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ANALYSIS OF TRANSIENT ELECTROMAGENTIC SCATTERING FROM A THREE-DIMENSIONAL OPEN CAVITY*

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Abstract. This paper is concerned with the mathematical analysis of the time-domain Maxwell equations in a three-dimensional open cavity. An exact transparent boundary condition is developed to reformulate the open cavity scattering problem in an unbounded domain, equivalently, into an initial-boundary value problem in a bounded domain. The well-posedness and stability are studied for the reduced problem. Moreover, an a priori estimate is established for the electric field with a minimum regularity requirement for the data.

Key words. time-domain Maxwell equations, transparent boundary conditions, three-dimensional open cavity scattering problem, well-posedness, stability, a priori estimates

AMS subject classifications. 35Q61, 78A25, 78M30

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1. Introduction.

This paper is concerned with the mathematical analysis of an electromagnetic open cavity scattering problem where the wave propagation is governed by the time-domain Maxwell equations. As shown in Figure 1, an open cavity refers to a compactly supported domain with its opening aligned with the infinite ground plane. Cavity scattering problems have significant industry and military applications such as the design of cavity-backed conformal antennas and deliberate control in the form of enhancement or reduction of a radar cross section.

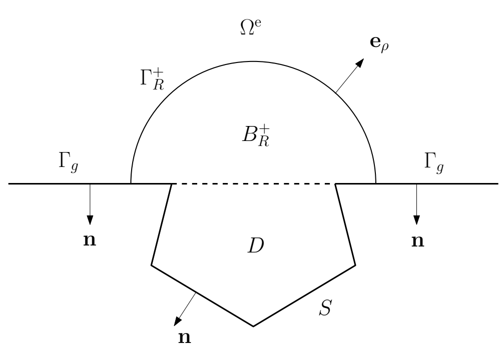

FIG. 1. *A schematic diagram of the open cavity problem geometry.*

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The time-harmonic problems have been widely investigated by numerous researchers in the engineering and mathematical communities [2, 3, 4, 7, 9, 10, 18, 25, 27, 33]. A large amount of information is available regarding their solutions for both the two-dimensional Helmholtz and the three-dimensional Maxwell equations [7, 8, 30, 32, 44]. We refer to [26] for a good introduction to the cavity scattering problem. One may consult [16, 17, 34, 35] for recent accounts of finite element methods and integral equation methods for general electromagnetic scattering problems.

The time-domain problems have attracted much attention due to their capability of capturing wide-band signals and modeling more general material and nonlinearity [11, 24, 29, 42, 43, 45]. However, rigorous mathematical analysis is very rare, especially for time-domain three-dimensional Maxwell equations. The transient cavity scattering problems have been mathematically studied in [22, 23, 37, 39, 40], where the focus was largely on (i) temporal discretization by the Newmark scheme; (ii) the reduction of the resulted system via the frequency-domain transparent boundary condition (TBC), and (iii) the analysis of the finite element method for the reduced problem. To the best of our knowledge, the theoretical analysis of the time-domain Maxwell cavity problem itself was left undone and is still lacking.

In this work, we aim to analyze the transient electromagnetic wave scattering by an open cavity which is embedded in a perfectly conducting infinite ground plane. The problem geometry is shown in Figure 1. The cavity is filled with an inhomogeneous, isotropic, and nondispersive medium which is allowed to protrude from its opening to above the ground plane, while the upper half space outside of the cavity is filled with a homogeneous, isotropic, and nondispersive medium. Therefore, the analysis in this work includes both planar and overfilled cavities. This problem is quite challenging due to the unbounded nature of the domain. In the frequency domain, various approaches have been investigated such as developing TBCs and designing perfectly matched layer techniques [1, 5, 12, 14, 20, 21, 28, 31, 32]. In contrast, much less study has addressed this issue in the time domain. In [13], the well-posedness and stability were shown for the electromagnetic obstacle scattering problem by using a time-domain TBC on the sphere. Here, we present a new time-domain TBC on a hemisphere enclosing the cavity and prove well-posedness and stability of the underlying scattering problem. The proofs are based on examining the well-posedness of the time-harmonic Maxwell equations with complex wavenumbers and applying the abstract inversion theorem of the Laplace transform. Moreover, an a priori estimate, featuring an explicit dependence on time and a minimum regularity requirement of the data, is established for the electric field by studying directly the time-domain Maxwell equations.

The outline of this paper is as follows. In section 2, we introduce the model problem and exploit the time-domain TBC to reduce the scattering problem to an initial-boundary value problem in a bounded domain. In section 3, we analyze two auxiliary problems pertinent to the reduced time-domain Maxwell equations to pave the way for the analysis of the main results in section 4. We study in section 4 the well-posedness and stability of the reduced Maxwell equations and derive an a priori estimate with a minimum requirement for the regularity of the data. The paper is concluded with some remarks and directions for future work in section 5. To avoid distraction from the main results, we collect in the appendices some necessary notation and useful results on the Laplace transform, spherical harmonics, and functional spaces.

2. Formulation and reduction of the problem. In this section, we introduce the mathematical model of interest and exploit the exact TBC to reduce the unbounded domain to a bounded one.

2.1. A model problem. We describe the setting of the cavity problem and define some necessary notation. As seen in Figure 1, denote by D the cavity embedded in the perfectly electrically conducting infinite ground plane Γ_g. Let $S = \partial D \cap \mathbb{R}^3_-$, the part of cavity wall below the ground, be Lipschitz continuous and perfectly electrically conducting. The medium in the cavity is characterized by the dielectric permittivity ε and the magnetic permeability μ, which satisfy

$$0 < \varepsilon_{\min} \le \varepsilon \le \varepsilon_{\max} < \infty, \quad 0 < \mu_{\min} \le \mu \le \mu_{\max} < \infty.$$

Here $\varepsilon_{\min}, \varepsilon_{\max}, \mu_{\min}$, and $\mu_{\max}$ are constants. Let B_R^+ and Γ_R^+ be the half-ball and hemisphere above the ground plane, where the radius R is large enough to completely contain the possibly overfilled portion of the cavity. The unbounded region $\mathbb{R}^3_+ \setminus \bar{B}_R^+$ is filled with a homogeneous, isotropic, and nondispersive medium with a constant permittivity ε_0 and a constant permeability μ_0. Throughout this paper, we assume for simplicity in exposition that $\varepsilon_0 = \mu_0 = 1$. Finally, we denote by $\Omega = B_R^+ \cup D$ the bounded domain in which our reduced initial-boundary value problem will be formulated. It is easy to note that $\partial\Omega = \Gamma_R^+ \cup S$ is Lipschitz continuous.

As is shown in Figure 1, it is evident that the problem geometry is applicable not only to the open cavity problem but also to a broader class of scattering problems where the surface S or a part of it may be above the ground plane.

Consider the system of time-domain Maxwell equations in $\mathbb{R}^3_+ \cup D$ for $t > 0$:

$$\begin{cases} \nabla \times \boldsymbol{E}(\boldsymbol{r}, t) + \mu \partial_t \boldsymbol{H}(\boldsymbol{r}, t) = \boldsymbol{0}, \\ \nabla \times \boldsymbol{H}(\boldsymbol{r}, t) - \varepsilon \partial_t \boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{J}(\boldsymbol{r}, t), \end{cases} \tag{2.1}$$

where $\boldsymbol{r} = (x, y, z) \in \mathbb{R}^3$, $\boldsymbol{E}$ is the electric field, $\boldsymbol{H}$ is the magnetic field, and $\boldsymbol{J}$ is the electric current density which is assumed to be compactly supported in D. The system is constrained by the initial conditions:

$$\boldsymbol{E}|_{t=0} = \boldsymbol{E}_0, \quad \boldsymbol{H}|_{t=0} = \boldsymbol{H}_0 \quad \text{in } \mathbb{R}^3_+ \cup D, \tag{2.2}$$

where $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ are also assumed to be compactly supported in D. We consider the perfectly electrical conducting boundary condition on the ground plane and cavity wall:

$$\boldsymbol{n} \times \boldsymbol{E} = \boldsymbol{0} \quad \text{on } \Gamma_g \cup S, \quad t > 0, \tag{2.3}$$

where $\boldsymbol{n}$ is the unit outward normal vector on $\Gamma_g \cup S$. In addition, we impose the Silver–Müller radiation condition:

$$\hat{\boldsymbol{r}} \times (\partial_t \boldsymbol{E} \times \hat{\boldsymbol{r}}) + \hat{\boldsymbol{r}} \times \partial_t \boldsymbol{H} = o(|\boldsymbol{r}|^{-1}), \quad \text{as } |\boldsymbol{r}| \to \infty, \quad t > 0, \tag{2.4}$$

where $\hat{\boldsymbol{r}} = \boldsymbol{r}/|\boldsymbol{r}|$.

The purpose of this paper is to study the well-posedness and establish the stability for the time-domain electromagnetic cavity scattering problem (2.1)–(2.4). Hereafter, the expression "$a \lesssim b$" stands for $a \le Cb$, where C is a generic positive constant independent of any function and important parameters, which are clear from the context.

2.2. Transparent boundary condition. We introduce a TBC to reformulate the electromagnetic wave propagation problem into an equivalent initial-boundary value problem in a bounded domain. The essential idea is to design a boundary operator which maps the tangential component of the electric field to the tangential trace of the magnetic field.

More precisely, we consider the reduced initial-boundary value problem:

$$\begin{cases} \nabla \times \boldsymbol{E} + \mu \partial_t \boldsymbol{H} = 0, \quad \nabla \times \boldsymbol{H} - \varepsilon \partial_t \boldsymbol{E} = \boldsymbol{J} & \text{in } \Omega, \ \ t > 0, \\ \boldsymbol{E}\,|_{t=0} = \boldsymbol{E}_0, \quad \boldsymbol{H}\,|_{t=0} = \boldsymbol{H}_0 & \text{in } \Omega, \\ \boldsymbol{n} \times \boldsymbol{E} = 0 & \text{on } S, \ \ t > 0, \\ \mathscr{T}[\boldsymbol{E}_{\Gamma_R^+}] = \boldsymbol{H} \times \boldsymbol{n} & \text{on } \Gamma_R^+, \ \ t > 0, \end{cases} \tag{2.5}$$

where $\boldsymbol{E}_{\Gamma_R^+}$ is the tangential trace of $\boldsymbol{E}$ on Γ_R^+, and $\mathscr{T}$ is the time-domain electric-to-magnetic (EtM) Calderon operator, as the counterpart of the time-harmonic setting, for instance, with a spherical boundary (cf. [17, 34]).

In what follows, we derive the formulation of the operator $\mathscr{T}$ and analyze its important properties. Equivalently, we aim to prove the well-posedness and stability of the reduced problem (2.5). In particular, an a priori estimate is established with a minimum requirement for the regularity of the data. The proofs are based on the abstract inversion theorem of the Laplace transform and the a priori estimates for the time-harmonic Maxwell equations with a complex wavenumber. These will be the main topics of the forthcoming sections.

