = \sum_{i=1}^{j} T\tilde{u}_i^{(k)} + \sum_{i=j+1}^{n} T\tilde{u}_i^{(k-1)} + g \quad \text{on } \Gamma_j, \tag{5.7}$$

Therefore, we only need to solve a well-posed single cavity scattering problems (5.5)–(5.7) for the block Gauss–Seidel method at each iteration.

5.2. Transparent boundary condition

Based on the boundary operator (2.11), the transparent boundary conditions (2.12), (3.7), and (4.5) are useful to carry mathematical analysis for the boundary value problems. However, they are not convenient to be implemented numerically. In practice, we adopt an alternative and equivalent transparent boundary condition [38].

Let

$$G(\mathbf{x}, \mathbf{x}') = \frac{\mathrm{i}}{4} \left[H_0^{(1)}(\kappa_0 \rho) - H_0^{(1)}(\kappa_0 \bar{\rho}) \right]$$

be the Green function of the Helmholtz equation in the upper half space, where $H_0^{(1)}$ is the Hankel function of the first kind with order zero, $\mathbf{x} = (x, y), \mathbf{x}' = (x', y'), \rho = |\mathbf{x} - \mathbf{x}'|, \bar{\rho} = |\mathbf{x} - \bar{\mathbf{x}}'|$, and $\bar{\mathbf{x}}' = (x', -y')$ is the image of $\mathbf{x}'$ with respect to the real axis. By the Green's theorem and the radiation condition, we obtain

$$\partial_y u^s(x, 0) = \frac{\mathrm{i}\kappa_0}{2} \int_\Gamma \frac{1}{|x - x'|} H_1^{(1)}(\kappa_0 |x - x'|) u^s(x', 0) dx',$$

where $H_1(1)$ is the Hankel function of the first kind with order one. Hence, the alternative boundary condition is

$$\partial_{\mathbf{n}} u = Tu + g \quad \text{on } \Gamma, \tag{5.8}$$

where the boundary operator T is defined as

$$Tu = \frac{\mathrm{i}\kappa_0}{2} \int_\Gamma \frac{1}{|x - x'|} H_1^{(1)}(\kappa_0 |x - x'|) u(x', 0) dx'. \tag{5.9}$$

Here the integral is understood in the sense of Hadamard finite-part. For multiple cavities with apertures given as $\Gamma_1 \cup \ldots \cup \Gamma_n$, the boundary operator is defined as

$$Tu = \frac{\mathrm{i}\kappa_0}{2} \sum_{j=1}^{n} \int_{\Gamma_j} \frac{1}{|x - x'|} H_1^{(1)}(\kappa_0 |x - x'|) u(x', 0) dx'.$$

The boundary operator (5.9) can be approximated by

$$Tu(x_i) \approx \sum_{k=1}^{m} g_{ik} u(x_k, 0),$$

where

$$\begin{aligned} \mathrm{Re}\, g_{ik} &= -t_{ik} \frac{\kappa_0 |x_i - x_k|}{2} Y_1(\kappa_0 |x_i - x_k|), \\ \mathrm{Im}\, g_{ik} &= \frac{\kappa_0 h_x}{2} \frac{J_1(\kappa_0 |x_i - x_k|)}{|x_i - x_k|} \end{aligned}$$

and

$$t_{ik} = \begin{cases} \frac{1}{h_x}(1 - \ln 2) & \text{for } |i-k| = 1, \\ -\frac{2}{h_x} & \text{for } |i-k| = 0, \\ \frac{1}{h_x} \ln \frac{|i-k|^2}{|i-k|^2 - 1} & \text{for } |i-k| \geqslant 2, \end{cases}$$

where h_x is the step size of the partition for the cavity aperture Γ. Therefore, the boundary integral $\langle Tu, v\rangle$ in the weak formulation for the cavity scattering problem can be approximated by any numerical quadratures.

5.3. Numerical examples

In this section, two examples of the cavity scattering model problems are presented to illustrate the proposed method. The physical parameter of interest is the radar cross section (RCS), which is defined by

$$\sigma = \frac{4}{\kappa_0} |P(\varphi)|^2.$$

Here ϕ is the observation angle and P is the far-field coefficient given by

$$P(\varphi) = \frac{\kappa_0}{2} \sin\phi \int_\Gamma u(x, 0) e^{i\kappa_0 x \cos\varphi} dx.$$

When the incident and observation directions are the same, σ is called the backscatter RCS, which is defined by

$$\text{Backscatter RCS}(\varphi) = 10\log_{10}\sigma(\varphi)\,\text{dB}.$$

In the following two examples, the numerical results are obtained by using a linear finite element over triangles at the wavenumber $\kappa_0 = \pi$, which accounts for the wavelength $\lambda = 2$.

