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Convergence analysis in near-field imaging

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Abstract

This paper is devoted to the mathematical analysis of the direct and inverse modeling of the diffraction by a perfectly conducting grating surface in the near-field regime. It is motivated by our effort to analyze recent significant numerical results, in order to solve a class of inverse rough surface scattering problems in near-field imaging. In a model problem, the diffractive grating surface is assumed to be a small and smooth deformation of a plane surface. On the basis of the variational method, the direct problem is shown to have a unique weak solution. An analytical solution is introduced as a convergent power series in the deformation parameter by using the transformed field and Fourier series expansions. A local uniqueness result is proved for the inverse problem where only a single incident field is needed. On the basis of the analytic solution of the direct problem, an explicit reconstruction formula is presented for recovering the grating surface function with resolution beyond the Rayleigh criterion. Error estimates for the reconstructed grating surface are established with fully revealed dependence on such quantities as the surface deformation parameter, measurement distance, noise level of the scattering data, and regularity of the exact grating surface function.

Keywords: inverse diffraction grating, near-field imaging, error analysis

1. Introduction

Scattering problems are concerned with how an inhomogeneous medium scatters an incident wave. The direct scattering problem is that of determining the scattered field from a knowledge of the incident field and the differential equation governing the wave motion; the

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inverse scattering problem is that of determining the nature of the inhomogeneity, such as its geometry, from a knowledge of the scattered field. These problems play a fundamental role in many scientific areas such as radar and sonar, geophysical exploration, and medical imaging. According to the Rayleigh criterion, there is a resolution limit to the sharpness of details that can be observed by conventional far-field imaging: half of the wavelength, which is also referred to as the diffraction limit. Near-field imaging is an effective approach to breaking the diffraction limit and obtaining images with subwavelength resolution, which leads to exciting applications in broad areas of modern science and technology, including surface chemistry, biology, materials science, and information storage [23].

We consider, as a model problem, the diffraction when a time-harmonic electromagnetic plane wave is incident on a periodic structure, which is known as a diffractive grating [38]. There are two kinds of diffractive grating problems: given the periodic structure or the grating surface and the incident field, the direct problem is that of determining the diffracted field; the inverse problem is that of determining the grating surface from the measurement of the diffracted field, given the incident field. Recently, problems of scattering in periodic structures have received considerable attention in the applied mathematical community, and have been investigated extensively in both mathematical and numerical aspects. We refer the reader to [6, 9, 20, 21, 33, 34, 36, 37, 40] and references therein for the existence, uniqueness, and numerical approximations of solutions for the direct one-dimensional and two-dimensional grating problems. A comprehensive review can be found in [7] on diffractive optics technology and its mathematical modeling, as well as computational methods. The mathematical questions of uniqueness and stability for the inverse problems have been studied by many researchers [1, 5, 10, 16–18, 24, 28, 31, 41]. Computationally, a number of numerical methods have been developed for the reconstruction of perfectly conducting grating surfaces for the one-dimensional grating [4, 14, 15, 19, 25–27, 29, 30, 32, 35, 39]. These works addressed conventional far-field imaging, where the roles of evanescent wave components were ignored and the resolution of reconstructions was limited by Rayleigh's criterion. It is challenging to achieve a stable construction with subwavelength resolution due to the nonlinear and ill-posed nature of the inverse problem.

Recently, novel approaches have been developed for solving a class of inverse rough surface scattering problems in near-field imaging [8, 11, 12, 22]. Under the assumption that the scattering surface is a small and smooth deformation of a plane surface, the method begins with the transformed field expansion, to convert two-dimensional or three-dimensional boundary value problems into a successive sequence of one-dimensional two-point boundary value problems, which can be solved analytically. By keeping only the leading and linear terms in the power series expansion, the inverse problems are linearized and explicit reconstruction formulas are obtained. A spectral cutoff regularization is adopted to suppress the exponential growth of the noise. The method requires only a single incident field with one frequency and one incident direction, and is realized by using the fast Fourier transform. Numerical results show that the method is efficient and stable for reconstructing surfaces with subwavelength resolution.

In [11, 22], the authors presented the method as it was and showed numerical examples to illustrate the effectiveness of the proposed method. However, there was no justification as regards the mathematical issues such as the questions of the well-posedness of the model problem, convergence of the power series, uniqueness of the inverse problem, and error estimates for the inverse problem. This paper is devoted to the mathematical analysis of the model problem studied in [11, 22] and is intended to deal with all the mathematical issues raised above. For the direct problem, we give a criterion under which it has a unique weak solution by studying its variational form; for the power series, we show the convergence by

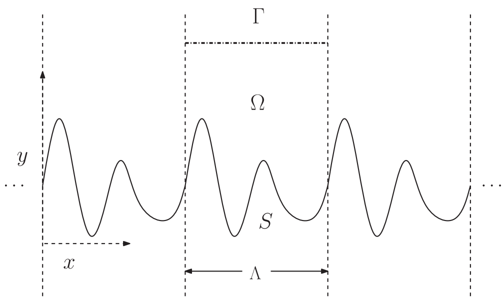

Figure 1. The problem geometry.

carefully studying the H^2 regularity of the solution for the recursive equation; for the inverse problem, we prove the uniqueness by estimating the first nonzero eigenvalue of the Dirichlet problem; for the reconstruction method, we show the error estimates for noiseless and noise data, illustrate the balance among resolution, stability, and accuracy, and, in particular, we give the best measurement height in terms of the perturbation parameter. Our results in this paper confirm those numerical observations, clarify the trade-off between the resolution and stability of reconstructions, and provide a deep understanding of the ill-posed nature of the inverse problem in near-field imaging. To the best of our knowledge, this is the first paper to mathematically analyze the near-field imaging of rough surfaces, and all the results are original contributions to this key area. Other related work may be found in [13] for an inverse surface scattering problem in the context of near-field imaging, and in [2, 3] for resolution and stability analysis of conductivity imaging.

The outline of the paper is as follows. Section 2 addresses the direct problem: the model problem is introduced for the diffraction of a plane wave by a grating surface; the direct problem is proved to have a unique weak solution by using the variational approach; an analytic solution is deduced for the direct problem by using the transformed field and Fourier expansions; the convergence of the power series is shown. The inverse problem is discussed in section 3: a local uniqueness result is described; an explicit reconstruction formula is presented; error estimates are established for the reconstruction method. The paper is concluded with some remarks and directions for future work in section 4.

2. Direct scattering

In this section, we introduce a model problem for the diffraction by a perfectly conducting grating. An analytic solution is deduced as a power series for the direct problem from the transformed field expansion together with the Fourier series expansion.

2.1. The model problem

Let us first specify the problem geometry. As seen in Figure 1, the problem may be restricted to a single period of Λ in x due to the periodicity of the grating surface. Without loss of generality, the period Λ is assumed to be 2π throughout the paper. Let the grating surface in one period be described by the curve

$$S = \Big\{ (x, y) \in \mathbb{R}^2 \colon y = f(x),\ 0 < x < 2\pi \Big\},$$

where $f \geqslant 0$ is a periodic function with period 2π and is assumed to take the form

$$f(x) = \varepsilon g(x), \quad g \in C^k(\mathbb{R}). \tag{2.1}$$

Here $2 \leqslant k \in \mathbb{N}$, ε is assumed to be a sufficiently small positive constant and is called the surface deformation parameter, and $g \geqslant 0$ is also a periodic function with period 2π. Define

$$M = \sup_{x\in\mathbb{R}} \left| \frac{\mathrm{d}^m}{\mathrm{d}x^m} g(x) \right|, \quad 0 \leqslant m \leqslant k, \tag{2.2}$$

which is one of the important parameters describing the error bound of the reconstructed grating surface. Let the space above S be filled with a homogeneous medium characterized by a positive constant wavenumber κ with an associated wavelength $\lambda = 2\pi/\kappa$. In the applications of near-field imaging, the wavelength is comparable in size to the period, i.e., $\lambda \sim \Lambda$. Hence, the wavenumber κ is of the order of 1, i.e., $\kappa = O(1)$.

Denote by

$$\Omega = \left\{ (x, y) \in \mathbb{R}^2 \colon f(x) < y < h,\ 0 < x < 2\pi \right\}$$

the domain bounded below by S and bounded above by

$$\Gamma = \left\{ (x, y) \in \mathbb{R}^2 \colon y = h,\ 0 < x < 2\pi \right\},$$

where h is a positive constant satisfying $h > \max_{x\in(0,\,2\pi)} f(x)$ and described as the measurement distance.

Let an incoming plane wave $u^{\mathrm{inc}}(x, y) = \mathrm{e}^{\mathrm{i}\kappa(x \sin\theta - y \cos\theta)}$ be incident on the grating surface from above, where $\theta \in (-\pi/2, \pi/2)$ is the incident angle. For normal incidence, i.e., $\theta = 0$, the incident field reduces to

$$u^{\mathrm{inc}}(x, y) = \mathrm{e}^{-\mathrm{i}\kappa y}. \tag{2.3}$$

For simplicity, we focus on the normal incidence from now on, since our method requires only a single incident wave with one wavenumber and one incident direction. We mention that the analysis works for general non-normal incidence with obvious modifications.