Since $\boldsymbol{J}$ is supported in D and $\varepsilon_0 = \mu_0 = 1$, the Maxwell equations (2.1) become

$$\nabla \times \boldsymbol{E} + \partial_t \boldsymbol{H} = \boldsymbol{0}, \quad \nabla \times \boldsymbol{H} - \partial_t \boldsymbol{E} = \boldsymbol{0} \quad \text{in } \Omega^e := \mathbb{R}^3_+ \setminus \bar{B}_R^+, \ \ t > 0. \tag{2.6}$$

Let $\breve{\boldsymbol{E}}(\boldsymbol{r}, s) = \mathscr{L}(\boldsymbol{E})$ and $\breve{\boldsymbol{H}}(\boldsymbol{r}, s) = \mathscr{L}(\boldsymbol{H})$ be the Laplace transforms of $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{H}(\boldsymbol{r}, t)$ with respect to t, respectively, where the complex variable

$$s = s_1 + \mathrm{i} s_2 \quad \text{with} \quad s_1, s_2 \in \mathbb{R}, \ \ s_1 > 0, \ \ \mathrm{i} = \sqrt{-1}\,, \tag{2.7}$$

as seen in Appendix A. Recall that

$$\mathscr{L}(\partial_t \boldsymbol{E}) = s\breve{\boldsymbol{E}} - \boldsymbol{E}_0, \quad \mathscr{L}(\partial_t \boldsymbol{H}) = s\breve{\boldsymbol{H}} - \boldsymbol{H}_0.$$

Taking the Laplace transform of (2.6), and noting that $\boldsymbol{E}_0, \boldsymbol{H}_0$ are supported in D, we obtain the time-harmonic Maxwell equations with complex parameters:

$$\nabla \times \breve{\boldsymbol{E}} + s\breve{\boldsymbol{H}} = \boldsymbol{0}, \quad \nabla \times \breve{\boldsymbol{H}} - s\breve{\boldsymbol{E}} = \boldsymbol{0} \quad \text{in } \Omega^e, \ \ s_1 > 0. \tag{2.8}$$

Let $h_n^{(1)}(z)$ be the spherical Hankel function of the first kind of order n (cf. [41]). We introduce the vector wave functions

$$\begin{cases} \boldsymbol{M}_n^m(\rho, \theta, \varphi) = \nabla \times \big(\boldsymbol{r} h_n^{(1)}(\mathrm{i}s\rho) X_n^m(\theta, \varphi)\big), \\ \boldsymbol{N}_n^m(\rho, \theta, \varphi) = -s^{-1} \nabla \times \boldsymbol{M}_n^m(\theta, \varphi), \end{cases} \tag{2.9}$$

where (ρ, θ, φ) is the spherical coordinates, and X_n^m is the rescaled spherical harmonic function defined in (B.2). We refer to Appendix B for the properties of spherical harmonics and the relevant calculus to be used throughout this paper.

The vector wave functions in (2.9) are the radiation solutions of (2.8) in $\mathbb{R}^3 \setminus \{0\}$ (cf. [34, Theorem 9.16]):

$$\nabla \times \boldsymbol{M}_n^m(\rho,\theta,\varphi) + s\boldsymbol{N}_n^m(\rho,\theta,\varphi) = \boldsymbol{0}, \quad \nabla \times \boldsymbol{N}_n^m(\rho,\theta,\varphi) - s\boldsymbol{M}_n^m(\rho,\theta,\varphi) = \boldsymbol{0}. \tag{2.10}$$

It can be verified from (2.9) that the vector wave functions satisfy

$$\boldsymbol{M}_n^m(\rho,\theta,\varphi) = h_n^{(1)}(\mathrm{i}s\rho)\nabla_\Gamma X_n^m(\theta,\varphi) \times \boldsymbol{e}_\rho \tag{2.11}$$

and

$$\begin{aligned}\boldsymbol{N}_n^m(\rho,\theta,\varphi) = &-\frac{\sqrt{n(n+1)}}{s\rho}\Big(h_n^{(1)}(\mathrm{i}s\rho) + \mathrm{i}s\rho(h_n^{(1)})'(\mathrm{i}s\rho)\Big)\boldsymbol{X}_n^m(\theta,\varphi)\\ &-\frac{n(n+1)}{s\rho}h_n^{(1)}(\mathrm{i}s\rho)Y_n^m(\theta,\varphi)\boldsymbol{e}_\rho.\end{aligned} \tag{2.12}$$

Once again, we refer to Appendix B for the notation and definition. A simple calculation yields

$$\begin{cases}\boldsymbol{e}_\theta \times \boldsymbol{M}_n^m(\rho,\pi/2,\varphi) = 0 & \text{for } |m| \le n,\ m+n = \text{even},\ n \in \mathbb{N},\\ \boldsymbol{e}_\theta \times \boldsymbol{N}_n^m(\rho,\pi/2,\varphi) = 0 & \text{for } |m| \le n,\ m+n = \text{odd},\ n \in \mathbb{N}.\end{cases}$$

Therefore, in the domain Ω^{e}, the solution of the electric field $\breve{\boldsymbol{E}}(\rho,\theta,\varphi,s)$, which satisfies the perfectly electric conducting condition $\boldsymbol{n} \times \breve{\boldsymbol{E}} = 0$ on Γ_g, i.e., $\boldsymbol{e}_\theta \times \breve{\boldsymbol{E}}(\rho,\pi/2,\varphi,s) = 0$, can be written in the series

$$\breve{\boldsymbol{E}}(\rho,\theta,\varphi,s) = \sum_{|m|\le n}^{\text{odd}} \alpha_n^m(s)\,\boldsymbol{N}_n^m(\rho,\theta,\varphi) + \sum_{|m|\le n}^{\text{even}} \beta_n^m(s)\,\boldsymbol{M}_n^m(\rho,\theta,\varphi), \tag{2.13}$$

which is uniformly convergent on compact subsets in Ω^{e}. The corresponding magnetic field $\breve{\boldsymbol{H}}$ is given by

$$\breve{\boldsymbol{H}} = -s^{-1}\nabla \times \breve{\boldsymbol{E}} = -\sum_{|m|\le n}^{\text{odd}} \alpha_n^m(s)\,\boldsymbol{M}_n^m(\rho,\theta,\varphi) + \sum_{|m|\le n}^{\text{even}} \beta_n^m(s)\,\boldsymbol{N}_n^m(\rho,\theta,\varphi). \tag{2.14}$$

Note that in the above, we compressed the summation notation as in Appendix B.

To deduce the explicit representation of the EtM Calderon operator, we need to express $\breve{\boldsymbol{E}}_{\Gamma_R^+} = -\boldsymbol{e}_\rho \times (\boldsymbol{e}_\rho \times \breve{\boldsymbol{E}})$ and $\breve{\boldsymbol{H}} \times \boldsymbol{e}_\rho$ on Γ_R^+ in terms of the coefficients α_n^m and β_n^m. From the definition (2.11), one verifies that

$$\begin{aligned}-\boldsymbol{e}_\rho \times (\boldsymbol{e}_\rho \times \boldsymbol{M}_n^m(\rho,\theta,\varphi)) &= -\sqrt{n(n+1)}\,h_n^{(1)}(\mathrm{i}s\rho)\boldsymbol{Y}_n^m(\theta,\varphi),\\ -\boldsymbol{e}_\rho \times (\boldsymbol{e}_\rho \times \boldsymbol{N}_n^m(\rho,\theta,\varphi)) &= -\frac{\sqrt{n(n+1)}}{s\rho}\Big(h_n^{(1)}(\mathrm{i}s\rho) + \mathrm{i}s\rho\, {h_n^{(1)}}'(\mathrm{i}s\rho)\Big)\boldsymbol{X}_n^m(\theta,\varphi),\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{e}_\rho \times \boldsymbol{M}_n^m(\rho,\theta,\varphi) &= \sqrt{n(n+1)}\,h_n^{(1)}(\mathrm{i}s\rho)\boldsymbol{X}_n^m(\theta,\varphi),\\ \boldsymbol{e}_\rho \times \boldsymbol{N}_n^m(\rho,\theta,\varphi) &= -\frac{\sqrt{n(n+1)}}{s\rho}\Big(h_n^{(1)}(\mathrm{i}s\rho) + \mathrm{i}s\rho\, {h_n^{(1)}}'(\mathrm{i}s\rho)\Big)\boldsymbol{Y}_n^m(\theta,\varphi).\end{aligned}$$

Therefore, by (2.13), the tangential component of the electric field along Γ_R^+ is

$$\breve{\boldsymbol{E}}_{\Gamma_R^+} = -\sum_{|m|\le n}^{\text{odd}} \frac{\sqrt{n(n+1)}}{sR}\left(h_n^{(1)}(\mathrm{i}sR) + \mathrm{i}sR\, {h_n^{(1)}}'(\mathrm{i}sR)\right)\alpha_n^m(s)\,\boldsymbol{X}_n^m(\theta,\varphi) + \sum_{|m|\le n}^{\text{even}} \sqrt{n(n+1)}\, h_n^{(1)}(\mathrm{i}sR)\beta_n^m(s)\,\boldsymbol{Y}_n^m(\theta,\varphi),$$

and similarly, by (2.14), the tangential trace of the magnetic field along Γ_R^+ is

$$\breve{\boldsymbol{H}}\times\boldsymbol{e}_\rho = \sum_{|m|\le n}^{\text{odd}} \sqrt{n(n+1)}\, h_n^{(1)}(\mathrm{i}sR)\alpha_n^m(s)\,\boldsymbol{X}_n^m(\theta,\varphi) + \sum_{|m|\le n}^{\text{even}} \frac{\sqrt{n(n+1)}}{sR}\left(h_n^{(1)}(\mathrm{i}sR) + \mathrm{i}sR {h_n^{(1)}}'(\mathrm{i}sR)\right)\beta_n^m(s)\,\boldsymbol{Y}_n^m(\theta,\varphi).$$

Consequently, we have the following explicit representation of the frequency domain EtM Calderon operator $\mathscr{B}$: given any tangential component of the electric field along Γ_R^+ with the expansion

$$\boldsymbol{u} = \sum_{|m|\le n}^{\text{odd}} \alpha_n^m\,\boldsymbol{X}_n^m + \sum_{|m|\le n}^{\text{even}} \beta_n^m\,\boldsymbol{Y}_n^m, \tag{2.15}$$

the tangential trace of the magnetic field on Γ_R^+ is

$$\mathscr{B}[\boldsymbol{u}] = -\sum_{|m|\le n}^{\text{odd}} \frac{sR}{1+r_n^{(1)}(\mathrm{i}sR)}\,\alpha_n^m\,\boldsymbol{X}_n^m - \sum_{|m|\le n}^{\text{even}} \frac{(1+r_n^{(1)}(\mathrm{i}sR))}{sR}\,\beta_n^m\,\boldsymbol{Y}_n^m, \tag{2.16}$$

where

$$r_n^{(1)}(z) = \frac{z\,{h_n^{(1)}}'(z)}{h_n^{(1)}(z)}. \tag{2.17}$$

We now analyze some properties of the EtM Calderon operator and refer to Appendix D for the definitions of the function spaces to be used hereafter.

LEMMA 2.1. *The Calderon operator* $\mathscr{B}: \boldsymbol{H}^{-1/2}(\mathrm{curl},\Gamma_R^+) \to \boldsymbol{H}^{-1/2}(\mathrm{div},\Gamma_R^+)$ *is continuous.*

Proof. For any $\boldsymbol{u},\boldsymbol{v}\in\boldsymbol{H}^{-1/2}(\mathrm{curl},\Gamma_R^+)$, we can expand

$$\boldsymbol{u} = \sum_{|m|\le n}^{\text{odd}} u_{1n}^m\,\boldsymbol{X}_n^m + \sum_{|m|\le n}^{\text{even}} u_{2n}^m\,\boldsymbol{Y}_n^m, \quad \boldsymbol{v} = \sum_{|m|\le n}^{\text{odd}} v_{1n}^m\,\boldsymbol{X}_n^m + \sum_{|m|\le n}^{\text{even}} v_{2n}^m\,\boldsymbol{Y}_n^m.$$

Then we have from (2.16) that

$$\mathscr{B}[\boldsymbol{u}] = -\sum_{|m|\le n}^{\text{odd}} \frac{sR\,u_{1n}^m}{1+r_n^{(1)}(\mathrm{i}sR)}\,\boldsymbol{X}_n^m - \sum_{|m|\le n}^{\text{even}} \frac{(1+r_n^{(1)}(\mathrm{i}sR))u_{2n}^m}{sR}\,\boldsymbol{Y}_n^m.$$

To prove the lemma, it is required to estimate

$$\langle \mathscr{B}[\boldsymbol{u}], \boldsymbol{v}\rangle = -\sum_{|m|\le n}^{\text{odd}} \frac{sR}{1+r_n^{(1)}(\mathrm{i}sR)}\, u_{1n}^m\, \bar v_{1n}^m - \sum_{|m|\le n}^{\text{even}} \frac{(1+r_n^{(1)}(\mathrm{i}sR))}{sR}\, u_{2n}^m\, \bar v_{2n}^m. \tag{2.18}$$