Example 1. Consider a plane wave scattering from the model problem of three identical rectangular cavities at normal incidence, i.e., $\theta = 0$. Each cavity is 0.5λ wide and 0.5λ deep; there is 0.5λ distance away from each other. Specifically, the domains are given as follows for the three rectangular cavities:

$$\begin{aligned} &\text{Cavity1}: \quad [-2.5, -1.5] \times [-1.0, 0.0], \\ &\text{Cavity2}: \quad [-0.5, 0.5] \times [-1.0, 0.0], \\ &\text{Cavity3}: \quad [1.5, 2.5] \times [-1.0, 0.0]. \end{aligned}$$

Each rectangular domain is divided into 100×100 small equal rectangles and then each small rectangle is subdivided into two equal triangles.

Table 1 lists the six types of combination of cavities, where $\varepsilon = 1$ means an empty or unfilled cavity and $\varepsilon = 1 + \mathrm{i}$ stands for a filled cavity with some homogeneous absorbing medium.

Figs. 4–9 show the magnitude and the phase of the total field on the apertures at the normal incidence and the backscatter RCS for all the six types of cavities. These numerical results are obtained by using the block Gauss–Seidel iterative method. To show the convergence of the iterative method, we define the error between two consecutive approximations

Table 1
Six types of combination of cavities for Example 1.

ε	Type 1	Type 2	Type 3	Type 4	Type 5	Type 6
Cavity 1	1.0	1.0	1.0	1.0 + i	1.0 + i	1.0 + i
Cavity 2	1.0	1.0	1.0 + i	1.0 + i	1.0	1.0 + i
Cavity 3	1.0	1.0 + i	1.0	1.0	1.0 + i	1.0 + i

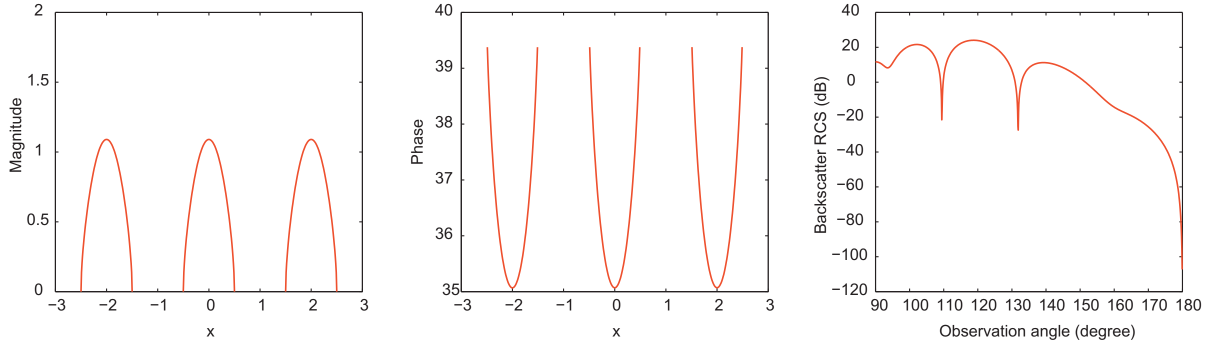

Fig. 4. The magnitude and the phase of the total field on the aperture at the normal incidence, and the backscatter RCS for Example 1 with type 1 cavity: (left) magnitude; (middle) phase; (right) backscatter RCS.