The diffraction of a time-harmonic electromagnetic wave in the transverse electric polarization can be modeled by the two-dimensional Helmholtz equation:

$$\Delta u + \kappa^2 u = 0 \quad \text{in } \Omega. \tag{2.4}$$

For a perfectly conducting grating, the total field u vanishes on the grating surface:

$$u = 0 \quad \text{on } S. \tag{2.5}$$

Motivated by uniqueness, we are interested in the periodic solution, i.e., u satisfies

$$u(x + 2\pi, y) = u(x, y).$$

For any periodic function $u(x)$ with period 2π, it has a Fourier series expansion

$$u(x) = \sum_{n\in\mathbb{Z}} u^{(n)} \mathrm{e}^{\mathrm{i}nx}, \quad u^{(n)} = \frac{1}{2\pi} \int_0^{2\pi} u(x) \mathrm{e}^{-\mathrm{i}nx}\, \mathrm{d}x.$$

Define an operator

$$(Tu)(x) = \sum_{n\in\mathbb{Z}} \mathrm{i}\beta_n u^{(n)} \mathrm{e}^{\mathrm{i}nx}, \tag{2.6}$$

where

$$\beta_n = \begin{cases} \left(\kappa^2 - n^2\right)^{1/2} & \text{for} \quad \kappa \geqslant |n|, \\ \mathrm{i}\left(n^2 - \kappa^2\right)^{1/2} & \text{for} \quad \kappa < |n|. \end{cases}$$

We point out that the assumption that $\beta_n \neq 0$, i.e., $\kappa \neq |n|$, for $n \in \mathbb{Z}$, is not necessary in this paper. The resonance $\beta_n = 0$ is included in our analysis. It follows from Rayleigh's expansion that u satisfies the transparent boundary condition

$$\partial_y u = Tu + \rho \quad \text{on } \Gamma, \tag{2.7}$$

where $\rho(x) = -2\mathrm{i}\kappa \mathrm{e}^{-\mathrm{i}\kappa h}$.

Given the surface S and the incident field u^{inc}, the direct diffractive grating problem is to find the periodic solution u of the boundary value problem (2.4)–(2.7). Given the normal incident field u^{inc}, the inverse diffractive grating problem is that of determining the grating surface S, i.e., the periodic function f, from the measurement of the noisy field u on Γ, i.e., $u^\delta(x, h)$, at a fixed wavenumber κ, where δ is the noise level. In particular, we are interested in the inverse problem in the near-field regime where the measurement distance h is much smaller than the wavelength λ.

We point out that the following two hypotheses are adopted in the paper:

$$\text{(H1) } \kappa h < 1 \quad \text{and} \quad \text{(H2) } M\varepsilon h^{-1} < 1.$$

The first hypothesis (H1) ensures the uniqueness and existence of a weak solution for the direct problem; while the second hypothesis guarantees the convergence of an analytic power series solution for the direct problem. Recall the grating surface function $f = \varepsilon g$ and the measurement distance $h > \max_x f(x) = \varepsilon \max_x g(x)$. For sufficiently small surface deformation parameter ε, the measurement distance h can also be taken as a sufficiently small positive number such that both of the hypotheses (H1) and (H2) can be satisfied. For instance, we may take $h = \varepsilon^{1/2}$ and then the hypotheses (H1) and (H2) reduce to $\kappa\varepsilon^{1/2} < 1$ and $M\varepsilon^{1/2} < 1$, which are satisfied for sufficiently small ε. Throughout the paper, $a \lesssim b$ stands for $a \leqslant Cb$, where C is a positive constant independent of ε, h, δ, and M.

2.2. The variational problem

To describe the boundary value problem and derive its variational formulation, we need to introduce some functional spaces.

Define the Sobolev spaces $H^s(\Omega) = \{u \colon D^\alpha u \in L^2(\Omega) \text{for all } |\alpha| \leqslant s\}$ and $H_S^1(\Omega) = \{u \in H^1(\Omega) \colon u = 0 \text{ on } S\}$, which are Banach spaces for the norm

$$\| u \|_{s,\Omega} = \left[\sum_{|\alpha| \leqslant s} \int_\Omega \left| D^\alpha u(x, y) \right|^2 \, \mathrm{d}x \, \mathrm{d}y \right]^{1/2}.$$

Introduce the periodic functional space

$$H^1_{S,\,\mathrm{p}}(\Omega) = \left\{ u \in H^1_S(\Omega) \colon u(0, y) = u(2\pi, y) \right\},$$

which is a subspace of $H_S^1(\Omega)$. For a periodic function u defined on Γ with Fourier coefficient $u^{(n)}$, we define the trace functional space

$$H^s(\Gamma) = \left\{ u \in L^2(\Gamma): \sum_{n\in\mathbb{Z}} \left(1 + n^2\right)^s \left|u^{(n)}\right|^2 < \infty \right\},$$

which is also a Banach space for the norm

$$\|u\|_{s,\Gamma} = \left[2\pi \sum_{n\in\mathbb{Z}} \left(1 + n^2\right)^s \left|u^{(n)}\right|^2 \right]^{1/2}.$$

It is clear that the dual space associated with $H^s(\Gamma)$ is the space $H^{-s}(\Gamma)$ with respect to the scalar product in $L^2(\Gamma)$ defined by

$$\langle u, v\rangle_\Gamma = \int_\Gamma u\bar{v}\, \mathrm{d}x = 2\pi \sum_{n\in\mathbb{Z}} u^{(n)}\bar{v}^{(n)}. \tag{2.8}$$

The following Poincaré inequality and trace regularity results are useful in subsequent analysis.

Lemma 2.1. *The estimate*

$$\|u\|_{0,\Omega} \lesssim h\, \|\nabla u\|_{0,\Omega}$$

holds for any $u \in H_S^1(\Omega)$.

Proof. Define the rectangular domain

$$D = \left\{ (x, y) \in \mathbb{R}^2: 0 < x < 2\pi,\ 0 < y < h \right\} \supseteq \Omega.$$

For any $u \in H_S^1(\Omega)$, consider the zero extension of u to the domain D:

$$\tilde{u}(x, y) = \begin{cases} u(x, y) & \text{for } (x, y) \in \Omega, \\ 0 & \text{for } (x, y) \in D\backslash\Omega. \end{cases} \tag{2.9}$$

It is clearly seen that

$$\|u\|_{0,\Omega} = \|\tilde{u}\|_{0,D} \quad \text{and} \quad \|\nabla u\|_{0,\Omega} = \|\nabla\tilde{u}\|_{0,D}.$$

Simple calculation yields

$$\tilde{u}(x, y) = \int_0^y \partial_y\tilde{u}(x, y)\, \mathrm{d}y,$$

which implies, by applying the Cauchy–Schwarz inequality, that

$$\left|\tilde{u}(x, y)\right|^2 \leqslant h \int_0^h \left|\partial_y\tilde{u}(x, y)\right|^2 \mathrm{d}y.$$

Hence we have

$$\|\tilde{u}\|_{0,D} \lesssim h\, \|\nabla\tilde{u}\|_{0,D},$$

which leads to the assertion of the lemma. □

Lemma 2.2. *The estimate*

$$\|u\|_{1/2,\Gamma} \lesssim \|u\|_{1,\Omega}$$

holds for any $u \in H^1_{S,\,\mathrm{p}}(\Omega)$.

Proof. For any $u \in H^1_{S,\,\mathrm{p}}(\Omega)$, let $\tilde{u}$ be its zero extension to the domain D defined in (2.9). It is easy to see that

$$u(x, h) = \tilde{u}(x, h) = \sum_{n\in\mathbb{Z}} \tilde{u}^{(n)}(h)\mathrm{e}^{\mathrm{i}nx}$$

and

$$\| u \|^2_{1/2,\,\Gamma} = \| \tilde{u} \|^2_{1/2,\,\Gamma} = 2\pi \sum_{n\in\mathbb{Z}} \left(1 + n^2\right)^{1/2} \left| \tilde{u}^{(n)}(h) \right|^2.$$

It follows from the Cauchy–Schwarz inequality that

$$\begin{aligned}\left|\tilde{u}^{(n)}(h)\right|^2 &= \int_0^h \frac{\mathrm{d}}{\mathrm{d}y} \left|\tilde{u}^{(n)}(y)\right|^2 \,\mathrm{d}y \leqslant \int_0^h 2\left|\tilde{u}^{(n)}(y)\right| \, \left|\frac{\mathrm{d}}{\mathrm{d}y}\tilde{u}^{(n)}(y)\right| \,\mathrm{d}y\\ &\leqslant \left(1 + n^2\right)^{1/2} \int_0^h \left|\tilde{u}^{(n)}(y)\right|^2 \mathrm{d}y + \left(1 + n^2\right)^{-1/2} \int_0^h \left|\frac{\mathrm{d}}{\mathrm{d}y}\tilde{u}^{(n)}(y)\right|^2, \mathrm{d}y,\end{aligned} \tag{2.10}$$

which gives

$$\left(1 + n^2\right)^{1/2} \left|\tilde{u}^{(n)}(h)\right|^2 \leqslant \left(1 + n^2\right) \int_0^h \left|\tilde{u}^{(n)}(y)\right|^2 \,\mathrm{d}y + \int_0^h \left|\frac{\mathrm{d}}{\mathrm{d}y}\tilde{u}^{(n)}(y)\right|^2 \,\mathrm{d}y.$$

Using the Fourier series expansion of $\tilde{u}$, we can verify that

$$\| u \|^2_{1,\,\Omega} = \| \tilde{u} \|^2_{H^1(D)} = 2\pi \sum_{n\in\mathbb{Z}} \left(1 + n^2\right) \int_0^h \left|\tilde{u}^{(n)}(y)\right|^2 \,\mathrm{d}y + \int_0^h \left|\frac{\mathrm{d}}{\mathrm{d}y}\tilde{u}^{(n)}(y)\right|^2 \,\mathrm{d}y.$$

Combining the above estimates yields the result. □

Lemma 2.3. *The estimate*

$$\|\partial^2_{xx} u\|_{-1/2,\Gamma} \lesssim \|u\|_{2,\Omega}$$

holds for any $u \in H^2_{S,\,\mathrm{p}}(\Omega)$.