It follows from the Cauchy–Schwarz inequality that

$$\begin{aligned}|\langle \mathscr{B}[\boldsymbol{u}], \boldsymbol{v}\rangle| \le{}& \left[\sum_{|m|\le n}^{\text{odd}} \frac{\sqrt{1+n(n+1)}}{|1+r_n^{(1)}(\mathrm{i}sR)|^2}\, |sR|^2\, |u_{1n}^m|^2 + \sum_{|m|\le n}^{\text{even}} \frac{|1+r_n^{(1)}(\mathrm{i}sR)|^2}{\sqrt{1+n(n+1)}} \frac{|u_{2n}^m|^2}{|sR|^2}\right]^{1/2}\\ &\times \left[\sum_{|m|\le n}^{\text{odd}} \frac{1}{\sqrt{1+n(n+1)}} |v_{1n}^m|^2 + \sum_{|m|\le n}^{\text{even}} \sqrt{1+n(n+1)}|v_{2n}^m|^2\right]^{1/2}.\end{aligned}$$

By Lemma C.3, we have

$$\begin{aligned}\frac{\sqrt{1+n(n+1)}}{|1+r_n^{(1)}(\mathrm{i}sR)|^2}\, |sR|^2\, |u_{1n}^m|^2 &= \frac{1}{\sqrt{1+n(n+1)}} \frac{1+n(n+1)}{|1+r_n^{(1)}(\mathrm{i}sR)|^2} |sR|^2 |u_{1n}^m|^2\\ &\lesssim \frac{1}{\sqrt{1+n(n+1)}}\, |u_{1n}^m|^2,\end{aligned}$$

and

$$\begin{aligned}\frac{|1+r_n^{(1)}(\mathrm{i}sR)|^2}{\sqrt{1+n(n+1)}} \frac{|u_{2n}^m|^2}{|sR|^2} &= \sqrt{1+n(n+1)} \frac{|1+r_n^{(1)}(\mathrm{i}sR)|^2}{1+n(n+1)} \frac{|u_{2n}^m|^2}{|sR|^2}\\ &\lesssim \sqrt{1+n(n+1)}\, |u_{2n}^m|^2.\end{aligned}$$

Combining the above estimates and using the expressions of the norms in Appendix D yields

$$|\langle \mathscr{B}[\boldsymbol{u}], \boldsymbol{v}\rangle| \lesssim \|\boldsymbol{u}\|_{\boldsymbol{H}^{-1/2}(\mathrm{curl},\,\Gamma_R^+)}\, \|\boldsymbol{v}\|_{\boldsymbol{H}^{-1/2}(\mathrm{curl},\,\Gamma_R^+)},$$

which completes the proof. ∎

Another important property of the EtM Calderon operator is stated as follows.

LEMMA 2.2. *It holds that*

$$\mathrm{Re}\,\langle \mathscr{B}[\boldsymbol{u}], \boldsymbol{u}\rangle \ge 0 \quad \forall\, \boldsymbol{u} \in \boldsymbol{H}^{-1/2}(\mathrm{curl}, \Gamma_R^+). \tag{2.19}$$

Proof. From (2.18), we obtain

$$-\langle \mathscr{B}[\boldsymbol{u}], \boldsymbol{u}\rangle = \sum_{|m|\le n}^{\text{odd}} \frac{sR}{1+r_n^{(1)}(\mathrm{i}sR)}\, |u_{1n}^m|^2 + \sum_{|m|\le n}^{\text{even}} \frac{1+r_n^{(1)}(\mathrm{i}sR)}{sR}\, |u_{2n}^m|^2. \tag{2.20}$$

By Lemmas C.1 and C.2,

$$s_1\mathrm{Re}\big(1+r_n^{(1)}(\mathrm{i}sR)\big) \le 0, \quad s_2\mathrm{Im}\big(1+r_n^{(1)}(\mathrm{i}sR)\big) \le 0.$$

Taking the real part of (2.20) gives

$$\begin{aligned}-\mathrm{Re}\,\langle \mathscr{B}[\boldsymbol{u}], \boldsymbol{u}\rangle ={}& \sum_{|m|\le n}^{\text{odd}} \frac{R\big(s_1\mathrm{Re}(1+r_n^{(1)}(\mathrm{i}sR)) + s_2\mathrm{Im}(1+r_n^{(1)}(\mathrm{i}sR))\big)}{|1+r_n^{(1)}(\mathrm{i}sR)|^2}\, |u_{1n}^m|^2\\ &+ \sum_{|m|\le n}^{\text{even}} \frac{\big(s_1\mathrm{Re}(1+r_n^{(1)}(\mathrm{i}sR)) + s_2\mathrm{Im}(1+r_n^{(1)}(\mathrm{i}sR))\big)}{|s|^2R} |u_{2n}^m|^2 \le 0,\end{aligned}$$

which completes the proof. $\square$

With the aid of the frequency domain EtM Calderon operator, we obtain the following TBC imposed on the hemisphere Γ_R^+ in the s-domain:

$$\mathscr{B}[\breve{\boldsymbol{E}}_{\Gamma_R^+}] = \breve{\boldsymbol{H}} \times \boldsymbol{e}_\rho, \tag{2.21}$$

which maps the tangential component of the electric field to the tangential trace of the magnetic field. Taking the inverse Laplace transform of (2.21) yields the TBC in the time domain:

$$\mathscr{T}[\boldsymbol{E}_{\Gamma_R^+}] = \boldsymbol{H} \times \boldsymbol{e}_\rho, \quad \text{where } \mathscr{T} := \mathscr{L}^{-1} \circ \mathscr{B} \circ \mathscr{L}. \tag{2.22}$$

Equivalently, we may eliminate the magnetic field and obtain an alternative TBC in the s-domain:

$$s^{-1}(\nabla \times \breve{\boldsymbol{E}}) \times \boldsymbol{n} + \mathscr{B}[\breve{\boldsymbol{E}}_{\Gamma_R^+}] = 0 \quad \text{on } \Gamma_R^+. \tag{2.23}$$

Correspondingly, by taking the inverse Laplace transform of (2.23), we may derive an alternative TBC in the time domain:

$$(\nabla \times \boldsymbol{E}) \times \boldsymbol{n} + \mathscr{C}[\boldsymbol{E}_{\Gamma_R^+}] = 0 \quad \text{on } \Gamma_R^+, \quad \text{where } \mathscr{C} = \mathscr{L}^{-1} \circ s\mathscr{B} \circ \mathscr{L}. \tag{2.24}$$

3. Analysis of two auxiliary problems. In this section, we make necessary preparations for the proof of the main results by considering two auxiliary problems pertinent to (2.5).

3.1. Time-harmonic Maxwell equations with a complex wavenumber. This subsection is devoted to the mathematical study of a time-harmonic Maxwell scattering problem with a complex wavenumber, which may be viewed as a frequency version of the initial-boundary value problem of the Maxwell equations under the Laplace transform.

Consider the auxiliary boundary value problem:

$$\begin{cases} \nabla \times \big((s\mu)^{-1}\nabla \times \boldsymbol{u}\big) + s\varepsilon \boldsymbol{u} = \boldsymbol{j} & \text{in } \Omega, \\ \boldsymbol{n} \times \boldsymbol{u} = 0 & \text{on } S, \\ s^{-1}(\nabla \times \boldsymbol{u}) \times \boldsymbol{n} + \mathscr{B}[\boldsymbol{u}_\Gamma] = 0 & \text{on } \Gamma_R^+, \end{cases} \tag{3.1}$$

where $s = s_1 + \mathrm{i}s_2$ with $s_1, s_2 \in \mathbb{R}, s_1 > 0$, and the applied current density $\boldsymbol{j}$ is assumed to be supported in $D \subset \Omega$.

By multiplying a test function $\boldsymbol{v} \in \boldsymbol{H}_S(\mathrm{curl}, \Omega)$, which is defined in Appendix D, and integrating by parts, we arrive at the variational formulation of (3.1): find $\boldsymbol{u} \in \boldsymbol{H}_S(\mathrm{curl}, \Omega)$ such that

$$a_{\mathrm{TH}}(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{j} \cdot \bar{\boldsymbol{v}} \, \mathrm{d}\boldsymbol{r} \quad \forall \ \boldsymbol{v} \in \boldsymbol{H}_S(\mathrm{curl}, \Omega), \tag{3.2}$$

where the sesquilinear form

$$a_{\mathrm{TH}}(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega (s\mu)^{-1}(\nabla \times \boldsymbol{u}) \cdot (\nabla \times \bar{\boldsymbol{v}}) \, \mathrm{d}\boldsymbol{r} + \int_\Omega s\varepsilon \, \boldsymbol{u} \cdot \bar{\boldsymbol{v}} \, \mathrm{d}\boldsymbol{r} + \langle \mathscr{B}[\boldsymbol{u}_{\Gamma_R^+}], \boldsymbol{v}_{\Gamma_R^+} \rangle. \tag{3.3}$$

THEOREM 3.1. *The variational problem* (3.2) *has a unique solution* $\boldsymbol{u} \in \boldsymbol{H}_S(\mathrm{curl}, \Omega)$ *which satisfies*

$$\|\nabla \times \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} + \|s\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \lesssim s_1^{-1} \|s\boldsymbol{j}\|_{\boldsymbol{L}^2(\Omega)},$$

where $s = s_1 + \mathrm{i}s_2$ *with* $s_1, s_2 \in \mathbb{R}$ *and* $s_1 > 0$.

Proof. It suffices to show the coercivity of a_{TH}, since its continuity follows directly from the Cauchy–Schwarz inequality, Lemma 2.1, and Lemma D.1. A simple calculation yields

$$a_{\mathrm{TH}}(\boldsymbol{u}, \boldsymbol{u}) = \int_\Omega (s\mu)^{-1} |\nabla \times \boldsymbol{u}|^2 \, \mathrm{d}\boldsymbol{r} + \int_\Omega s\varepsilon |\boldsymbol{u}|^2 \, \mathrm{d}\boldsymbol{r} + \langle \mathscr{B}[\boldsymbol{u}_{\Gamma_R^+}], \boldsymbol{u}_{\Gamma_R^+} \rangle. \tag{3.4}$$

Taking the real part of (3.4) and using Lemma 2.2, we get

$$\mathrm{Re}\{a_{\mathrm{TH}}(\boldsymbol{u}, \boldsymbol{u})\} \gtrsim \frac{s_1}{|s|^2} \left(\|\nabla \times \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|s\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} \right). \tag{3.5}$$

It follows from the Lax–Milgram lemma that the variational problem (3.2) has a unique solution $\boldsymbol{u} \in \boldsymbol{H}_S(\mathrm{curl}, \Omega)$. Moreover, we have from (3.2) that

$$|a_{\mathrm{TH}}(\boldsymbol{u}, \boldsymbol{u})| \le |s|^{-1} \|\boldsymbol{j}\|_{\boldsymbol{L}^2(\Omega)} \|s\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}. \tag{3.6}$$

Combining (3.5)–(3.6) leads to

$$\|\nabla \times \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|s\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} \lesssim s_1^{-1} \|s\boldsymbol{j}\|_{\boldsymbol{L}^2(\Omega)} \|s\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)},$$

which completes the proof after applying the Cauchy–Schwarz inequality. ∎

It is noteworthy that Theorem 3.1 gives the stability estimate with an explicit dependence on the complex wavenumber s corresponding to a lossy medium with $s_1 > 0$. However, it is challenging to obtain a similar estimate for a lossless medium with $s_1 = 0$.

3.2. A second auxiliary problem. Before considering the reduced problem (2.5), we need to study an auxiliary initial-boundary value problem and establish its well-posedness and stability. Consider the system of time-domain Maxwell equations:

$$\begin{cases} \nabla \times \boldsymbol{U} + \mu \partial_t \boldsymbol{V} = 0, \quad \nabla \times \boldsymbol{V} - \varepsilon \partial_t \boldsymbol{U} = 0 & \text{in } \Omega, \ t > 0, \\ \boldsymbol{n} \times \boldsymbol{U} = 0 & \text{on } S \cup \Gamma_R^+, \ t > 0, \\ \boldsymbol{U}|_{t=0} = \boldsymbol{E}_0, \quad \boldsymbol{V}|_{t=0} = \boldsymbol{H}_0 & \text{in } \Omega, \end{cases} \tag{3.7}$$

where $\boldsymbol{E}_0, \boldsymbol{H}_0$ are assumed to be compactly supported in D as before.