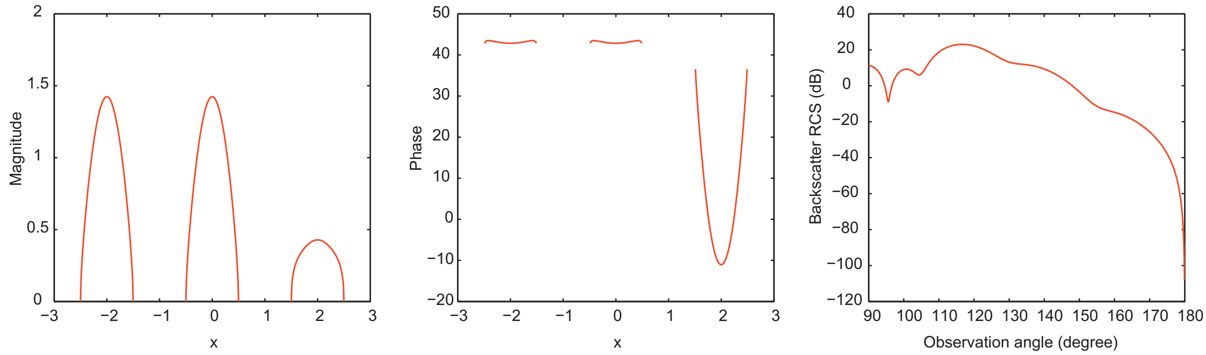

Fig. 5. The magnitude and the phase of the total field on the aperture at the normal incidence, and the backscatter RCS for Example 1 with type 2 cavity: (left) magnitude; (middle) phase; (right) backscatter RCS.

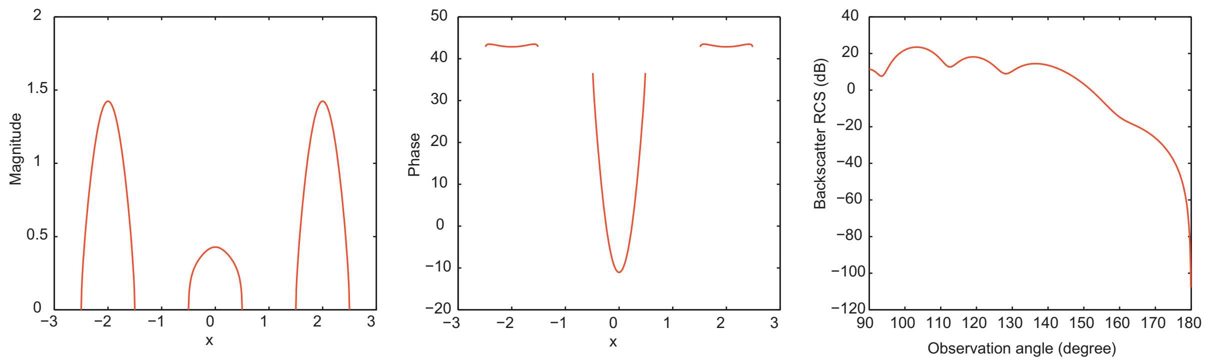

Fig. 6. The magnitude and the phase of the total field on the aperture at the normal incidence, and the backscatter RCS for Example 1 with type 3 cavity: (left) magnitude; (middle) phase; (right) backscatter RCS.

$$e_k = \max_{1\leqslant j\leqslant 3}\{\|u_j^{(k)} - u_j^{(k-1)}\|_{L^2(\Gamma_j)}\},$$

where k is the number of iteration. Fig. 10 shows the error e_k of the block Gauss–Seidel method for two consecutive approximations again the number of iterations for all three types of cavities. It can be seen from Fig. 10 that the type 1 cavity takes the most number of iterations to reach the same level of accuracy as the other types of cavities; the type 6 uses the least number of iterations to reach the same level of accuracy; the types 2 and 3 and the types 4 and 5 take almost the same number of iterations to reach the same level of accuracy, respectively, while the types 4 and 5 take less number of iterations than the types 2 and 3. The reason is that the types 2–6 cavities are filled with complex medium which accounts for the absorption of the energy and thus the damping of the amplitude of the field; the more number of cavities are filled with the absorption material, the less number of iterations are needed to reach certain level of accuracy.

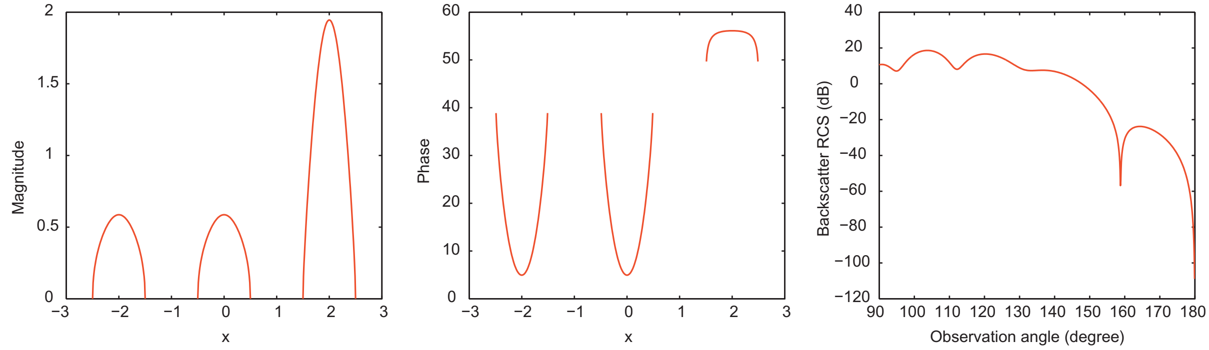

Fig. 7. The magnitude and the phase of the total field on the aperture at the normal incidence, and the backscatter RCS for Example 1 with type 4 cavity: (left) magnitude; (middle) phase; (right) backscatter RCS.