Proof. For any $u \in H^2_{S,\,\mathrm{p}}(\Omega)$, let $\tilde{u}$ be its zero extension to the domain D defined in (2.9). It is easy to verify that

$$\begin{aligned}\left\|\partial^2_{xx} u\right\|^2_{-1/2,\Gamma} &= \left\|\partial^2_{xx}\tilde{u}\right\|^2_{-1/2,\Gamma} = 2\pi \sum_{n\in\mathbb{Z}} n^4 \left(1 + n^2\right)^{-1/2} \left|\tilde{u}^{(n)}\right|^2\\ &\leqslant 2\pi \sum_{n\in\mathbb{Z}} \left(1 + n^2\right)^{3/2} \left|\tilde{u}^{(n)}\right|^2 = \left\|\tilde{u}\right\|^2_{3/2,\,\Gamma} = \| u \|^2_{3/2,\,\Gamma}.\end{aligned}$$

Using (2.10), we have

$$\left(1 + n^2\right)^{3/2} \left|\tilde{u}^{(n)}(h)\right|^2 \leqslant \left(1 + n^2\right)^2 \int_0^h \left|\tilde{u}^{(n)}(y)\right|^2 \,\mathrm{d}y + \left(1 + n^2\right) \int_0^h \left|\frac{\mathrm{d}}{\mathrm{d}y}\tilde{u}^{(n)}(y)\right|^2 \,\mathrm{d}y.$$

Simple calculations yields

$$\|u\|^2_{2,\,\Omega} = \|\tilde{u}\|^2_{2,\,D} = 2\pi \sum_{n\in\mathbb{Z}} \left(1 + n^2 + n^4\right) \int_0^h \left|\tilde{u}^{(n)}(y)\right|^2 \mathrm{d}y$$

$$+\left(1 + n^2\right) \int_0^h \left|\frac{\mathrm{d}}{\mathrm{d}y}\tilde{u}^{(n)}(y)\right|^2 \mathrm{d}y + \int_0^h \left|\frac{\mathrm{d}^2}{\mathrm{d}y^2}\tilde{u}^{(n)}(y)\right|^2 \mathrm{d}y.$$

Combining with the above inequality yields the result. □

The following two lemmas are concerned with the continuity and analyticity of the boundary operator, respectively.

Lemma 2.4. *The boundary operator T: $H^s(\Gamma) \to H^{s-1}(\Gamma)$ is continuous, i.e.,*

$$\|Tu\|_{s-1,\Gamma} \lesssim \|u\|_{s,\Gamma}$$

for any $u \in H^s(\Gamma)$.

Proof. For any $u \in H^s(\Gamma)$, it follows from (2.6) that

$$\|\, Tu\,\|^2_{s-1,\,\Gamma} = 2\pi \sum_{n\in\mathbb{Z}} \left(1 + n^2\right)^{s-1} \left|\beta_n\right|^2 \left|u^{(n)}\right|^2$$

$$= 2\pi \sum_{n\in\mathbb{Z}} \left(1 + n^2\right)^s \left(\frac{\left|\beta_n\right|}{\left(1 + n^2\right)^{1/2}}\right)^2 \left|u^{(n)}\right|^2.$$

To prove the lemma, it is necessary to estimate

$$F_n = \frac{\left|\beta_n\right|}{\left(1 + n^2\right)^{1/2}} = \frac{\left|\kappa^2 - n^2\right|^{1/2}}{\left(1 + n^2\right)^{1/2}}, \quad n \in \mathbb{Z}.$$

Define a function

$$F(\xi) = \frac{\left|\kappa^2 - \xi^2\right|^{1/2}}{\left(1 + \xi^2\right)^{1/2}}, \quad \xi \in \mathbb{R}.$$

It can be verified that the even function $F(\xi)$ decreases for $0 < \xi < \kappa$ and increases for $\xi > \kappa$. Hence a simple calculation yields

$$F_n \leqslant \max_{\xi\in\mathbb{R}} F(\xi) = \max\,\{F(0),\ F(\infty)\} = \max\,\{\kappa,\ 1\} = C,$$

which completes the proof. □

Lemma 2.5. *The estimates*

$$\mathrm{Re}\,\langle Tu,\, u\rangle_\Gamma \leqslant 0 \quad \text{and} \quad \mathrm{Im}\,\langle Tu,\, v\rangle_\Gamma \geqslant 0$$

hold for any $u \in H^{1/2}(\Gamma)$.

Proof. By the definitions of (2.6) and (2.8), we have for any $u \in H^{1/2}(\Gamma)$ that

$$\langle Tu, u \rangle_{\Gamma} = \mathrm{i}\, 2\pi \sum_{n \in \mathbb{Z}} \beta_n \left| u^{(n)} \right|^2.$$

Taking the real part gives

$$\mathrm{Re}\, \langle Tu, u \rangle_{\Gamma} = -\, 2\pi \sum_{|n| > \kappa} \left(n^2 - \kappa^2 \right)^{1/2} \left| u^{(n)} \right|^2 \leqslant 0$$

and taking the imaginary part gives

$$\mathrm{Im}\, \langle Tu, u \rangle_{\Gamma} = 2\pi \sum_{|n| < \kappa} \left(\kappa^2 - n^2 \right)^{1/2} \left| u^{(n)} \right|^2 \geqslant 0,$$

which completes the proof. □

We next present a variational formulation for the direct diffractive grating problem and give a proof of the well-posedness of this boundary value problem.

Multiplying (2.4) by the complex conjugate of a test function $v \in H^1_{S,\,\mathrm{p}}(\Omega)$, integrating over Ω, and using integration by parts, we deduce the variational formulation for the direct problem (2.4)–(2.7): find $u \in H^1_{S,\,\mathrm{p}}(\Omega)$ such that

$$a_{\Omega}(u, v) = \langle \rho, v \rangle_{\Gamma} \quad \text{for all } v \in H^1_{S,\,\mathrm{p}}(\Omega), \tag{2.11}$$

where the sesquilinear form

$$a_{\Omega}(u, v) = \int_{\Omega} \nabla u \cdot \nabla \bar{v}\, \mathrm{d}x\, \mathrm{d}y - \kappa^2 \int_{\Omega} u \bar{v}\, \mathrm{d}x\, \mathrm{d}y - \langle Tu, v \rangle_{\Gamma}. \tag{2.12}$$

Theorem 2.6. *The variational problem (2.11) has a unique weak solution u in $H^1_{S,\,\mathrm{p}}(\Omega)$, which satisfies the estimate*

$$\| u \|_{1,\Omega} \lesssim \| \rho \|_{-1/2,\Gamma}.$$

Proof. It suffices to prove the continuity and coercivity of the sesquilinear form a_{Ω}. The continuity follows directly from the Cauchy–Schwarz inequality, lemma 2.2, and lemma 2.4:

$$\begin{aligned} \left| a_{\Omega}(u, v) \right| &\lesssim \| \nabla u \|_{0,\Omega} \| \nabla v \|_{0,\Omega} + \| u \|_{0,\Omega} \| v \|_{0,\Omega} + \| Tu \|_{-1/2,\Gamma} \| v \|_{1/2,\Gamma} \\ &\lesssim \| \nabla u \|_{0,\Omega} \| \nabla v \|_{0,\Omega} + \| u \|_{0,\Omega} \| v \|_{0,\Omega} + \| u \|_{1/2,\Gamma} \| v \|_{1/2,\Gamma} \\ &\lesssim \| \nabla u \|_{0,\Omega} \| \nabla v \|_{0,\Omega} + \| u \|_{0,\Omega} \| v \|_{0,\Omega} + \| u \|_{1,\Omega} \| v \|_{1,\Omega} \\ &\lesssim \| u \|_{1,\Omega} \| v \|_{1,\Omega}. \end{aligned}$$

Replacing v by u in (2.12) yields

$$a_{\Omega}(u, u) = \int_{\Omega} |\nabla u|^2\, \mathrm{d}x\, \mathrm{d}y - \kappa^2 \int_{\Omega} |u|^2\, \mathrm{d}x\, \mathrm{d}y - \langle Tu, u \rangle_{\Gamma}.$$

Let $t = \left(1 - (\kappa h)^2 \right) \big/ (1 + h^{-2})$. By the hypothesis (H1), i.e., $\kappa h < 1$, we have $0 < t < 1$. Taking the real part, and applying lemma 2.5 and lemma 2.1, we obtain

$$
\begin{aligned}
\mathrm{Re}\ a_\Omega(u, u) &= \|\nabla u\|_{0,\Omega}^2 - \kappa^2 \|u\|_{0,\Omega}^2 - \mathrm{Re}\,\langle Tu, u\rangle_\Gamma \geqslant \|\nabla u\|_{0,\Omega}^2 - \kappa^2 \|u\|_{0,\Omega}^2 \\
&= t\ \|\nabla u\|_{0,\Omega}^2 + (1-t)\|\nabla u\|_{0,\Omega}^2 - \kappa^2 \|u\|_{0,\Omega}^2 \\
&\geqslant th^{-2}\ \|u\|_{0,\Omega}^2 + \left[1 - t - (\kappa h)^2\right] \|\nabla u\|_{0,\Omega}^2 \\
&= \left(\frac{1-(\kappa h)^2}{1+h^2}\right)\|u\|_{1,\Omega}^2 \gtrsim \|u\|_{1,\Omega}^2.
\end{aligned}
$$

It follows from the Lax–Milgram lemma that there exists a unique weak solution of the variational problem (2.11) in $H^1_{S,\,\mathrm{p}}(\Omega)$. Furthermore, we have from the coercivity of a_Ω and trace regularity in lemma 2.2 that the solution u satisfies

$$
\begin{aligned}
\|u\|_{1,\Omega}^2 &\lesssim |a(u, u)| = \left|\langle \rho, u\rangle_\Gamma\right| \\
&\leqslant \|\rho\|_{-1/2,\Gamma}\|u\|_{1/2,\Gamma} \lesssim \|\rho\|_{-1/2,\Gamma}\|u\|_{1,\Omega},
\end{aligned}
$$

which completes the proof. □

2.3. The analytic solution

We present the transformed field expansion for analytically deriving the solution of the direct problem. Consider the change of variables

$$
\tilde{x} = x, \qquad \tilde{y} = h\left(\frac{y-f}{h-f}\right),
$$

which maps the domain Ω to the rectangle D.