Let $\breve{\boldsymbol{U}} = \mathscr{L}(\boldsymbol{U})$ and $\breve{\boldsymbol{V}} = \mathscr{L}(\boldsymbol{V})$. Taking the Laplace transform of (3.7), we obtain the boundary value problem:

$$\begin{cases} \nabla \times \left((s\mu)^{-1} \nabla \times \breve{\boldsymbol{U}} \right) + s\varepsilon \breve{\boldsymbol{U}} = \breve{\boldsymbol{j}} & \text{in } \Omega, \\ \boldsymbol{n} \times \breve{\boldsymbol{U}} = 0 & \text{on } S \cup \Gamma_R^+, \end{cases} \tag{3.8}$$

where the current density function

$$\breve{\boldsymbol{j}} = \varepsilon \boldsymbol{E}_0 + s^{-1} \nabla \times \boldsymbol{H}_0.$$

The variational formulation for (3.8) is to find $\breve{\boldsymbol{U}} \in \boldsymbol{H}_0(\mathrm{curl}, \Omega)$ such that

$$a_{\mathrm{AP}}(\breve{\boldsymbol{U}}, \boldsymbol{v}) = \int_\Omega \breve{\boldsymbol{j}} \cdot \bar{\boldsymbol{v}}\, \mathrm{d}\boldsymbol{r} \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{curl}, \Omega), \tag{3.9}$$

where the sesquilinear form

$$a_{\mathrm{AP}}(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega (s\mu)^{-1} (\nabla \times \boldsymbol{u}) \cdot (\nabla \times \bar{\boldsymbol{v}})\, \mathrm{d}\boldsymbol{r} + \int_\Omega s\varepsilon \boldsymbol{u} \cdot \bar{\boldsymbol{v}}\, \mathrm{d}\boldsymbol{r}. \tag{3.10}$$

Following the same proof as in Theorem 3.1, we can show the well-posedness of the variational problem (3.9) and its stability, as stated below.

LEMMA 3.2. *The variational problem* (3.9) *has a unique solution* $\breve{\boldsymbol{U}} \in \boldsymbol{H}_0(\mathrm{curl}, \Omega)$ *which satisfies*

$$\|\nabla \times \breve{\boldsymbol{U}}\|_{\boldsymbol{L}^2(\Omega)} + \|s\breve{\boldsymbol{U}}\|_{\boldsymbol{L}^2(\Omega)} \lesssim s_1^{-1}|s| \|\boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega)} + s_1^{-1} \|\nabla \times \boldsymbol{H}_0\|_{\boldsymbol{L}^2(\Omega)}.$$

THEOREM 3.3. *The auxiliary problem* (3.7) *has a unique solution* $(\boldsymbol{U}, \boldsymbol{V})$, *which satisfies the stability estimates:*

$$\begin{aligned} \|\boldsymbol{U}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{V}\|_{\boldsymbol{L}^2(\Omega)} &\lesssim \|\boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{H}_0\|_{\boldsymbol{L}^2(\Omega)}, \\ \|\partial_t \boldsymbol{U}\|_{\boldsymbol{L}^2(\Omega)} + \|\partial_t \boldsymbol{V}\|_{\boldsymbol{L}^2(\Omega)} &\lesssim \|\nabla \times \boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla \times \boldsymbol{H}_0\|_{\boldsymbol{L}^2(\Omega)}, \\ \|\partial_t^2 \boldsymbol{U}\|_{\boldsymbol{L}^2(\Omega)} + \|\partial_t^2 \boldsymbol{V}\|_{\boldsymbol{L}^2(\Omega)} &\lesssim \|\nabla \times (\nabla \times \boldsymbol{E}_0)\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla \times (\nabla \times \boldsymbol{H}_0)\|_{\boldsymbol{L}^2(\Omega)}. \end{aligned}$$

Proof. Let $\breve{\boldsymbol{U}} = \mathscr{L}(\boldsymbol{U})$ and $\breve{\boldsymbol{V}} = \mathscr{L}(\boldsymbol{V})$ as before. Taking the Laplace transform of (3.7) leads to

$$\begin{cases} \nabla \times \breve{\boldsymbol{U}} + s\mu\, \breve{\boldsymbol{V}} = \mu\, \boldsymbol{H}_0, \ \ \nabla \times \breve{\boldsymbol{V}} - s\varepsilon\, \breve{\boldsymbol{U}} = -\varepsilon\, \boldsymbol{E}_0 & \text{in } \Omega, \\ \boldsymbol{n} \times \breve{\boldsymbol{U}} = 0 & \text{on } S \cap \Gamma_R^+. \end{cases} \tag{3.11}$$

It follows from Lemma 3.2 that

$$\|\nabla \times \breve{\boldsymbol{U}}\|_{\boldsymbol{L}^2(\Omega)} + \|s\breve{\boldsymbol{U}}\|_{\boldsymbol{L}^2(\Omega)} \lesssim s_1^{-1}|s| \|\boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega)} + s_1^{-1} \|\nabla \times \boldsymbol{H}_0\|_{\boldsymbol{L}^2(\Omega)}.$$

By (3.11), we have

$$\|\nabla \times \breve{\boldsymbol{V}}\|_{\boldsymbol{L}^2(\Omega)} + \|s\breve{\boldsymbol{V}}\|_{\boldsymbol{L}^2(\Omega)} \lesssim \big(1 + s_1^{-1}|s|\big) \|\boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{H}_0\|_{\boldsymbol{L}^2(\Omega)} + s_1^{-1} \|\nabla \times \boldsymbol{H}_0\|_{\boldsymbol{L}^2(\Omega)}.$$

It follows from Lemma A.2 that $\breve{\boldsymbol{U}}$ and $\breve{\boldsymbol{V}}$ are holomorphic functions of s, and the inverse Laplace transform of $\breve{\boldsymbol{U}}$ and $\breve{\boldsymbol{V}}$ exist and are supported in $[0, \infty)$.

We next prove the stability. Define the energy function

$$e_1(t) = \|\varepsilon^{1/2} \boldsymbol{U}(\cdot, t)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mu^{1/2} \boldsymbol{V}(\cdot, t)\|^2_{\boldsymbol{L}^2(\Omega)}.$$

It follows from (3.7) and integration by parts that

$$\begin{aligned} e_1(t) - e_1(0) = \int_0^t e_1'(\tau)\, \mathrm{d}\tau &= 2\mathrm{Re} \int_0^t \int_\Omega \big(\varepsilon \partial_t \boldsymbol{U} \cdot \bar{\boldsymbol{U}} + \mu \partial_t \boldsymbol{V} \cdot \bar{\boldsymbol{V}} \big)\, \mathrm{d}\boldsymbol{r} \mathrm{d}\tau \\ &= 2\mathrm{Re} \int_0^t \int_\Omega \big((\nabla \times \boldsymbol{V}) \cdot \bar{\boldsymbol{U}} - (\nabla \times \boldsymbol{U}) \cdot \bar{\boldsymbol{V}} \big)\, \mathrm{d}\boldsymbol{r} \mathrm{d}\tau \\ &= 2\mathrm{Re} \int_0^t \int_\Omega \big((\nabla \times \bar{\boldsymbol{U}}) \cdot \boldsymbol{V} - (\nabla \times \boldsymbol{U}) \cdot \bar{\boldsymbol{V}} \big)\, \mathrm{d}\boldsymbol{r} \mathrm{d}\tau = 0. \end{aligned}$$

Hence we have

$$\|\varepsilon^{1/2}\boldsymbol{U}(\cdot,t)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mu^{1/2}\boldsymbol{V}(\cdot,t)\|^2_{\boldsymbol{L}^2(\Omega)} = \|\varepsilon^{1/2}\boldsymbol{E}_0\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mu^{1/2}\boldsymbol{H}_0\|^2_{\boldsymbol{L}^2(\Omega)},$$

which implies

$$\|\boldsymbol{U}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{V}\|_{\boldsymbol{L}^2(\Omega)} \lesssim \|\boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{H}_0\|_{\boldsymbol{L}^2(\Omega)}.$$

Taking the first and second partial derivatives of (3.7) with respect to t yields

$$\begin{cases} \nabla\times\partial_t\boldsymbol{U} + \mu\partial_t^2\boldsymbol{V} = \boldsymbol{0}, \;\; \nabla\times\partial_t\boldsymbol{V} - \varepsilon\partial_t^2\boldsymbol{U} = 0 & \text{in } \Omega, \;\; t>0, \\ \boldsymbol{n}\times\partial_t\boldsymbol{U} = \boldsymbol{0} & \text{on } S\cup\Gamma_R^+, \;\; t>0, \\ \partial_t\boldsymbol{U}|_{t=0} = \varepsilon^{-1}(\nabla\times\boldsymbol{H}_0), \;\; \partial_t\boldsymbol{V}|_{t=0} = -\mu^{-1}(\nabla\times\boldsymbol{E}_0) & \text{in } \Omega, \end{cases}$$

and

$$\begin{cases} \nabla\times\partial_t^2\boldsymbol{U} + \mu\partial_t^3\boldsymbol{V} = 0, \;\; \nabla\times\partial_t^2\boldsymbol{V} - \varepsilon\partial_t^3\boldsymbol{U} = 0 & \text{in } \Omega, \;\; t>0, \\ \boldsymbol{n}\times\partial_t^2\boldsymbol{U} = 0 & \text{on } S\cup\Gamma_R^+, \; t>0, \\ \partial_t^2\boldsymbol{U}|_{t=0} = -(\varepsilon\mu)^{-1}(\nabla\times(\nabla\times\boldsymbol{E}_0)) & \text{in } \Omega, \\ \partial_t^2\boldsymbol{V}|_{t=0} = -(\varepsilon\mu)^{-1}(\nabla\times(\nabla\times\boldsymbol{H}_0)) & \text{in } \Omega. \end{cases}$$

Considering the energy functions

$$e_2(t) = \|\varepsilon^{1/2}\partial_t\boldsymbol{U}(\cdot,t)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mu^{1/2}\partial_t\boldsymbol{V}(\cdot,t)\|^2_{\boldsymbol{L}^2(\Omega)}$$

and

$$e_3(t) = \|\varepsilon^{1/2}\partial_t^2\boldsymbol{U}(\cdot,t)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mu^{1/2}\partial_t^2\boldsymbol{V}(\cdot,t)\|^2_{\boldsymbol{L}^2(\Omega)}$$

for the above two problems, respectively, we can follow the same steps for proving the first inequality to derive the other two inequalities. ∎

4. The reduced Maxwell equations. In this section, we derive the main results of this work, which include the well-posedness of the reduced problem (2.5) and the related a priori estimates.