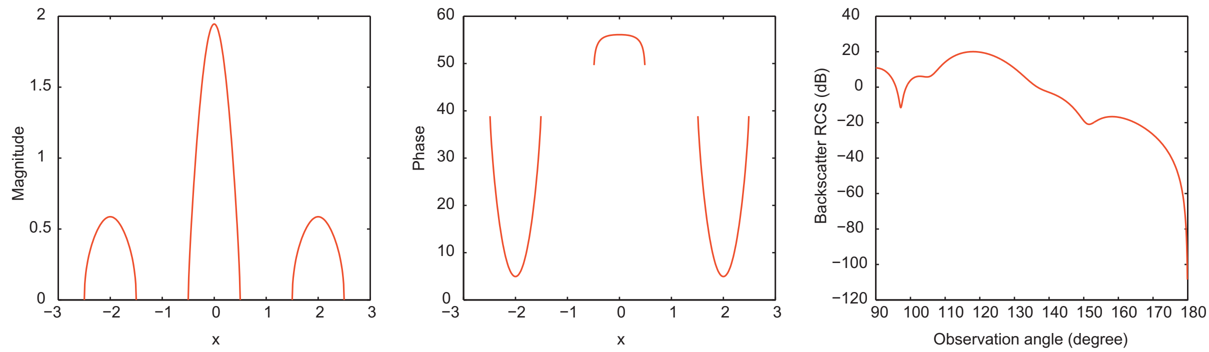

Fig. 8. The magnitude and the phase of the total field on the aperture at the normal incidence, and the backscatter RCS for Example 1 with type 5 cavity: (left) magnitude; (middle) phase; (right) backscatter RCS.

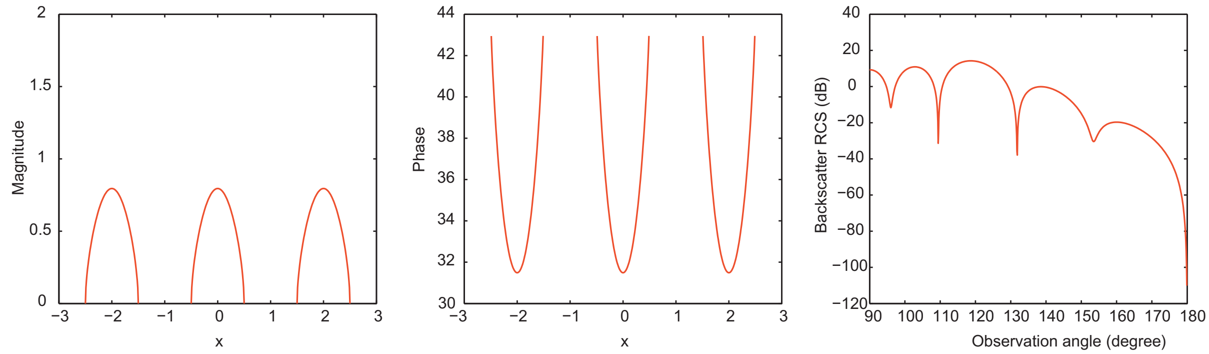

Fig. 9. The magnitude and the phase of the total field on the aperture at the normal incidence, and the backscatter RCS for Example 1 with type 6 cavity: (left) magnitude; (middle) phase; (right) backscatter RCS.