Introduce a new function $\tilde{u}(\tilde{x}, \tilde{y}) = u(x, y)$ under the transformation. Dropping the tilde for simplicity of notation, we can verify from (2.4)–(2.7) that u satisfies the following boundary value problem:

$$
\begin{cases}
c_1\partial_{xx}^2 u + c_2\partial_{yy}^2 u + c_3\partial_{xy}^2 u + c_4\partial_y u + c_1\kappa^2 u = 0 & \text{in } D, \\
u = 0 & \text{on } y = 0, \\
\partial_y u = \left(1 - h^{-1}f\right)(Tu + \rho) & \text{on } y = h,
\end{cases}
\tag{2.13}
$$

where

$$
\begin{aligned}
c_1 &= (h-f)^2, \\
c_2 &= [f'(h-y)]^2 + h^2, \\
c_3 &= -2f'(h-y)(h-f), \\
c_4 &= -(h-y)\left[f''(h-f) + 2(f')^2\right].
\end{aligned}
$$

Following the boundary perturbation method, we consider a formal expansion of u in a power series in ε:

$$
u(x, y; \varepsilon) = \sum_{m=0}^{\infty} u_m(x, y)\varepsilon^m. \tag{2.14}
$$

Substituting $f = \varepsilon g$ into c_j and inserting (2.14) into (2.13), we may derive a recursion equation for u_m:

$$
\Delta u_m + \kappa^2 u_m = v_m \quad \text{in } D, \tag{2.15}
$$

where

$$v_m = h^{-1}\Big[2g\partial^2_{xx}u_{m-1} + 2\,g'(h-y)\partial^2_{xy}u_{m-1} + g''(h-y)\partial_y u_{m-1} + 2\kappa^2\,gu_{m-1}\Big]$$
$$-h^{-2}\Big[g^2\partial^2_{xx}u_{m-2} - (g')^2(h-y)^2\partial^2_{yy}u_{m-2} - 2gg'(h-y)\partial^2_{xy}u_{m-2}$$
$$+\Big(2(g')^2 - gg''\Big)(h-y)\partial_y u_{m-2} - \kappa^2\,g^2u_{m-2}\Big]. \qquad (2.16)$$

In addition, u_m satisfies the boundary conditions

$$\begin{cases} u_m(x,y) = 0 & \text{on } y = 0 \\ \partial_y u_m - Tu_m = \rho_m & \text{on } y = h, \end{cases} \qquad (2.17)$$

where

$$\rho_0 = \rho, \quad \rho_m = -\,gh^{-1}Tu_{m-1}, \;\; m \geqslant 1. \qquad (2.18)$$

Next we deduce an analytic solution to the problem (2.15)–(2.18). Since u_m, v_m, ρ_m are all periodic functions in x with period 2π, they follow the Fourier series expansions

$$u_m(x,y) = \sum_{m\in\mathbb{Z}} u_m^{(n)}(y)\mathrm{e}^{\mathrm{i}nx},$$
$$v_m(x,y) = \sum_{m\in\mathbb{Z}} v_m^{(n)}(y)\mathrm{e}^{\mathrm{i}nx},$$
$$\rho_m(x) = \sum_{m\in\mathbb{Z}} \rho_m^{(n)}\mathrm{e}^{\mathrm{i}nx}.$$

Substituting these expansions into (2.15)–(2.17), we obtain a two-point boundary value problem for the Fourier coefficient $u_m^{(n)}$:

$$\begin{cases} \dfrac{\mathrm{d}^2u_m^{(n)}}{\mathrm{d}y^2} + \beta_n^2 u_m^{(n)} = v_m^{(n)}, & 0 < y < h, \\ u_m^{(n)} = 0 & \text{at } y = 0, \\ \dfrac{\mathrm{d}u_m^{(n)}}{\mathrm{d}y} - \mathrm{i}\beta_n u_m^{(n)} = \rho_m^{(n)} & \text{at } y = h. \end{cases} \qquad (2.19)$$

It follows from [11, 22] that we may obtain an explicit solution of (2.19).

Theorem 2.7. *The two−point boundary value problem (2.19) has a unique solution, which is given by*

$$u_m^{(n)}(y) = K_1^{(n)}(y)\rho_m^{(n)} - \int_0^h K_2^{(n)}(y,z)v_m^{(n)}(z)\mathrm{d}z, \qquad (2.20)$$

where

$$K_1^{(n)}(y) = \frac{e^{i\beta_n h}}{2i\beta_n}\left(e^{i\beta_n y} - e^{-i\beta_n y}\right) \tag{2.21}$$

and

$$K_2^{(n)}(y, z) = \begin{cases} \dfrac{e^{i\beta_n y}}{2i\beta_n}\left(e^{i\beta_n z} - e^{-i\beta_n z}\right), & z < y, \\ \dfrac{e^{i\beta_n z}}{2i\beta_n}\left(e^{i\beta_n y} - e^{-i\beta_n y}\right), & z > y. \end{cases} \tag{2.22}$$

Remark 2.8. The solution representation (2.20) is still valid for $\beta_n = 0$. In this case, $K_1^{(n)}$ and $K_2^{(n)}$ reduce to

$$K_1^{(n)}(y) = y, \quad K_2^{(n)}(y, z) = \begin{cases} z, & z < y, \\ y, & z > y. \end{cases}$$

In particular, we may derive a compact form for the leading term u_0. Recalling (2.16) and (2.18), we have

$$v_0 = 0, \quad \rho_0 = -\,2i\kappa e^{-i\kappa h},$$

whose Fourier coefficients are

$$v_0^{(n)} = 0, \quad \rho_0^{(n)} = \begin{cases} -2i\kappa e^{-i\kappa h}, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

Using the solution representation (2.20) and noting that $\beta_0 = \beta$, we get

$$\begin{aligned} u_0^{(n)}(y) = K_1^{(n)}(y)\rho_0^{(n)} &= \frac{e^{i\kappa h}}{2i\kappa}\left(e^{i\kappa y} - e^{-i\kappa y}\right)\rho_0^{(n)} \\ &= \begin{cases} e^{-i\kappa y} - e^{i\kappa y}, & n = 0, \\ 0, & n \neq 0, \end{cases} \end{aligned}$$

which yields

$$u_0(x, y) = \sum_{n\in\mathbb{Z}} u_0^{(n)}(y) e^{i\alpha_n x} = e^{-i\kappa y} - e^{i\kappa y}. \tag{2.23}$$

It is clearly seen that the leading term u_0 consists of the incident field $e^{-i\kappa y}$ and the reflected field $-e^{i\kappa y}$, which arise from the interaction of the incident wave and the plane surface $y = 0$.

2.4. Convergence

In this section, we prove the well-posedness of and present an energy estimate of the solution for the recursion problem (2.15)–(2.18) in order to show the convergence of the power series (2.14).

Introduce the Banach space

$$H^1_{0,\,\mathrm{p}}(D) = \left\{ u \in H^1(D)\colon u(0, y) = u(2\pi, y),\ \ u(x, 0) = 0,\ \ 0 < x < 2\pi \right\}.$$

Multiplying (2.15) by a test function $w \in H^1_{0,\,\mathrm{p}}(D)$ and using integration by parts, we may deduce the variational formulation for the problem (2.15)–(2.18): find $u_m \in H^1_{0,\,\mathrm{p}}(D)$ such that

$$a_D(u_m, w) = \left\langle \rho_m, w \right\rangle_\Gamma - \left\langle v_m, w \right\rangle_D \quad \text{for all } w \in H^1_{0,\,\mathrm{p}}(D), \tag{2.24}$$

where the sesquilinear form

$$a_D(u_m, w) = \int_D \nabla u_m \cdot \nabla \bar{w}\, \mathrm{d}x\, \mathrm{d}y - \kappa^2 \int_D u_m \bar{w}\, \mathrm{d}x\, \mathrm{d}y - \left\langle T u_m, w \right\rangle_\Gamma, \tag{2.25}$$

and the linear functional

$$\left\langle v_m, w \right\rangle_D = \int_D v_m \bar{w}\, \mathrm{d}x\, \mathrm{d}y.$$

Theorem 2.9. *The variational problem (2.24) has a unique weak solution u_m in $H^1_{0,\,\mathrm{p}}(\Omega)$, which satisfies the estimate*

$$\| u_m \|_{1,D} \lesssim \| \rho_m \|_{-1/2,\Gamma} + \| v_m \|_{0,D}.$$

Proof. Following a proof analogous to that of theorem 2.6, we may show that the sesquilinear form a_D is bounded and coercive. Therefore, the variational problem (2.24) has a unique weak solution u_m in $H^1_{0,\,\mathrm{p}}(D)$. In addition, the solution u_m satisfies

$$\begin{aligned}
\| u_m \|^2_{1,\,D} \lesssim \left| a_D(u_m, u_m) \right| &= \left| \left\langle \rho_m, u_m \right\rangle_\Gamma - \left\langle v_m, u_m \right\rangle_D \right| \\
&\leqslant \| \rho_m \|_{-1/2,\Gamma} \| u_m \|_{1/2,\Gamma} + \| v_m \|_{0,D} \| u_m \|_{0,D} \\
&\lesssim \left(\| \rho_m \|_{-1/2,\Gamma} + \| v_m \|_{0,D} \right) \| u_m \|_{1,D},
\end{aligned}$$

which yields the energy estimate and completes the proof. □

The following lemmas help to prove the convergence of the power series (2.14).