4.1. Well-posedness. Let $\boldsymbol{e} = \boldsymbol{E} - \boldsymbol{U}$ and $\boldsymbol{h} = \boldsymbol{H} - \boldsymbol{V}$. It follows from (2.5) and (3.7) that $\boldsymbol{e}$ and $\boldsymbol{h}$ satisfy the following system:

$$(4.1) \qquad \begin{cases} \nabla\times\boldsymbol{e} + \mu\partial_t\boldsymbol{h} = 0, \;\; \nabla\times\boldsymbol{h} - \varepsilon\partial_t\boldsymbol{e} = \boldsymbol{J} & \text{in } \Omega, \; t>0, \\ \boldsymbol{n}\times\boldsymbol{e} = 0 & \text{on } S, \;\; t>0, \\ \boldsymbol{e}|_{t=0} = 0, \quad \boldsymbol{h}|_{t=0} = 0 & \text{in } \Omega, \\ \mathscr{T}\boldsymbol{e}_{\Gamma_R^+} = \boldsymbol{h}\times\boldsymbol{n} + \boldsymbol{V}\times\boldsymbol{n} & \text{on } \Gamma_R^+, \; t>0. \end{cases}$$

Let $\breve{\boldsymbol{e}} = \mathscr{L}(\boldsymbol{e})$ and $\breve{\boldsymbol{h}} = \mathscr{L}(\boldsymbol{h})$. Taking the Laplace transform of (4.1) and eliminating $\breve{\boldsymbol{h}}$, we obtain

$$(4.2) \qquad \begin{cases} \nabla\times((s\mu)^{-1})\nabla\times\breve{\boldsymbol{e}}) + s\varepsilon\breve{\boldsymbol{e}} = -\breve{\boldsymbol{J}} & \text{in } \Omega, \\ \boldsymbol{n}\times\breve{\boldsymbol{e}} = 0 & \text{on } S, \\ s^{-1}(\nabla\times\breve{\boldsymbol{e}})\times\boldsymbol{n} + \mathscr{B}\breve{\boldsymbol{e}}_{\Gamma_R^+} = \breve{\boldsymbol{V}}\times\boldsymbol{n} & \text{on } \Gamma_R^+. \end{cases}$$

LEMMA 4.1. *The problem* (4.2) *has a unique weak solution* $\breve{\boldsymbol{e}} \in \boldsymbol{H}_S(\mathrm{curl},\,\Omega)$ *which satisfies*

$$\|\nabla\times\breve{\boldsymbol{e}}\|^2_{\boldsymbol{L}^2(\Omega)} + \|s\breve{\boldsymbol{e}}\|^2_{\boldsymbol{L}^2(\Omega)} \lesssim s_1^{-1}\Big(\|s\breve{\boldsymbol{J}}\|_{\boldsymbol{L}^2(\Omega)} + \|s\breve{\boldsymbol{V}}\times\boldsymbol{n}\|_{\boldsymbol{H}^{-1/2}(\mathrm{div},\,\Gamma_R^+)} + \||s|^2\breve{\boldsymbol{V}}\times\boldsymbol{n}\|_{\boldsymbol{H}^{-1/2}(\mathrm{div},\,\Gamma_R^+)}\Big).$$

Proof. The well-posedness of the solution $\breve{\boldsymbol{e}} \in \boldsymbol{H}_S(\mathrm{curl},\,\Omega)$ follows directly from Theorem 3.1. Moreover, we have

$$a_{\mathrm{TH}}(\breve{\boldsymbol{e}},\breve{\boldsymbol{e}}) = \langle\breve{\boldsymbol{V}}\times\boldsymbol{n},\breve{\boldsymbol{e}}_{\Gamma_R^+}\rangle - \int_\Omega \breve{\boldsymbol{J}}\cdot\bar{\breve{\boldsymbol{e}}}\,\mathrm{d}\boldsymbol{r}.$$

By the coercivity of a_{TH} in (3.5) and the trace theorem in Lemma D.1,

$$\begin{aligned}\frac{s_1}{|s|^2}\Big(\|\nabla\times\breve{\boldsymbol{e}}\|^2_{\boldsymbol{L}^2(\Omega)} + \|s\breve{\boldsymbol{e}}\|^2_{\boldsymbol{L}^2(\Omega)}\Big) &\lesssim \|s^{-1}\breve{\boldsymbol{J}}\|_{\boldsymbol{L}^2(\Omega)}\|s\breve{\boldsymbol{e}}\|_{\boldsymbol{L}^2(\Omega)} \\ &\quad + \|\breve{\boldsymbol{V}}\times\boldsymbol{n}\|_{\boldsymbol{H}^{-1/2}(\mathrm{div},\,\Gamma_R^+)}\|\breve{\boldsymbol{e}}\|_{\boldsymbol{H}(\mathrm{curl},\,\Omega)} \\ &\lesssim \|s^{-1}\breve{\boldsymbol{J}}\|_{\boldsymbol{L}^2(\Omega)}\|s\breve{\boldsymbol{e}}\|_{\boldsymbol{L}^2(\Omega)} + \|\breve{\boldsymbol{V}}\times\boldsymbol{n}\|_{\boldsymbol{H}^{-1/2}(\mathrm{div},\,\Gamma_R^+)}\|\nabla\times\breve{\boldsymbol{e}}\|_{\boldsymbol{L}^2(\Omega)} \\ &\quad + \|s^{-1}\breve{\boldsymbol{V}}\times\boldsymbol{n}\|_{\boldsymbol{H}^{-1/2}(\mathrm{div},\,\Gamma_R^+)}\|s\breve{\boldsymbol{e}}\|_{\boldsymbol{L}^2(\Omega)},\end{aligned}$$

which completes the proof. ∎

To show the well-posedness of the reduced problem (2.5), we make the following assumptions for the initial and boundary data:

$$\boldsymbol{E}_0, \boldsymbol{H}_0 \in \boldsymbol{H}(\mathrm{curl},\,\Omega), \quad \boldsymbol{J}\in\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega)), \quad \boldsymbol{J}|_{t=0}=0. \tag{4.3}$$

THEOREM 4.2. *The problem has a unique solution* $(\boldsymbol{E},\boldsymbol{H})$ *which satisfies*

$$\begin{aligned}&\boldsymbol{E}\in\boldsymbol{L}^2(0,T;\boldsymbol{H}_S(\mathrm{curl},\,\Omega))\cap\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega)),\\ &\boldsymbol{H}\in\boldsymbol{L}^2(0,T;\boldsymbol{H}(\mathrm{curl},\,\Omega))\cap\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega)),\end{aligned}$$

and the stability estimate,

$$\begin{aligned}&\max_{t\in[0,T]}\big(\|\partial_t\boldsymbol{E}\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla\times\boldsymbol{E}\|_{\boldsymbol{L}^2(\Omega)} + \|\partial_t\boldsymbol{H}\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla\times\boldsymbol{H}\|_{\boldsymbol{L}^2(\Omega)}\big)\\ &\quad\lesssim \|\boldsymbol{E}_0\|_{\boldsymbol{H}(\mathrm{curl},\Omega)} + \|\boldsymbol{H}_0\|_{\boldsymbol{H}(\mathrm{curl},\Omega)} + \|\boldsymbol{J}\|_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))}.\end{aligned}$$

Proof. Recall the decomposition $\boldsymbol{E}=\boldsymbol{U}+\boldsymbol{e}$ and $\boldsymbol{H}=\boldsymbol{V}+\boldsymbol{h}$, where $(\boldsymbol{U},\boldsymbol{V})$ satisfy (3.7) and $(\boldsymbol{e},\boldsymbol{h})$ satisfy (4.1). Since

$$\begin{aligned}&\int_0^T\Big(\|\nabla\times\boldsymbol{e}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\partial_t\boldsymbol{e}\|^2_{\boldsymbol{L}^2(\Omega)}\Big)\,\mathrm{d}t\\ &\le\int_0^T e^{-2s_1(t-T)}\Big(\|\nabla\times\boldsymbol{e}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\partial_t\boldsymbol{e}\|^2_{\boldsymbol{L}^2(\Omega)}\Big)\,\mathrm{d}t\\ &= e^{2s_1T}\int_0^T e^{-2s_1t}\Big(\|\nabla\times\boldsymbol{e}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\partial_t\boldsymbol{e}\|^2_{\boldsymbol{L}^2(\Omega)}\Big)\,\mathrm{d}t\\ &\lesssim\int_0^\infty e^{-2s_1t}\Big(\|\nabla\times\boldsymbol{e}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\partial_t\boldsymbol{e}\|^2_{\boldsymbol{L}^2(\Omega)}\Big)\,\mathrm{d}t,\end{aligned}$$

it suffices to estimate the integral

$$\int_0^\infty e^{-2s_1 t}\left(\|\nabla\times\boldsymbol{e}\|^2_{\boldsymbol{L}^2(\Omega)}+\|\partial_t\boldsymbol{e}\|^2_{\boldsymbol{L}^2(\Omega)}\right)\mathrm{d}t.$$

Taking the Laplace transform of (4.1) yields

$$\begin{cases}\nabla\times\breve{\boldsymbol{e}}+s\mu\breve{\boldsymbol{h}}=0,\quad \nabla\times\breve{\boldsymbol{h}}-s\varepsilon\breve{\boldsymbol{e}}=\breve{\boldsymbol{J}} & \text{in }\ \Omega,\\ \boldsymbol{n}\times\breve{\boldsymbol{e}}=0 & \text{on }\ S,\\ \mathscr{B}\breve{\boldsymbol{e}}_{\Gamma_R^+}=\breve{\boldsymbol{h}}\times\boldsymbol{n}+\breve{\boldsymbol{V}}\times\boldsymbol{n} & \text{on }\ \Gamma_R^+.\end{cases}\tag{4.4}$$

By Lemma 4.1,

$$\begin{aligned}\|\nabla\times\breve{\boldsymbol{e}}\|^2_{\boldsymbol{L}^2(\Omega)}+\|s\breve{\boldsymbol{e}}\|^2_{\boldsymbol{L}^2(\Omega)}\lesssim s_1^{-1}\Big(&\|s\breve{\boldsymbol{J}}\|_{\boldsymbol{L}^2(\Omega)}+\|s\breve{\boldsymbol{V}}\times\boldsymbol{n}\|_{\boldsymbol{H}^{-1/2}(\mathrm{div},\,\Gamma_R^+)}\\ &+\||s|^2\breve{\boldsymbol{V}}\times\boldsymbol{n}\|_{\boldsymbol{H}^{-1/2}(\mathrm{div},\,\Gamma_R^+)}\Big).\end{aligned}\tag{4.5}$$

By (4.4),

$$\begin{aligned}\|\nabla\times\breve{\boldsymbol{h}}\|^2_{\boldsymbol{L}^2(\Omega)}+\|s\breve{\boldsymbol{h}}\|^2_{\boldsymbol{L}^2(\Omega)}\lesssim s_1^{-1}\Big(&\|\breve{\boldsymbol{J}}\|_{\boldsymbol{L}^2(\Omega)}+\|s\breve{\boldsymbol{J}}\|_{\boldsymbol{L}^2(\Omega)}+\|s\breve{\boldsymbol{V}}\times\boldsymbol{n}\|_{\boldsymbol{H}^{-1/2}(\mathrm{div},\,\Gamma_R^+)}\\ &+\||s|^2\breve{\boldsymbol{V}}\times\boldsymbol{n}\|_{\boldsymbol{H}^{-1/2}(\mathrm{div},\,\Gamma_R^+)}\Big).\end{aligned}\tag{4.6}$$

It follows from [36, Lemma 44.1] that $(\breve{\boldsymbol{e}},\breve{\boldsymbol{h}})$ are holomorphic functions of s on the half plane $s_1>\gamma>0$, where γ is any positive constant. Hence we have from Lemma A.2 that the inverse Laplace transform of $\breve{\boldsymbol{e}}$ and $\breve{\boldsymbol{h}}$ exist and are supported in $[0,\infty]$.