Example 2. Consider the scattering of a quintuple cavity model. Let a plane wave be incident onto five identical rectangular cavities at the normal direction. Each cavity is 0.5λ wide and 0.5λ deep; there is 0.5λ distance away from each other. The domains for the five rectangular cavities are given as follows:

$$
\begin{aligned}
&\text{Cavity 1}: && [-4.5, -3.5] \times [-1.0, 0.0],\\
&\text{Cavity 2}: && [-2.5, -1.5] \times [-1.0, 0.0],\\
&\text{Cavity 3}: && [-0.5, 0.5] \times [-1.0, 0.0],\\
&\text{Cavity 4}: && [1.5, 2.5] \times [-1.0, 0.0],\\
&\text{Cavity 5}: && [3.5, 4.5] \times [-1.0, 0.0].
\end{aligned}
$$

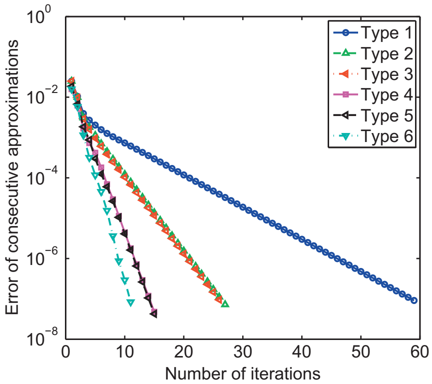

Fig. 10. The error of two consecutive approximations again the number of iterations for all six types of cavities in Example 1.

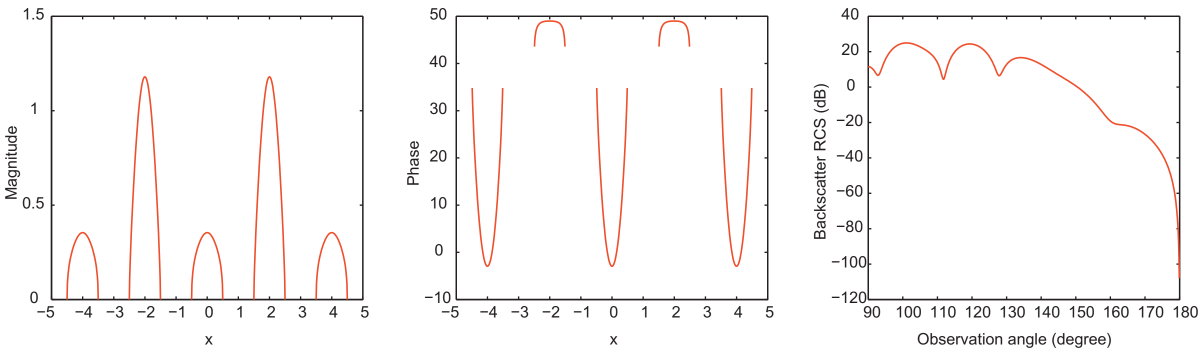

Fig. 11. The magnitude and the phase of the total field on the aperture at the normal incidence, and the backscatter RCS for Example 1: (left) magnitude; (middle) phase; (right) backscatter RCS.

Again, each rectangular domain is divided into 100×100 small equal rectangles and then each small rectangle is subdivided into two equal triangles. The cavity 2 and 4 are empty cavities with $\varepsilon = 1.0$, and the cavities 1, 3, and 5 are filled with the same homogeneous medium with $\varepsilon = 1.0 + i$. Fig. 11 shows the magnitude and the phase of the total field on the apertures at the normal incidence and the backscatter RCS.

6. Concluding remarks

The problem of electromagnetic scattering by cavities embedded in the infinite two-dimensional ground plane is an important area of research. This paper marks the first known mathematical investigation into the phenomenon of scattering by multiple cavities. We solve the problem by reducing the overall scattering problem to coupled single cavity scattering problems via the introduction of a novel transparent boundary condition over the cavity apertures. Uniqueness and existence of the variational formulation for the multiple cavity scattering problem is achieved. Numerical experiments of model problems demonstrate the efficiency and accuracy of our numerical methods. Our numerical approach gives rise to the wide spread applicability of numerical solvers developed for single cavity models to the general multiple cavity setting. Numerical examples indicate that the convergence of the Gauss–Seidel iterative method depends on the wavenumber κ_0 and the dielectric permittivity ε_j inside the cavities; it does not depend on the separation distance between cavities even for the distance which is smaller than half the wavelength. When the number of cavities is large, it does take a lot more time for the method to go over all the cavities in order to account for the multiple interaction among them. Future research includes the rigorous analysis of the convergence of the Gauss–Seidel iterative method, which requires the stability analysis of the cavity scattering problem (see [12] for a single cavity model). Multiple over-filled cavity models are another natural consideration. Finally, we note that transient scattering problems involving multiple cavities can be particularly interesting and challenging.

Acknowledgments

The authors thank Weiwei Sun of City University of Hong Kong and Kui Du of Xiamen University for helpful discussions.

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