Lemma 2.10. *The estimate*

$$\| g w \|_{1/2,\Gamma} \lesssim M \| w \|_{1/2,\Gamma}$$

holds for any $w \in H^{1/2}(\Gamma)$.

Proof. Using an equivalent norm in $H^{1/2}(\Gamma)$ and the mean value theorem, we have

$$\begin{aligned}
\| gw\|_{1/2,\,\Gamma}^2 &= \| gw\|_{0,\,\Gamma}^2 + \int_\Gamma\int_\Gamma \frac{|g(t)w(t) - g(s)w(s)|^2}{|t-s|^2}\,\mathrm{d}t\,\mathrm{d}s \\
&\lesssim M^2\,\| w\|_{0,\,\Gamma}^2 + \int_\Gamma\int_\Gamma \frac{|w(t) - w(s)|^2}{|t-s|^2}|g(s)|^2\,\mathrm{d}t\,\mathrm{d}s \\
&\quad + \int_\Gamma\int_\Gamma \frac{|g(t) - g(s)|^2}{|t-s|^2}|w(t)|^2\,\mathrm{d}t\,\mathrm{d}s \\
&\lesssim M^2\,\| w\|_{0,\,\Gamma}^2 + M^2\,\| w\|_{1/2,\,\Gamma}^2 + M^2\int_\Gamma\int_\Gamma |w(t)|^2\,\mathrm{d}t\,\mathrm{d}s \\
&\lesssim M^2\,\| w\|_{0,\,\Gamma}^2 + M^2\,\| w\|_{1/2,\,\Gamma}^2 + M^2\,\| w\|_{0,\,\Gamma}^2 \\
&\lesssim M^2\,\| w\|_{1/2,\,\Gamma}^2 .
\end{aligned}$$

The proof is completed after taking the square root on both sides of the above inequality. □

Lemma 2.11. *The estimate*

$$\|\rho_m\|_{-1/2,\Gamma} \lesssim Mh^{-1}\,\| u_{m-1}\|_{2,D}$$

holds.

Proof. It follows from (2.16) and lemma 2.10 that

$$\begin{aligned}
\|\rho_m\|_{-1/2,\Gamma} &= h^{-1}\,\| g\,Tu_{m-1}\|_{-1/2,\Gamma} = h^{-1}\max_{w\in H^{1/2}(\Gamma)} \frac{\left|\left\langle g\,Tu_{m-1},\ w\right\rangle_\Gamma\right|}{\| w\|_{1/2,\Gamma}} \\
&= h^{-1}\max_{w\in H^{1/2}(\Gamma)} \frac{\left|\left\langle Tu_{m-1},\ gw\right\rangle_\Gamma\right|}{\| w\|_{1/2,\Gamma}} \\
&\leqslant h^{-1}\max_{w\in H^{1/2}(\Gamma)} \frac{\| Tu_{m-1}\|_{-1/2,\Gamma}\ \| gw\|_{1/2,\Gamma}}{\| w\|_{1/2,\Gamma}} \\
&\lesssim Mh^{-1}\,\| u_{m-1}\|_{1/2,\Gamma} \lesssim Mh^{-1}\,\| u_{m-1}\|_{1,D} \lesssim Mh^{-1}\,\| u_{m-1}\|_{2,D},
\end{aligned}$$

which completes the proof. □

Lemma 2.12. *The estimate*

$$\| v_m\|_{0,\,D}^2 \lesssim \left(Mh^{-1}\right)^2\,\| u_{m-1}\|_{2,\,D}^2 + \left(Mh^{-1}\right)^4\,\| u_{m-2}\|_{2,\,D}^2$$

holds.

Proof. Let

$$w_1 = h^{-1}\Big[\,2\,g\partial_{xx}^2 u_{m-1} + 2\,g'(h-y)\partial_{xy}^2 u_{m-1} + g''(h-y)\partial_y u_{m-1} + 2\kappa^2\,g u_{m-1}\Big]$$

and

$$w_2 = h^{-2}\Big[g^2\partial^2_{xx}u_{m-2} + (g')^2 (h-y)^2\partial^2_{yy}u_{m-2} + 2gg'(h-y)\partial^2_{xy}u_{m-2} - \left(2(g')^2 - gg''\right)(h-y)\partial_y u_{m-2} + \kappa^2 g^2 u_{m-2}\Big].$$

It follows from (2.16) that we have

$$v_m = w_1 - w_2 \quad \text{and} \quad \left|v_m\right|^2 \lesssim \left|w_1\right|^2 + \left|w_2\right|^2.$$

Using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\left|w_1\right|^2 &\leqslant h^{-2}\left(|2\,g|^2 + |2\,g'|^2 + |g''|^2 + \left|2\kappa^2\,g\right|^2\right) \times \left(\left|\partial^2_{xx}u_{m-1}\right|^2 \right.\\ &\quad \left. + |h-y|^2\left|\partial^2_{xy}u_{m-1}\right|^2 + |h-y|^2\left|\partial_y u_{m-1}\right|^2 + \left|u_{m-1}\right|^2\right)\\ &\lesssim \left(Mh^{-1}\right)^2 \sum_{|\alpha|\leqslant 2}\left|D^\alpha u_{m-1}\right|^2\end{aligned}$$

and

$$\begin{aligned}\left|w_2\right|^2 &\leqslant h^{-4}\left(|g|^4 + |g'|^4 + |2\,gg'|^2 + \left|2(g')^2 - gg''\right|^2 + \kappa^4|g|^4\right) \times \left(\left|\partial^2_{xx}u_{m-2}\right|^2\right.\\ &\quad \left. + |h-y|^4\left|\partial^2_{yy}u_{m-2}\right|^2 + |h-y|^2\left|\partial^2_{xy}u_{m-2}\right|^2 + |h-y|^2\left|\partial_y u_{m-2}\right|^2 + \left|u_{m-2}\right|^2\right)\\ &\lesssim \left(Mh^{-1}\right)^4 \sum_{|\alpha|\leqslant 2}\left|D^\alpha u_{m-2}\right|^2.\end{aligned}$$

Combining the above estimates yields

$$\| v_m \|^2_{0,\,D} \lesssim \| w_1 \|^2_{0,\,D} + \| w_2 \|^2_{0,\,D} \lesssim \left(Mh^{-1}\right)^2 \| u_{m-1} \|^2_{2,\,D} + \left(Mh^{-1}\right)^4 \| u_{m-2} \|^2_{2,\,D},$$

which completes the proof after taking the square root. □

The following H^2 estimate plays an important role in the convergence of the power series.

Lemma 2.13. *The solution u_m of the variational problem (2.24) satisfies the estimate*

$$\| u_m \|_{2,D} \lesssim \left(Mh^{-1}\right)^m, \quad m \geqslant 0.$$

Proof. The proof is based on the method of induction. Clearly, the explicit solution of the leading term u_0 in (2.23) verifies the assertion for $m = 0$. Assuming that the assertion is true for all $0, 1, \ldots, m-1$, we next show that the assertion is also true for m.

Using integration by parts and the periodicity of the solution, we obtain from a straightforward calculation that

$$\int_D |\Delta u_m|^2 \, \mathrm{d}x \, \mathrm{d}y = \int_D \left|\partial_{xx}^2 u_m + \partial_{yy}^2 u_m\right|^2 \, \mathrm{d}x \, \mathrm{d}y$$
$$= \int_D \left|\partial_{xx}^2 u_m\right|^2 \, \mathrm{d}x \, \mathrm{d}y + \int_D \left|\partial_{yy}^2 u_m\right|^2 \, \mathrm{d}x \, \mathrm{d}y + 2 \, \operatorname{Re} \int_D \partial_{xx}^2 \bar{u}_m \partial_{yy}^2 u_m \, \mathrm{d}x \, \mathrm{d}y$$
$$= \int_D \left|\partial_{xx}^2 u_m\right|^2 \, \mathrm{d}x \, \mathrm{d}y + \int_D \left|\partial_{yy}^2 u_m\right|^2 \, \mathrm{d}x \, \mathrm{d}y + 2 \int_D \left|\partial_{xy}^2 u_m\right|^2 \, \mathrm{d}x \, \mathrm{d}y$$
$$+2 \, \operatorname{Re} \left\langle \partial_y u_m, \, \partial_{xx}^2 u_m \right\rangle_\Gamma.$$

Using the boundary condition (2.17), we have

$$\left\langle \partial_y u_m, \, \partial_{xx}^2 u_m \right\rangle_\Gamma = \left\langle T u_m + \rho_m, \, \partial_{xx}^2 u_m \right\rangle_\Gamma = \left\langle T u_m, \, \partial_{xx}^2 u_m \right\rangle_\Gamma + \left\langle \rho_m, \, \partial_{xx}^2 u_m \right\rangle_\Gamma$$

Let $u_m(x, h)$ have the following Fourier series expansion:

$$u_m(x, h) = \sum_{n \in \mathbb{Z}} u_m^{(n)}(h) \mathrm{e}^{\mathrm{i}nx}.$$

Simple calculation yields

$$\operatorname{Re} \left\langle T u_m, \, \partial_{xx}^2 u_m \right\rangle_\Gamma = 2\pi \sum_{|n| > \kappa} n^2 \left(n^2 - \kappa^2\right)^{1/2} \left| u_m^{(n)}(h) \right|^2 \geqslant 0.$$