Denote by $\boldsymbol{e}=\mathscr{L}^{-1}(\breve{\boldsymbol{e}})$ and $\boldsymbol{h}=\mathscr{L}^{-1}(\breve{\boldsymbol{h}})$. Since $\breve{\boldsymbol{e}}=\mathscr{L}(\boldsymbol{e})=\mathscr{F}(e^{-s_1t}\boldsymbol{e})$, where $\mathscr{F}$ is the Fourier transform in s_2, we have from the Parseval identity and (4.5) that

$$\begin{aligned}\int_0^\infty e^{-2s_1t}\left(\|\nabla\times\boldsymbol{e}\|^2_{\boldsymbol{L}^2(\Omega)}+\|\partial_t\boldsymbol{e}\|^2_{\boldsymbol{L}^2(\Omega)}\right)\mathrm{d}t&=2\pi\int_{-\infty}^{\infty}\left(\|\nabla\times\breve{\boldsymbol{e}}\|^2_{\boldsymbol{L}^2(\Omega)}+\|s\breve{\boldsymbol{e}}\|^2_{\boldsymbol{L}^2(\Omega)}\right)\mathrm{d}s_2\\ \lesssim s_1^{-2}\int_{-\infty}^{\infty}\|s\breve{\boldsymbol{J}}\|^2_{\boldsymbol{L}^2(\Omega)}\mathrm{d}s_2+s_1^{-2}\int_{-\infty}^{\infty}\Big(&\|s\breve{\boldsymbol{V}}\times\boldsymbol{n}\|^2_{\boldsymbol{H}^{-1/2}(\mathrm{div},\,\Gamma_R^+)}\\ &+\||s|^2\breve{\boldsymbol{V}}\times\boldsymbol{n}\|^2_{\boldsymbol{H}^{-1/2}(\mathrm{div},\,\Gamma_R^+)}\Big)\mathrm{d}s_2.\end{aligned}$$

Since $\boldsymbol{J}|_{t=0}=0$ in Ω, $\boldsymbol{V}\times\boldsymbol{n}|_{t=0}=\partial_t(\boldsymbol{V}\times\boldsymbol{n})|_{t=0}=0$ on Γ_R^+, we have $\mathscr{L}(\partial_tJ)=s\breve{\boldsymbol{J}}$ in Ω and $\mathscr{L}(\partial_t(\boldsymbol{V}\times\boldsymbol{n}))=s\breve{\boldsymbol{V}}\times\boldsymbol{n}$ on Γ_R^+. It is easy to note that

$$|s|^2\breve{\boldsymbol{V}}\times\boldsymbol{n}=(2s_1-s)s\breve{\boldsymbol{V}}\times\boldsymbol{n}=2s_1\mathscr{L}(\partial_t(\boldsymbol{V}\times\boldsymbol{n}))-\mathscr{L}(\partial_t^2(\boldsymbol{V}\times\boldsymbol{n}))\quad\text{on }\ \Gamma_R^+.$$

Hence we have

$$\begin{aligned}&\int_0^\infty e^{-2s_1t}\left(\|\nabla\times\boldsymbol{e}\|^2_{\boldsymbol{L}^2(\Omega)}+\|\partial_t\boldsymbol{e}\|^2_{\boldsymbol{L}^2(\Omega)}\right)\mathrm{d}t\\ &\quad\lesssim s_1^{-2}\int_{-\infty}^{\infty}\left(\|\mathscr{L}(\partial_t\boldsymbol{J})\|^2_{\boldsymbol{L}^2(\Omega)}+\|\mathscr{L}(\partial_t^2\boldsymbol{V}\times\boldsymbol{n})\|^2_{\boldsymbol{H}^{-1/2}(\mathrm{div},\,\Gamma_R^+)}\right)\mathrm{d}s_2\\ &\qquad+(1+s_1^{-2})\int_{-\infty}^{\infty}\|\mathscr{L}(\partial_t\boldsymbol{V}\times\boldsymbol{n})\|^2_{\boldsymbol{H}^{-1/2}(\mathrm{div},\,\Gamma_R^+)}\mathrm{d}s_2.\end{aligned}$$

Using the Parseval identity again gives

$$\begin{aligned}\int_0^\infty e^{-2s_1 t}\left(\|\nabla\times\boldsymbol{e}\|^2_{\boldsymbol{L}^2(\Omega)}+\|\partial_t\boldsymbol{e}\|^2_{\boldsymbol{L}^2(\Omega)}\right)\mathrm{d}t \\ \lesssim s_1^{-2}\int_0^\infty e^{-2s_1 t}\left(\|\partial_t\boldsymbol{J}\|^2_{\boldsymbol{L}^2(\Omega)}+\|\partial_t^2\boldsymbol{V}\times\boldsymbol{n}\|^2_{\boldsymbol{H}^{-1/2}(\mathrm{div},\,\Gamma_R^+)}\right)\mathrm{d}t \\ +(1+s_1^{-2})\int_0^\infty e^{-2s_1 t}\|\partial_t\boldsymbol{V}\times\boldsymbol{n}\|^2_{\boldsymbol{H}^{-1/2}(\mathrm{div},\,\Gamma_R^+)}\mathrm{d}t,\end{aligned}$$

which shows that

$$\boldsymbol{e}\in\boldsymbol{L}^2(0,T;\boldsymbol{H}_S(\mathrm{curl},\,\Omega))\cap\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega)).$$

Similarly, we can show from (4.6) that

$$\boldsymbol{h}\in\boldsymbol{L}^2(0,T;\boldsymbol{H}(\mathrm{curl},\Omega))\cap\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega)).$$

We next prove the stability. Let $\tilde{\boldsymbol{E}}$ be the extension of $\boldsymbol{E}$ with respect to t in $\mathbb{R}$ such that $\tilde{\boldsymbol{E}}=0$ outside the interval $[0,t]$. By Lemmas A.1 and 2.2, we get

$$\begin{aligned}\mathrm{Re}\int_0^t e^{-2s_1 t}\int_{\Gamma_R^+}\mathscr{T}[\boldsymbol{E}_{\Gamma_R^+}]\cdot\bar{\boldsymbol{E}}_{\Gamma_R^+}\,\mathrm{d}\gamma\mathrm{d}t &= \mathrm{Re}\int_{\Gamma_R^+}\int_0^\infty e^{-2s_1 t}\mathscr{T}[\tilde{\boldsymbol{E}}_{\Gamma_R^+}]\cdot\bar{\tilde{\boldsymbol{E}}}_{\Gamma_R^+}\,\mathrm{d}t\mathrm{d}\gamma \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty \mathrm{Re}\langle\mathscr{B}[\check{\tilde{\boldsymbol{E}}}_{\Gamma_R^+}](s),\check{\tilde{\boldsymbol{E}}}_{\Gamma_R^+}(s)\rangle\,\mathrm{d}s_2\geq 0,\end{aligned}$$

which yields after taking $s_1\to 0$ that

$$\mathrm{Re}\int_0^t\int_{\Gamma_R^+}\mathscr{T}[\boldsymbol{E}_{\Gamma_R^+}]\cdot\bar{\boldsymbol{E}}_{\Gamma_R^+}\,\mathrm{d}\gamma\mathrm{d}t\geq 0. \tag{4.7}$$

For any $0<t<T$, consider an energy function

$$e(t)=\|\varepsilon^{1/2}\boldsymbol{E}(\cdot,t)\|^2_{\boldsymbol{L}^2(\Omega)}+\|\mu^{1/2}\boldsymbol{H}(\cdot,t)\|^2_{\boldsymbol{L}^2(\Omega)}.$$

It is easy to note that

$$\begin{aligned}\int_0^t u'(t)\mathrm{d}t &= \left(\|\varepsilon^{1/2}\boldsymbol{E}(\cdot,t)\|^2_{\boldsymbol{L}^2(\Omega)}+\|\mu^{1/2}\boldsymbol{H}(\cdot,t)\|^2_{\boldsymbol{L}^2(\Omega)}\right) \\ &\quad -\left(\|\varepsilon^{1/2}\boldsymbol{E}_0\|^2_{\boldsymbol{L}^2(\Omega)}+\|\mu^{1/2}\boldsymbol{H}_0\|^2_{\boldsymbol{L}^2(\Omega)}\right).\end{aligned} \tag{4.8}$$

On the other hand, we have from (2.5) and (4.7) that

$$\begin{aligned}\int_0^t e'(t)\mathrm{d}t &= 2\mathrm{Re}\int_0^t\int_\Omega\left(\varepsilon\partial_t\boldsymbol{E}\cdot\bar{\boldsymbol{E}}+\mu\partial_t\boldsymbol{H}\cdot\bar{\boldsymbol{H}}\right)\,\mathrm{d}\boldsymbol{r}\mathrm{d}t \\ &= 2\mathrm{Re}\int_0^t\int_\Omega\left((\nabla\times\boldsymbol{H})\cdot\bar{\boldsymbol{E}}-(\nabla\times\boldsymbol{E})\cdot\bar{\boldsymbol{H}}\right)\,\mathrm{d}\boldsymbol{r}\mathrm{d}t-2\mathrm{Re}\int_0^t\int_\Omega\boldsymbol{J}\cdot\bar{\boldsymbol{E}}\,\mathrm{d}\boldsymbol{r}\mathrm{d}t \\ &= -2\mathrm{Re}\int_0^t\int_\Gamma\mathscr{T}[\boldsymbol{E}_{\Gamma_R^+}]\cdot\bar{\boldsymbol{E}}_{\Gamma_R^+}\,\mathrm{d}\gamma\mathrm{d}t-2\mathrm{Re}\int_0^t\int_\Omega\boldsymbol{J}\cdot\bar{\boldsymbol{E}}\,\mathrm{d}\boldsymbol{r}\mathrm{d}t \\ &\leq -2\mathrm{Re}\int_0^t\int_\Omega\boldsymbol{J}\cdot\bar{\boldsymbol{E}}\,\mathrm{d}\boldsymbol{r}\mathrm{d}t\leq 2\max_{t\in[0,T]}\|\boldsymbol{E}(\cdot,t)\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{J}\|_{\boldsymbol{L}^1(0,T;\boldsymbol{L}^2(\Omega))}.\end{aligned} \tag{4.9}$$

Taking the derivative of (2.5) with respect to t, we know that $(\partial_t\boldsymbol{E},\partial_t\boldsymbol{H})$ satisfy the same set of equations with the source and the initial condition replaced by $\partial_t\boldsymbol{J}$, $\partial_t\boldsymbol{E}|_{t=0}=\varepsilon^{-1}\nabla\times\boldsymbol{H}_0$, $\partial_t\boldsymbol{H}|_{t=0}=-\mu^{-1}\nabla\times\boldsymbol{H}_0$. Hence we may follow the same steps as above to obtain (4.9) for $(\partial_t\boldsymbol{E},\partial_t\boldsymbol{H})$, which completes the proof after combining the above estimates. $\square$

4.2. An a priori estimate. In what follows, we shall take a different route and study the Maxwell equations directly in the time domain. The goal is to derive an a priori stability estimate for the electric field with a minimum regularity requirement for the data and an explicit dependence on the time variable.

Eliminating the magnetic field in (2.1) and using the TBC (2.24), we consider the initial-boundary value problem in a bounded domain:

$$\begin{cases} \varepsilon\partial_t^2 \boldsymbol{E} = -\nabla\times(\mu^{-1}\nabla\times\boldsymbol{E}) - \boldsymbol{F} & \text{in } \Omega, \ \ t>0, \\ \boldsymbol{E}|_{t=0} = \boldsymbol{E}_0, \quad \partial_t\boldsymbol{E}|_{t=0} = \boldsymbol{E}_1 & \text{in } \Omega, \\ \boldsymbol{n}\times\boldsymbol{E} = 0 & \text{on } S, \ \ t>0, \\ (\nabla\times\boldsymbol{E})\times\boldsymbol{n} + \mathscr{C}[\boldsymbol{E}_{\Gamma_R^+}] = 0 & \text{on } \Gamma_R^+, \ \ t>0, \end{cases} \tag{4.10}$$

where

$$\boldsymbol{F} = \partial_t\boldsymbol{J}, \quad \boldsymbol{E}_1 = \varepsilon^{-1}(\nabla\times\boldsymbol{H}_0 - \boldsymbol{J}_0).$$

The variational problem is to find $\boldsymbol{E}\in\boldsymbol{H}_S(\mathrm{curl},\Omega)$ for all $t>0$ such that

$$\begin{aligned} \int_\Omega \varepsilon\partial_t^2\boldsymbol{E}\cdot\bar{\boldsymbol{w}}\,\mathrm{d}\boldsymbol{r} = &-\int_\Omega \mu^{-1}(\nabla\times\boldsymbol{E})\cdot(\nabla\times\bar{\boldsymbol{w}})\,\mathrm{d}\boldsymbol{r} \\ &- \langle\mathscr{C}[\boldsymbol{E}_{\Gamma_R^+}], \boldsymbol{w}_{\Gamma_R^+}\rangle - \int_\Omega \boldsymbol{F}\cdot\bar{\boldsymbol{w}}\,\mathrm{d}\boldsymbol{r} \quad \forall\ \ \boldsymbol{w}\in\boldsymbol{H}_S(\mathrm{curl},\Omega). \end{aligned} \tag{4.11}$$

To show the stability of its solution, we follow the argument in [36] but with a careful study of the TBC. The following lemma is useful for the subsequent analysis.