Using lemma 2.3 and lemma 2.10, we get

$$\left| \left\langle \rho_m, \, \partial_{xx}^2 u_m \right\rangle_\Gamma \right| = h^{-1} \left| \left\langle g T u_{m-1}, \, \partial_{xx}^2 u_m \right\rangle_\Gamma \right|$$
$$\leqslant h^{-1} \, \| g T u_{m-1} \|_{1/2,\Gamma} \| \partial_{xx}^2 u_m \|_{-1/2,\Gamma}$$
$$\leqslant M h^{-1} \, \| T u_{m-1} \|_{1/2,\Gamma} \, \| \partial_{xx}^2 u_m \|_{-1/2,\Gamma}$$
$$\leqslant M h^{-1} \, \| u_{m-1} \|_{3/2,\Gamma} \, \| \partial_{xx}^2 u_m \|_{-1/2,\Gamma}$$
$$\leqslant M h^{-1} \, \| u_{m-1} \|_{2,D} \, \| u_m \|_{2,D},$$

which yields after using the Cauchy–Schwarz inequality that

$$2 \left| \left\langle \rho_m, \, \partial_{xx}^2 u_m \right\rangle_\Gamma \right| \leqslant 2 \left(M h^{-1} \right)^2 \| u_{m-1} \|_{2, D}^2 + \frac{1}{2} \| u_m \|_{2, D}^2.$$

Consequently we have

$$\| u_m \|_{2, D}^2 \leqslant \| \Delta u_m \|_{0, D}^2 + \| u_m \|_{1, D}^2 - 2 \, \operatorname{Re} \left\langle \partial_y u_m, \, \partial_{xx}^2 u_m \right\rangle_\Gamma$$
$$\leqslant \| v_m - \kappa^2 u_m \|_{0, D}^2 + \| u_m \|_{1, D}^2 - 2 \, \operatorname{Re} \left\langle \rho_m, \, \partial_{xx}^2 u_m \right\rangle_\Gamma$$
$$\lesssim \| v_m \|_{0, D}^2 + \| u_m \|_{1, D}^2 + 2 \left| \left\langle \rho_m, \, \partial_{xx}^2 u_m \right\rangle_\Gamma \right|$$
$$\lesssim \| v_m \|_{0, D}^2 + \| u_m \|_{1, D}^2 + 2 \left(M h^{-1} \right)^2 \| u_{m-1} \|_{2, D}^2 + \frac{1}{2} \| u_m \|_{2, D}^2,$$

which yields

$$\| u_m \|_{2, D}^2 \lesssim \| v_m \|_{0, D}^2 + \| u_m \|_{1, D}^2 + \left(M h^{-1} \right)^2 \| u_{m-1} \|_{2, D}^2. \tag{2.26}$$

Using (2.26) and theorem 2.9, we obtain

$$
\begin{aligned}
\|u_m\|^2_{2,D} &\lesssim \|v_m\|^2_{0,D} + \|\rho_m\|^2_{-1/2,\Gamma} + \left(Mh^{-1}\right)^2 \|u_{m-1}\|^2_{2,D} \\
&\lesssim \left(Mh^{-1}\right)^2 \|u_{m-1}\|^2_{2,D} + \left(Mh^{-1}\right)^4 \|u_{m-2}\|^2_{2,D} \\
&\lesssim \left(Mh^{-1}\right)^2 \left(Mh^{-1}\right)^{2(m-1)} + \left(Mh^{-1}\right)^4 \left(Mh^{-1}\right)^{2(m-2)} \\
&\lesssim \left(Mh^{-1}\right)^{2m},
\end{aligned}
$$

which completes the proof. □

Theorem 2.14. *The power series (2.15) converges strongly.*

Proof. It suffices to prove that the power series (2.15) is dominated by a convergent geometric series. It follows from theorem 2.13 that

$$
\|u\|_{2,D} = \left\| \sum_{m=0}^{\infty} u_m \varepsilon^m \right\|_{2,D} \leqslant \sum_{m=0}^{\infty} \|u_m\|_{2,D} \quad \varepsilon^m \lesssim \sum_{m=0}^{\infty} \left(M\varepsilon h^{-1}\right)^m,
$$

which converges under the hypothesis (H2), i.e., $M\varepsilon h^{-1} < 1$. □

In theorem 2.6, it is shown that the direct problem (2.4)–(2.7) has a unique weak solution u in $H^1_{S,\mathrm{p}}(\Omega)$. In this section, the convergence analysis gives an indirect proof of the regularity of u, which is in $H^2_{S,\mathrm{p}}(\Omega)$.

3. Inverse scattering

In this section, we give a simple proof of uniqueness for the inverse diffraction grating problem, present an explicit inversion formula for the grating surface, and show error estimates for the reconstructed grating surface.

3.1. Uniqueness

The following local uniqueness result only requires a single incident field with one frequency and one incident direction. The proof is based on a combination of Holmgren's uniqueness and unique continuation theorems.

Let $G \subset \mathbb{R}^2$ be an bounded set with Lipschitz boundary ∂G. Define the depth of domain G by

$$
\operatorname{dep}(G) = \sup \left| y_1 - y_2 \right| \qquad \text{for any } (x_1, y_1), (x_2, y_2) \in G.
$$

Lemma 3.1. *The boundary value problem*

$$
\begin{cases} \Delta u + \kappa^2 u = 0 & \text{in } G, \\ u = 0 & \text{on } \partial G, \end{cases}
$$

has only a trivial solution if $\kappa \operatorname{dep}(G) < 1$.

Proof. It is easy to verify that the solution u satisfies

$$\|\nabla u\|_{0,G} = \kappa \|u\|_{0,G}.$$

Following the same method of proof as for lemma 2.1, we have

$$\|u\|_{0,G} \leqslant \mathrm{dep}(G) \|\nabla u\|_{0,G}.$$

Combining the above estimates yields $\kappa\, \mathrm{dep}(G) \geqslant 1$, which contradicts the assumption. □

Theorem 3.2. *Assume $f_j = \varepsilon g_j$, $g_j \in C^k(\mathbb{R})$, $j = 1, 2$, to be a periodic function with period 2π. Define $\Omega_j = \{(x, y) \in \mathbb{R}^2\colon 0 < x < 2\pi,\ f_j < y < h\}$. Let u_j be the unique weak solution of (2.11) in Ω_j. If $u_1 = u_2$ on Γ, then $f_1 = f_2$.*

Proof. Define $S_j = \{(x, y) \in \mathbb{R}\colon y = f_j(x), 0 < x < 2\pi\}$. Assume that $f_1 \neq f_2$. Then $\Omega_1 \backslash (\Omega_1 \cap \Omega_2)$ or $\Omega_2 \backslash (\Omega_1 \cap \Omega_2)$ is a non-empty set. Without loss of generality, we assume that $G = \Omega_1 \backslash (\Omega_1 \cap \Omega_2) \neq \varnothing$. It is easy to see that

$$\mathrm{dep}(G) \leqslant \varepsilon \max \left\{ \max_{x \in (0,\, 2\pi)} g_1(x), \quad \max_{x \in (0,\, 2\pi)} g_2(x) \right\},$$

which gives $\kappa\, \mathrm{dep}(G) < 1$ for sufficiently small ε. Denote ∂G by $C_1 \cup C_2$ with $C_j \subset S_j$. Since $u_1 - u_2 = 0$ on Γ, it follows from (2.7) that $\partial_y u_1 - \partial_y u_2 = 0$ on Γ. An application of Holmgren's uniqueness theorem yields $u_1 - u_2 = 0$ above Γ. By unique continuation, we get $u_1 - u_2 = 0$ in $\overline{\Omega_1 \cap \Omega_2}$ and, in particular, $u_1 - u_2 = 0$ on C_2. It follows from $u_2 = 0$ on C_2 that we have $u_1 = 0$ on C_2 and the problem

$$\begin{cases} \Delta u_1 + \kappa^2 u_1 = 0 & \text{in } G, \\ u_1 = 0 & \text{on } \partial G. \end{cases}$$

According to lemma 3.1, the above boundary value problem only has a trivial solution $u_1 = 0$ in G. An application of unique continuation again gives $u_1 = 0$ in Ω. But this contradicts the transparent boundary condition (2.7) since ρ is a nonzero function involving the incoming plane wave. □

The uniqueness result indicates that two grating surfaces will be identical if the diffracted fields are identical and if the two surfaces are close to a plane surface or, essentially, the area between the two surfaces is sufficiently small.

3.2. The reconstruction formula

We briefly describe an explicit reconstruction formula in this section. The details may be found in [11, 22].

Rewrite power series (2.14) as

$$u(x, y) = u_0(x, y) + \varepsilon u_1(x, y) + r(x, y), \tag{3.1}$$

where the remainder

$$r(x, y) = \sum_{m=2}^{\infty} u_m(x, y) \varepsilon^m.$$

Lemma 3.3. *The remainder satisfies the estimate*

$$\|r\|_{0,\Gamma} \lesssim \left(M\varepsilon h^{-1}\right)^2.$$

Proof. It follows from theorem 2.14 that

$$\begin{aligned}\|r\|_{0,\Gamma} &= \left\|\sum_{m=2}^{\infty} u_m\ \varepsilon^m\right\|_{0,\Gamma} \leqslant \sum_{m=2}^{\infty} \|u_m\|_{0,\Gamma}\ \ \varepsilon^m \leqslant \sum_{m=2}^{\infty} \|u_m\|_{1/2,\Gamma}\ \ \varepsilon^m \\ &\leqslant \sum_{m=2}^{\infty} \|u_m\|_{1,D}\ \ \varepsilon^m \leqslant \sum_{m=2}^{\infty} \|u_m\|_{2,D}\ \ \varepsilon^m \lesssim \sum_{m=2}^{\infty} \left(M\varepsilon h^{-1}\right)^m \lesssim \left(M\varepsilon h^{-1}\right)^2,\end{aligned}$$

where the hypothesis (H2) and the property of the convergent geometric series are used in the last inequality. □

Evaluating (3.1) at $y = h$ we have

$$u(x, h) = u_0(x, h) + \varepsilon u_1(x, h) + r(x, h).$$

Rearranging yields

$$\varepsilon u_1(x, h) = u(x, h) - u_0(x, h) - r(x, h). \tag{3.2}$$

In [11, 22], we showed a key identity

$$u_1^{(n)}(h) = 2\mathrm{i}\kappa \mathrm{e}^{\mathrm{i}\beta_n h} g_n, \tag{3.3}$$

where $u_1^{(n)}(h)$ and $g^{(n)}$ are the Fourier coefficients of $u_1(x, y)$ and $g(x)$, respectively. Let $f^{(n)}$ be the Fourier coefficient of f. Combining (3.2) and (3.3) and noting that $f^{(n)} = \varepsilon g^{(n)}$, we obtain

$$f^{(n)} = -\frac{\mathrm{i}}{2\kappa}\left[u^{(n)}(h) - u_0^{(n)}(h) - r^{(n)}(h)\right]\mathrm{e}^{-\mathrm{i}\beta_n h}, \tag{3.4}$$

where $u^{(n)}(h), u_0^{(n)}(h), r^{(n)}(h)$ are the Fourier coefficients of $u(x, h), u_0(x, h), r(x, h)$, respectively. Explicitly, we have

$$u_0^{(n)}(h) = \begin{cases} \mathrm{e}^{-\mathrm{i}\kappa h} - \mathrm{e}^{\mathrm{i}\kappa h}, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

Using the Fourier coefficient (3.4), the grating surface function can be written as

$$f(x) = \sum_{n\in\mathbb{Z}} f^{(n)} \mathrm{e}^{\mathrm{i}nx}. \tag{3.5}$$

Neglecting the remainder $r^{(n)}(h)$ in (3.4), we obtain

$$f_\varepsilon^{(n)} = -\frac{\mathrm{i}}{2\kappa}\left[u^{(n)}(h) - u_0^{(n)}(h)\right]\mathrm{e}^{-\mathrm{i}\beta_n h}, \tag{3.6}$$

which is the reconstruction formula for the linearized inverse problem with noise-free data. In practice, in the scattering data $u(x, h)$ there is always a certain level of noise contained. Let $u_\delta(x, h)$ be the noise data, satisfying

$$\|u_\delta - u\|_{0,\Gamma} = \left[2\pi \sum_{n\in\mathbb{Z}} \left|u_\delta^{(n)}(h) - u^{(n)}(h)\right|^2\right]^{1/2} \leqslant \delta, \tag{3.7}$$

where $0 < \delta < 1$ represents the noise level and $u_\delta^{(n)}(h)$ is the Fourier coefficient of the noise data $u_\delta(x, h)$. Replacing $u^{(n)}(h)$ with $u_\delta^{(n)}(h)$ in (3.6) yields an explicit reconstruction formula for the Fourier coefficient of the grating surface function with noise data:

$$f_{\varepsilon,\delta}^{(n)} = -\frac{\mathrm{i}}{2\kappa}\left[u_\delta^{(n)}(h) - u_0^{(n)}(h)\right]\mathrm{e}^{-\mathrm{i}\beta_n h}. \tag{3.8}$$

It follows from the definition of β_n and (3.8) that it is well-posed to reconstruct those Fourier coefficients $f^{(n)}$ with $|n| < \kappa$, since the small variations of the measured data will not be amplified and lead to large errors in the reconstruction, but the resolution of the reconstructed function f is restricted by the given wavenumber κ. In contrast, it is severely ill-posed to reconstruct those Fourier coefficients $f^{(n)}$ with $|n| > \kappa$, since the small variations in the data will be exponentially enlarged and lead to huge errors in the reconstruction, but they contribute to the superresolution of the reconstructed function f.

To obtain a stable and superresolved reconstruction, we may adopt a regularization to suppress the exponential growth of the reconstruction errors. We consider the spectral cutoff regularization. For fixed wavenumber κ and measurement distance h, the cutoff frequency ω is chosen in such a way that

$$\mathrm{e}^{(\omega^2-\kappa^2)^{1/2}h} = N,$$

where N is called the frequency cutoff criterion or the signal-to-noise ratio (SNR) [13]. More explicitly, we have

$$\omega = \left[\kappa^2 + \left(h^{-1}\log N\right)^2\right]^{1/2}, \tag{3.9}$$

which indicates that $\omega > \kappa$ as long as $N > 1$, as is natural to assume, and superresolution may be achieved.

Taking into account the frequency cutoff and using the Fourier coefficient (3.8), we may have the regularized reconstruction formula for noise-free scattering data:

$$f_\varepsilon(x) = \sum_{n\in\mathbb{Z}} f_\varepsilon^{(n)}\chi_{[-\omega,\omega]}(n)\mathrm{e}^{\mathrm{i}nx}, \tag{3.10}$$

and the regularized reconstruction formula for noisy scattering data:

$$f_{\varepsilon,\delta}(x) = \sum_{n\in\mathbb{Z}} f_{\varepsilon,\delta}^{(n)}\chi_{[-\omega,\omega]}(n)\mathrm{e}^{\mathrm{i}nx}, \tag{3.11}$$

where the characteristic function

$$\chi_{[-\omega,\omega]}(n) = \begin{cases} 1 & \text{for } |n| \leqslant \omega, \\ 0 & \text{for } |n| > \omega. \end{cases}$$

Therefore, the cutoff frequency ω determines the highest Fourier mode which can be recovered from the reconstruction formulas (3.10) and (3.11). As shown in (3.9), the cutoff frequency ω is an increasing function of N and a decreasing function of the measurement distance h; larger N or smaller h may help to achieve better resolution of the reconstruction. However, larger N or smaller h makes the method less stable and may bring about a larger approximation error. Therefore, there is a trade-off between the resolution and the stability of the reconstruction, which is clarified by the error estimates.

3.3. The error estimate

In this section, we present two error estimates of the reconstructed grating surface functions f_ε and $f_{\varepsilon,\delta}$, corresponding to the noise-free and noise scattering data, respectively.

The following lemma is a standard result about the regularity of a function and the decay rate of its Fourier coefficient.

Lemma 3.4. *Let $g \in C^k$ and $g^{(n)}$ be its Fourier coefficient. The estimate*

$$\left| g^{(n)} \right| \leqslant \frac{M}{|n|^k}, \qquad n \in \mathbb{Z},$$

holds.

Proof. By the periodicity of g, we have from integration by parts that

$$g^{(n)} = \frac{1}{2\pi} \int_0^{2\pi} g(x) \mathrm{e}^{-\mathrm{i}nx} \, \mathrm{d}x = \frac{1}{\mathrm{i}n} \frac{1}{2\pi} \int_0^{2\pi} g'(x) \mathrm{e}^{-\mathrm{i}nx} \mathrm{d}x.$$

Repeating the integration by parts yields

$$g^{(n)} = \frac{1}{(\mathrm{i}n)^k} \frac{1}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}^k}{\mathrm{d}x^k} g(x) \mathrm{e}^{-\mathrm{i}nx} \, \mathrm{d}x,$$

which completes the proof on noting the definition of M. □

We begin with an error estimate for the noise-free scattering data.

Theorem 3.5. *Let f and f_ε be the exact and reconstructed grating surface functions in (3.5) and (3.10), respectively. The error estimate*

$$\|f - f_\varepsilon\|_{0,\Gamma} \lesssim N \left(M\varepsilon h^{-1} \right)^2 + M\varepsilon \left(h^{-1} \log N \right)^{-(2k-1)/2}$$

holds.

Proof. It follows from (3.5), (3.6), and (3.10) that we have

$$\|f - f_\varepsilon\|_{0,\Gamma}^2 = 2\pi \sum_{|n| \leqslant \omega} \left| f^{(n)} - f_\varepsilon^{(n)} \right|^2 + 2\pi \sum_{|n| > \omega} \left| f^{(n)} \right|^2. \tag{3.12}$$

Using (3.4), (3.6), (3.9), and lemma 3.3 gives

$$\begin{aligned} \sum_{|n| \leqslant \omega} \left| f^{(n)} - f_\varepsilon^{(n)} \right|^2 &= \sum_{|n| < \kappa} \left| f^{(n)} - f_\varepsilon^{(n)} \right|^2 + \sum_{\kappa < |n| \leqslant \omega} \left| f^{(n)} - f_\varepsilon^{(n)} \right|^2 \\ &\leqslant \left(\frac{1}{2\kappa} \right)^2 \sum_{|n| < \kappa} \left| r^{(n)}(h) \right|^2 \left| \mathrm{e}^{-\mathrm{i} \left(\kappa^2 - n^2 \right)^{1/2} h} \right|^2 \\ &\quad + \left(\frac{1}{2\kappa} \right)^2 \sum_{\kappa < |n| \leqslant \omega} \left| r^{(n)}(h) \right|^2 \left| \mathrm{e}^{\left(n^2 - \kappa^2 \right)^{1/2} h} \right|^2 \end{aligned}$$

$$\leqslant \left(\frac{1}{2\kappa}\right)^2 \sum_{|n|<\kappa} \left|r^{(n)}(h)\right|^2 + \left(\frac{N}{2\kappa}\right)^2 \sum_{\kappa<|n|\leqslant\omega} \left|r^{(n)}(h)\right|^2$$

$$\leqslant \left(\frac{N}{2\kappa}\right)^2 \sum_{|n|\leqslant\omega} \left|r^{(n)}(h)\right|^2 \leqslant \left(\frac{N}{2\kappa}\right)^2 \sum_{n\in\mathbb{Z}} \left|r^{(n)}(h)\right|^2$$

$$\lesssim N^2 \| \ r \ \|_{0,\,\Gamma}^2 \lesssim N^2 \left(M\varepsilon h^{-1}\right)^4. \tag{3.13}$$