LEMMA 4.3. *Given $\xi\ge 0$ and $\boldsymbol{E}\in\boldsymbol{L}^2(0,\xi;\boldsymbol{H}^{-1/2}(\mathrm{curl},\Gamma))$, it holds that*

$$\mathrm{Re}\int_0^\xi\int_{\Gamma_R^+}\left(\int_0^t \mathscr{C}[\boldsymbol{E}_{\Gamma_R^+}](\tau)\mathrm{d}\tau\right)\cdot\bar{\boldsymbol{E}}_{\Gamma_R^+}(t)\,\mathrm{d}\gamma\mathrm{d}t \ge 0.$$

Proof. Let $\tilde{\boldsymbol{E}}$ be the extension of $\boldsymbol{E}$ with respect to t in $\mathbb{R}$ such that $\tilde{\boldsymbol{E}}=0$ outside the interval $[0,\xi]$. We obtain from (A.1), Lemma A.2, and Lemma 2.2 that

$$\begin{aligned} &\int_{\Gamma_R^+}\int_0^\xi e^{-2s_1 t}\left(\int_0^t \mathscr{C}[\boldsymbol{E}_{\Gamma_R^+}](\tau)\mathrm{d}\tau\right)\cdot\bar{\boldsymbol{E}}_{\Gamma_R^+}(t)\,\mathrm{d}t\mathrm{d}\gamma \\ &\quad= \int_{\Gamma_R^+}\int_0^\infty e^{-2s_1 t}\left(\int_0^t \mathscr{C}[\tilde{\boldsymbol{E}}_{\Gamma_R^+}](\tau)\mathrm{d}\tau\right)\cdot\bar{\tilde{\boldsymbol{E}}}_{\Gamma_R^+}(t)\,\mathrm{d}t\mathrm{d}\gamma \\ &\quad= \int_{\Gamma_R^+}\int_0^\infty e^{-2s_1 t}\left(\int_0^t \mathscr{L}^{-1}\circ s\mathscr{B}\circ\mathscr{L}\tilde{\boldsymbol{E}}_{\Gamma_R^+}(\tau)\mathrm{d}\tau\right)\cdot\bar{\tilde{\boldsymbol{E}}}_{\Gamma_R^+}(t)\,\mathrm{d}t\mathrm{d}\gamma \\ &\quad= \frac{1}{2\pi}\int_{-\infty}^\infty\int_{\Gamma_R^+}\mathscr{B}\circ\mathscr{L}\tilde{\boldsymbol{E}}_{\Gamma_R^+}(s)\cdot\mathscr{L}(\bar{\tilde{\boldsymbol{E}}}_{\Gamma_R^+})(s)\,\mathrm{d}\gamma\mathrm{d}s_2 \\ &\quad= \frac{1}{2\pi}\int_{-\infty}^\infty\left\langle\mathscr{B}[\breve{\tilde{\boldsymbol{E}}}_{\Gamma_R^+}],\, \breve{\tilde{\boldsymbol{E}}}_{\Gamma_R^+}\right\rangle\mathrm{d}s_2 \ge 0. \end{aligned}$$

The proof is completed by taking $s_1\to 0$ in the above inequality. ∎

THEOREM 4.4. *Let $\boldsymbol{E}\in\boldsymbol{H}_S(\mathrm{curl},\Omega)$ be the solution of* (4.11)*. Given $\boldsymbol{E}_0,\boldsymbol{E}_1\in\boldsymbol{L}^2(\Omega)$ and $\boldsymbol{F}\in\boldsymbol{L}^1(0,T;\boldsymbol{L}^2(\Omega))$, for any $T>0$, it holds that*

$$\|\boldsymbol{E}\|_{\boldsymbol{L}^\infty(0,T;\boldsymbol{L}^2(\Omega))} \lesssim \|\boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega)} + T\|\boldsymbol{E}_1\|_{\boldsymbol{L}^2(\Omega)} + T\|\boldsymbol{F}\|_{\boldsymbol{L}^1(0,T;\boldsymbol{L}^2(\Omega))} \tag{4.12}$$

and

$$\|\boldsymbol{E}\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))} \lesssim T^{1/2}\|\boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega)} + T^{3/2}\|\boldsymbol{E}_1\|_{\boldsymbol{L}^2(\Omega)} + T^{3/2}\|\boldsymbol{F}\|_{\boldsymbol{L}^1(0,T;\boldsymbol{L}^2(\Omega))}. \tag{4.13}$$

Proof. Let $0 < \xi < T$ and define an auxiliary function

$$\boldsymbol{\psi}(\boldsymbol{r}, t) = \int_t^\xi \boldsymbol{E}(\boldsymbol{r}, \tau)\,\mathrm{d}\tau, \quad \boldsymbol{r} \in \Omega, \quad 0 \le t \le \xi. \tag{4.14}$$

It is clear that

$$\boldsymbol{\psi}(\boldsymbol{r}, \xi) = 0, \quad \partial_t \boldsymbol{\psi}(\boldsymbol{r}, t) = -\boldsymbol{E}(\boldsymbol{r}, t). \tag{4.15}$$

For any $\boldsymbol{\phi}(\boldsymbol{r}, t) \in \boldsymbol{L}^2(0, \xi; \boldsymbol{L}^2(\Omega))$, we have

$$\int_0^\xi \boldsymbol{\phi}(\boldsymbol{r}, t) \cdot \bar{\boldsymbol{\psi}}(\boldsymbol{r}, t)\mathrm{d}t = \int_0^\xi \left(\int_0^t \boldsymbol{\phi}(\boldsymbol{r}, \tau)\mathrm{d}\tau\right) \cdot \bar{\boldsymbol{E}}(\boldsymbol{r}, t)\mathrm{d}t. \tag{4.16}$$

Indeed, using integration by parts and (4.15), we have

$$\begin{aligned}
\int_0^\xi \boldsymbol{\phi}(\boldsymbol{r}, t) \cdot \bar{\boldsymbol{\psi}}(\boldsymbol{r}, t)\mathrm{d}t &= \int_0^\xi \left(\boldsymbol{\phi}(\boldsymbol{r}, t) \cdot \int_t^\xi \bar{\boldsymbol{E}}(\boldsymbol{r}, \tau)\mathrm{d}\tau\right) \mathrm{d}t \\
&= \int_0^\xi \int_t^\xi \bar{\boldsymbol{E}}(\boldsymbol{r}, \tau)\mathrm{d}\tau \cdot \mathrm{d}\left(\int_0^t \boldsymbol{\phi}(\boldsymbol{r}, \tau)\mathrm{d}\tau\right) \\
&= \int_0^t \int_0^t \boldsymbol{\phi}(\boldsymbol{r}, \tau)\mathrm{d}\tau \cdot \int_t^\xi \bar{\boldsymbol{E}}(\boldsymbol{r}, \tau)\mathrm{d}\tau\Big|_0^\xi + \int_0^\xi \left(\int_0^t \boldsymbol{\phi}(\boldsymbol{r}, \tau)\mathrm{d}\tau\right) \cdot \bar{\boldsymbol{E}}(\boldsymbol{r}, t)\,\mathrm{d}t \\
&= \int_0^\xi \left(\int_0^t \boldsymbol{\phi}(\boldsymbol{r}, \tau)\mathrm{d}\tau\right) \cdot \bar{\boldsymbol{E}}(\boldsymbol{r}, t)\,\mathrm{d}t.
\end{aligned}$$

Next, we take the test function $\boldsymbol{w} = \boldsymbol{\psi}$ in (4.11) and get

$$\begin{aligned}
\int_\Omega \varepsilon \partial_t^2 \boldsymbol{E} \cdot \bar{\boldsymbol{\psi}}\,\mathrm{d}\boldsymbol{r} = &- \int_\Omega \mu^{-1}(\nabla \times \boldsymbol{E}) \cdot (\nabla \times \bar{\boldsymbol{\psi}})\,\mathrm{d}\boldsymbol{r} \\
&- \int_{\Gamma_R^+} \mathscr{C}[\boldsymbol{E}_{\Gamma_R^+}] \cdot \bar{\boldsymbol{\psi}}_{\Gamma_R^+}\,\mathrm{d}\gamma - \int_\Omega \boldsymbol{F} \cdot \bar{\boldsymbol{\psi}}\,\mathrm{d}\boldsymbol{r}.
\end{aligned} \tag{4.17}$$

It follows from (4.15) that

$$\begin{aligned}
\mathrm{Re}\int_0^\xi \int_\Omega \partial_t^2 \boldsymbol{E} \cdot \bar{\boldsymbol{\psi}}\,\mathrm{d}\boldsymbol{r}\mathrm{d}t &= \mathrm{Re}\int_\Omega \int_0^\xi \left(\partial_t(\partial_t \boldsymbol{E} \cdot \bar{\boldsymbol{\psi}}) + \partial_t \boldsymbol{E} \cdot \bar{\boldsymbol{E}}\right) \mathrm{d}t\mathrm{d}\boldsymbol{r} \\
&= \mathrm{Re}\int_\Omega \left((\partial_t \boldsymbol{E} \cdot \bar{\boldsymbol{\psi}})\Big|_0^\xi + \frac{1}{2}|\boldsymbol{E}|^2\Big|_0^\xi\right) \mathrm{d}\boldsymbol{r} \\
&= \frac{1}{2}\|\boldsymbol{E}(\cdot, \xi)\|^2_{\boldsymbol{L}^2(\Omega)} - \frac{1}{2}\|\boldsymbol{E}_0\|^2_{\boldsymbol{L}^2(\Omega)} - \mathrm{Re}\int_\Omega \boldsymbol{E}_1(\boldsymbol{r}) \cdot \bar{\boldsymbol{\psi}}(\boldsymbol{r}, 0)\,\mathrm{d}\boldsymbol{r}.
\end{aligned} \tag{4.18}$$

Integrating (4.17) from $t = 0$ to $t = \xi$ and taking the real parts yields

$$\begin{aligned}
&\frac{\varepsilon}{2}\|\boldsymbol{E}(\cdot, \xi)\|^2_{\boldsymbol{L}^2(\Omega)} - \frac{\varepsilon}{2}\|\boldsymbol{E}_0\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{1}{2}\int_\Omega \mu^{-1}\left|\int_0^\xi \nabla \times \boldsymbol{E}(\boldsymbol{r}, t)\mathrm{d}t\right|^2 \mathrm{d}\boldsymbol{r} \\
&\quad = \varepsilon\mathrm{Re}\int_\Omega \boldsymbol{E}_1(\boldsymbol{r}) \cdot \bar{\boldsymbol{\psi}}(\boldsymbol{r}, 0)\,\mathrm{d}\boldsymbol{r} - \mathrm{Re}\int_0^\xi \int_\Omega \boldsymbol{F} \cdot \bar{\boldsymbol{\psi}}\,\mathrm{d}\boldsymbol{r} \\
&\qquad - \mathrm{Re}\int_0^\xi \int_{\Gamma_R^+} \mathscr{C}[\boldsymbol{E}_{\Gamma_R^+}] \cdot \bar{\boldsymbol{\psi}}_{\Gamma_R^+}\,\mathrm{d}\gamma\mathrm{d}t.
\end{aligned} \tag{4.19}$$

In what follows, we estimate the three terms on the right-hand side of (4.19) separately.