It follows from lemma 3.4 and the integral test that

$$\sum_{|n|>\omega} \left|f^{(n)}\right|^2 = \varepsilon^2 \sum_{|n|>\omega} \left|g^{(n)}\right|^2 \leqslant (M\varepsilon)^2 \sum_{|n|>\omega} |n|^{-2k}$$

$$\lesssim (M\varepsilon)^2 \int_{\omega}^{\infty} t^{-2k} \mathrm{d}t \lesssim (M\varepsilon)^2 \omega^{-2k+1}$$

$$\lesssim (M\varepsilon)^2 \left[\kappa^2 + \left(h^{-1}\log N\right)^2\right]^{-(2k-1)/2}$$

$$\lesssim (M\varepsilon)^2 \left(h^{-1}\log N\right)^{-(2k-1)}. \tag{3.14}$$

Combining (3.12)–(3.14), we obtain

$$\|f - f_\varepsilon\|_{0,\,\Gamma}^2 \lesssim N^2 \left(M\varepsilon h^{-1}\right)^4 + (M\varepsilon)^2 \left(h^{-1}\log N\right)^{-(2k-1)}.$$

Taking the square root on both sides of the above estimate yields the result. □

It is clearly seen from theorem 3.5 that the error consists of two parts: the first part arises from the linearization on dropping higher order terms in the power series expansion; the second part comes from the truncation of the Fourier series expansion of the diffraction surface function. For a sufficiently smooth surface function, i.e., where k is large, the error is dominated by the first part, which is small for small enough ε.

Though it is preferred to choose a large N in order to recover more Fourier modes of the diffraction surface function, i.e., to achieve a better resolution, the estimate shows that the error grows almost linearly with respect to N. Therefore, the frequency cutoff criterion N is a delicate quantity which should be chosen to balance the resolution and the error. For noise-free data, N can be chosen as a power function of the surface deformation parameter ε, i.e.,

$$N = \varepsilon^{-p}, \tag{3.15}$$

where $0 < p < 1$ is a user-specified parameter. Plugging (3.15) into the error estimate in theorem 3.5, we obtain an estimate

$$\|f - f_\varepsilon\|_{0,\Gamma} \lesssim \left(Mh^{-1}\right)^2 \varepsilon^{2-p} + M\varepsilon \left(h^{-1}|\log\varepsilon|\right)^{-(2k-1)/2}. \tag{3.16}$$

Clearly, we have $\|f - f_\varepsilon\|_{0,\Gamma} \to 0$ as $\varepsilon \to 0$ fixed h. However, the measurement distance h cannot be chosen as an arbitrarily small number, as the first part in the error estimate (3.16) is a decreasing function of h and dominates the overall error for sufficiently small h. As we mentioned at the end of section 2.1, we may take

$$h = \varepsilon^{1/2}, \tag{3.17}$$

and then the error estimate (3.16) reduces to

$$\|f - f_\varepsilon\|_{0,\Gamma} \lesssim M^2 \varepsilon^{1-p} + M\varepsilon^{(2k+3)/4} (|\log\varepsilon|)^{-(2k-1)/2}.$$

The above estimate is completely characterized only by the two intrinsic parameters M and ε, which are associated with the problem itself.

Next we consider an error estimate for the noisy scattering data, which is the main result of the paper.

Theorem 3.6. *Let f and $f_{\varepsilon,\delta}$ be the exact and reconstructed grating surface functions in (3.5) and (3.11), respectively. The error estimate*

$$\|f - f_{\varepsilon,\delta}\|_{0,\Gamma} \lesssim N\left(M\varepsilon h^{-1}\right)^2 + N\delta + M\varepsilon\left(h^{-1}\log N\right)^{-(2k-1)/2}$$

holds.

Proof. It follows from (3.5), (3.11), and the triangle inequality that

$$\begin{aligned}\|f - f_{\varepsilon,\delta}\|^2_{0,\Gamma} &= 2\pi \sum_{|n|\leqslant\omega}\left|f^{(n)} - f^{(n)}_{\varepsilon,\delta}\right|^2 + 2\pi\sum_{|n|>\omega}\left|f^{(n)}\right|^2\\ &\lesssim \sum_{|n|\leqslant\omega}\left|f^{(n)} - f^{(n)}_{\varepsilon}\right|^2 + \sum_{|n|\leqslant\omega}\left|f^{(n)}_{\varepsilon} - f^{(n)}_{\varepsilon,\delta}\right|^2 + \sum_{|n|>\omega}\left|f^{(n)}\right|^2. \qquad (3.18)\end{aligned}$$

It suffices to estimate the middle term in (3.18). Using (3.6) and (3.8) yields

$$\begin{aligned}\sum_{|n|\leqslant\omega}\left|f^{(n)}_{\varepsilon} - f^{(n)}_{\varepsilon,\delta}\right|^2 &= \sum_{|n|<\kappa}\left|f^{(n)}_{\varepsilon} - f^{(n)}_{\varepsilon,\delta}\right|^2 + \sum_{\kappa<|n|\leqslant\omega}\left|f^{(n)}_{\varepsilon} - f^{(n)}_{\varepsilon,\delta}\right|^2\\ &\leqslant \left(\frac{1}{2\kappa}\right)^2\sum_{|n|<\kappa}\left|u^{(n)} - u^{(n)}_{\delta}\right|^2\left|\mathrm{e}^{-\mathrm{i}\left(\kappa^2-n^2\right)^{1/2}h}\right|^2\\ &\quad+\left(\frac{1}{2\kappa}\right)^2\sum_{\kappa<|n|\leqslant\omega}\left|u^{(n)} - u^{(n)}_{\delta}\right|^2\left|\mathrm{e}^{\left(n^2-\kappa^2\right)^{1/2}h}\right|^2\\ &\leqslant \left(\frac{1}{2\kappa}\right)^2\sum_{|n|<\kappa}\left|u^{(n)} - u^{(n)}_{\delta}\right|^2 + \left(\frac{N}{2\kappa}\right)^2\sum_{\kappa<|n|\leqslant\omega}\left|u^{(n)} - u^{(n)}_{\delta}\right|^2\\ &\leqslant \left(\frac{N}{2\kappa}\right)^2\sum_{|n|\leqslant\omega}\left|u^{(n)} - u^{(n)}_{\delta}\right|^2\\ &\leqslant \left(\frac{N}{2\kappa}\right)^2\sum_{n\in\mathbb{Z}}\left|u^{(n)} - u^{(n)}_{\delta}\right|^2 = \left(\frac{N}{2\kappa}\right)^2\|u_\delta - u\|^2_{0,\Gamma} \lesssim (N\delta)^2. \qquad (3.19)\end{aligned}$$

Combining (3.18)–(3.19) and the estimates in theorem 3.5, we obtain

$$\|f - f_{\varepsilon,\delta}\|^2_{0,\Gamma} \lesssim N^2\left(M\varepsilon h^{-1}\right)^4 + (N\delta)^2 + (M\varepsilon)^2\left(h^{-1}\log N\right)^{-(2k-1)}.$$

Taking the square root on both sides of the above estimate yields the result. □

As is shown in theorem 3.6, the error consists of three parts: besides the linearization and the truncation errors, which are the same as in theorem 3.5, an extra error arises from the noise of the scattering data.

Again, we take $h = \varepsilon^{1/2}$ in (3.17). The error estimate in theorem 3.6 reduces to

$$\|f - f_{\varepsilon,\delta}\|_{0,\Gamma} \lesssim NM^2\varepsilon + N\delta + M\varepsilon^{(2k+3)/4}(\log N)^{-(2k-1)/2}. \qquad (3.20)$$

For noise data, the frequency cutoff criterion N can be chosen as

$$N = \min \left\{ \varepsilon^{-p},\ \ \delta^{-q} \right\}, \tag{3.21}$$

where $0 < q < 1$ is also a user-specified parameter. We consider two cases:

(i) $\varepsilon^{-p} < \delta^{-q}$. It follows from (3.21) that $N = \varepsilon^{-p}$ and $\delta \leqslant \varepsilon^{p/q}$. Plugging N and δ into (3.20), we obtain

$$\|f - f_{\varepsilon,\delta}\|_{0,\Gamma} \lesssim M^2 \varepsilon^{1-p} + \varepsilon^{p(1-q)/q} + M\varepsilon^{(2k+3)/4} (|\log \varepsilon|)^{-(2k-1)/2}.$$

(ii) $\varepsilon^{-p} > \delta^{-q}$. It follows from (3.21) that $N = \delta^{-q}$ and $\varepsilon < \delta^{q/p}$. Plugging N and ε into (3.20), we get

$$\|f - f_{\varepsilon,\delta}\|_{0,\Gamma} \lesssim M^2 \delta^{q(1-p)/p} + \delta^{1-q} + M\delta^{q(2k+3)/4p} (|\log \delta|)^{-(2k-1)/2}.$$

It is clearly seen that the above two estimates are also completely characterized by the intrinsic parameters of the problem itself, either M and ε or M and δ.

4. Conclusions

We have shown an error analysis of the inverse diffractive grating problem in the application of near-field imaging. An error estimate is established for the noisy scattering data with explicit dependence on intrinsic parameters. The analysis raises several questions for future research including that of an error estimate beyond the linearization and extensions of the results to the two-dimensional grating problem.

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