We derive from the Cauchy–Schwarz inequality that

$$
\begin{aligned}
&\mathrm{Re}\int_\Omega \boldsymbol{E}_1(\boldsymbol{r})\cdot\bar{\boldsymbol{\psi}}(\boldsymbol{r},0)\,\mathrm{d}\boldsymbol{r} = \mathrm{Re}\int_\Omega \boldsymbol{E}_1(\boldsymbol{r})\cdot\Big(\int_0^\xi \bar{\boldsymbol{E}}(\boldsymbol{r},t)\,\mathrm{d}t\Big)\,\mathrm{d}\boldsymbol{r}\\
&= \mathrm{Re}\int_0^\xi\int_\Omega \boldsymbol{E}_1(\boldsymbol{r})\cdot\bar{\boldsymbol{E}}(\boldsymbol{r},t)\,\mathrm{d}\boldsymbol{r}\mathrm{d}t \le \|\boldsymbol{E}_1\|_{\boldsymbol{L}^2(\Omega)}\int_0^\xi \|\boldsymbol{E}(\cdot,t)\|_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t.
\end{aligned}
\tag{4.20}
$$

For $0\le t\le\xi\le T$, we have from (4.16) that

$$
\begin{aligned}
\mathrm{Re}\int_0^\xi\int_\Omega \boldsymbol{F}\cdot\bar{\boldsymbol{\psi}}\,\mathrm{d}\boldsymbol{r}\mathrm{d}t &= \mathrm{Re}\int_\Omega\int_0^\xi\left(\int_0^t \boldsymbol{F}(\boldsymbol{r},\tau)\mathrm{d}\tau\right)\cdot\bar{\boldsymbol{E}}(\boldsymbol{r},t)\,\mathrm{d}t\mathrm{d}\boldsymbol{r}\\
&= \mathrm{Re}\int_0^\xi\int_0^t\int_\Omega \boldsymbol{F}(\boldsymbol{r},\tau)\cdot\bar{\boldsymbol{E}}(\boldsymbol{r},t)\,\mathrm{d}\boldsymbol{r}\mathrm{d}\tau\mathrm{d}t\\
&\le \int_0^\xi\left(\int_0^t \|\boldsymbol{F}(\cdot,\tau)\|_{\boldsymbol{L}^2(\Omega)}\mathrm{d}\tau\right)\|\boldsymbol{E}(\cdot,t)\|_{\boldsymbol{L}^2(\Omega)}\mathrm{d}t\\
&\le \int_0^\xi\left(\int_0^\xi \|\boldsymbol{F}(\cdot,t)\|_{\boldsymbol{L}^2(\Omega)}\mathrm{d}t\right)\|\boldsymbol{E}(\cdot,t)\|_{\boldsymbol{L}^2(\Omega)}\mathrm{d}t\\
&\le \left(\int_0^\xi \|\boldsymbol{F}(\cdot,t)\|_{\boldsymbol{L}^2(\Omega)}\mathrm{d}t\right)\left(\int_0^\xi \|\boldsymbol{E}(\cdot,t)\|_{\boldsymbol{L}^2(\Omega)}\mathrm{d}t\right).
\end{aligned}
\tag{4.21}
$$

Using Lemma 4.3 and (4.16), we obtain

$$
\begin{aligned}
&\mathrm{Re}\int_0^\xi\int_\Gamma \mathscr{C}\left(\boldsymbol{E}_\Gamma\right)\cdot\bar{\boldsymbol{\psi}}_\Gamma\mathrm{d}\gamma\mathrm{d}t\\
&= \mathrm{Re}\int_{\Gamma_R^+}\int_0^\xi\left(\int_0^t \mathscr{C}[\boldsymbol{E}_{\Gamma_R^+}](\boldsymbol{r},\tau)\mathrm{d}\tau\right)\cdot\bar{\boldsymbol{E}}_{\Gamma_R^+}(\boldsymbol{r},t)\mathrm{d}t\mathrm{d}\gamma\ge 0.
\end{aligned}
\tag{4.22}
$$

Substituting (4.20)–(4.22) into (4.19), we have for any $\xi\in[0,T]$ that

$$
\begin{aligned}
&\frac{\varepsilon}{2}\|\boldsymbol{E}(\cdot,\xi)\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{1}{2}\int_\Omega \mu^{-1}\Big|\int_0^\xi \nabla\times\boldsymbol{E}(\boldsymbol{r},t)\mathrm{d}t\Big|^2\mathrm{d}\boldsymbol{r}\\
&\le \frac{\varepsilon}{2}\|\boldsymbol{E}_0\|^2_{\boldsymbol{L}^2(\Omega)} + \left(\int_0^\xi \|\boldsymbol{F}(\cdot,t)\|_{\boldsymbol{L}^2(\Omega)}\mathrm{d}t + \varepsilon\|\boldsymbol{E}_1\|_{\boldsymbol{L}^2(\Omega)}\right)\int_0^\xi \|\boldsymbol{E}(\cdot,t)\|_{\boldsymbol{L}^2(\Omega)}\mathrm{d}t.
\end{aligned}
\tag{4.23}
$$

Taking the L^∞-norm with respect to ξ on both sides of (4.23) yields

$$
\|\boldsymbol{E}\|^2_{\boldsymbol{L}^\infty(0,T;\boldsymbol{L}^2(\Omega))} \lesssim \|\boldsymbol{E}_0\|^2_{\boldsymbol{L}^2(\Omega)} + T\left(\|\boldsymbol{F}\|_{\boldsymbol{L}^1(0,T;\boldsymbol{L}^2(\Omega))} + \|\boldsymbol{E}_1\|_{\boldsymbol{L}^2(\Omega)}\right)\|\boldsymbol{E}\|_{\boldsymbol{L}^\infty(0,T;\boldsymbol{L}^2(\Omega))},
$$

which gives the estimate (4.12) after applying the Cauchy–Schwarz inequality.

Integrating (4.23) with respect to ξ from 0 to T and using the Cauchy–Schwarz inequality, we obtain

$$
\|\boldsymbol{E}\|^2_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))} \lesssim T\|\boldsymbol{E}_0\|^2_{\boldsymbol{L}^2(\Omega)} + T^{3/2}\left(\|\boldsymbol{F}\|_{\boldsymbol{L}^1(0,T;\boldsymbol{L}^2(\Omega))} + \|\boldsymbol{E}_1\|_{\boldsymbol{L}^2(\Omega)}\right)\|\boldsymbol{E}\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))},
$$

which implies the estimate (4.13) by using the Cauchy–Schwarz inequality again. ∎

In Theorem 4.4, it is required that $\boldsymbol{E}_0, \boldsymbol{E}_1 \in \boldsymbol{L}^2(\Omega)$, and $\boldsymbol{F} \in \boldsymbol{L}^1(0,T;\boldsymbol{L}^2(\Omega))$, which can be satisfied if the data satisfy

$$\boldsymbol{E}_0 \in \boldsymbol{L}^2(\Omega), \quad \boldsymbol{H}_0 \in \boldsymbol{H}(\text{curl}, \Omega), \quad \boldsymbol{J} \in \boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega)). \tag{4.24}$$

It is important to point out that the estimates in Theorem 4.2 were derived from a usual energy method, while the results in Theorem 4.4 were obtained by using different test functions (cf. (4.14)).

5. Concluding remarks. In this paper, we studied the time-domain Maxwell equations in a three-dimensional open cavity. The scattering problem was reduced to an initial-boundary value problem in a bounded domain by using the exact TBC. The reduced problem was shown to have a unique solution and its stability was also presented. The proofs were based on the examination of the time-harmonic Maxwell equations with a complex wavenumber and the abstract inversion theorem of the Laplace transform. Moreover, by directly considering the variational problem of the time-domain Maxwell equations, an a priori estimate was derived with an explicit dependence on time for the electric field. Computationally, the variational approach leads naturally to a class of finite element methods. As a time-dependent problem, a fast and accurate marching technique shall be developed to deal with the temporal convolution in the TBC. We will report the work on its numerical analysis and computation in a forthcoming paper.

Appendix A. Laplace transform. For any $s = s_1 + \mathrm{i}s_2$ with $s_1, s_2 \in \mathbb{R}, s_1 > 0$, define by $\breve{\boldsymbol{u}}(s)$ the Laplace transform of the vector field $\boldsymbol{u}(t)$, i.e.,

$$\breve{\boldsymbol{u}}(s) = \mathscr{L}(\boldsymbol{u})(s) = \int_0^\infty e^{-st}\boldsymbol{u}(t)\,\mathrm{d}t.$$

It can be verified from the integration by parts that

$$\int_0^t \boldsymbol{u}(\tau)\,\mathrm{d}\tau = \mathscr{L}^{-1}\left(s^{-1}\breve{\boldsymbol{u}}(s)\right), \tag{A.1}$$

where $\mathscr{L}^{-1}$ is the inverse Laplace transform. One verifies from the formula of the inverse Laplace transform that

$$\boldsymbol{u}(t) = \mathscr{F}^{-1}\big(e^{s_1 t}\mathscr{L}(\boldsymbol{u})(s_1 + \mathrm{i}s_2)\big), \tag{A.2}$$

where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform with respect to s_2.

Recall the Plancherel or Parseval identity for the Laplace transform (cf. [15, (2.46)]).

Lemma A.1. *If $\breve{\boldsymbol{u}} = \mathscr{L}(\boldsymbol{u})$ and $\breve{\boldsymbol{v}} = \mathscr{L}(\boldsymbol{v})$, then*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \breve{\boldsymbol{u}}(s)\cdot\breve{\boldsymbol{v}}(s)\mathrm{d}s_2 = \int_0^\infty e^{-2s_1 t}\boldsymbol{u}(t)\cdot\boldsymbol{v}(t)\mathrm{d}t$$

for all $s_1 > \lambda$, where λ is the abscissa of convergence for the Laplace transform of $\boldsymbol{u}$ and $\boldsymbol{v}$.

The following theorem (cf. [36, Theorem 43.1]) is an analogue of the Paley–Wiener–Schwarz theorem for the Fourier transform of the distributions with compact support in the case of Laplace transform.

Lemma A.2. *Let $\breve{\boldsymbol{w}}(s)$ denote a holomorphic function in the half plane $s_1 > \sigma_0$, valued in the Banach space $\mathbb{E}$. The following statements are equivalent:*

1. *there is a distribution* $\boldsymbol{w} \in \mathcal{D}'_+(\mathbb{E})$ *whose Laplace transform is equal to* $\breve{\boldsymbol{w}}(s)$*;*
2. *there is a* σ_1 *with* $\sigma_0 \leq \sigma_1 < \infty$ *and an integer* $m \geq 0$ *such that for all complex numbers* s *with* $s_1 > \sigma_1$*, it holds that* $\|\breve{\boldsymbol{w}}(s)\|_{\mathbb{E}} \lesssim (1+|s|)^m$*,*

where $\mathcal{D}'_+(\mathbb{E})$ *is the space of distributions on the real line which vanish identically in the open negative half line.*

Appendix B. Spherical harmonics on hemisphere. The spherical coordinates (ρ, θ, φ) are related to the Cartesian coordinates $\boldsymbol{r} = (x_1, x_2, x_3)$ by $x_1 = \rho \sin\theta \cos\varphi$, $x_2 = \rho \sin\theta \sin\varphi$, $x_3 = \rho\cos\theta$, with the local orthonormal basis $\{\boldsymbol{e}_\rho, \boldsymbol{e}_\theta, \boldsymbol{e}_\varphi\}$:

$$\begin{aligned} \boldsymbol{e}_\rho &= (\sin\theta\cos\varphi,\, \sin\theta\sin\varphi,\, \cos\theta), \\ \boldsymbol{e}_\theta &= (\cos\theta\cos\varphi,\, \cos\theta\sin\varphi,\, -\sin\theta), \\ \boldsymbol{e}_\varphi &= (-\sin\varphi,\, \cos\varphi,\, 0), \end{aligned}$$

where θ and φ are the Euler angles. Let $\Gamma = \{\boldsymbol{r} : \rho = 1\}$, $\Gamma^+ = \{\boldsymbol{r} : \rho = 1,\, x_3 \geq 0\}$, $\Gamma^- = \{\boldsymbol{r} : \rho = 1,\, x_3 \leq 0\}$ be the unit sphere, upper unit hemisphere, and lower unit hemisphere, respectively. Denote by $\Gamma_R = \{\boldsymbol{r} : \rho = R\}$, $\Gamma_R^+ = \{\boldsymbol{r} : \rho = R,\, x_3 \geq 0\}$, $\Gamma_R^- = \{\boldsymbol{r} : \rho = R,\, x_3 \leq 0\}$ the whole sphere, upper hemisphere, and lower hemisphere with radius R, respectively.

Let $\{Y_n^m(\theta,\varphi),\, |m| \leq n,\, n = 0, 1, 2, \dots\}$ be an orthonormal sequence of spherical harmonics of order n on the unit sphere Γ that satisfies

$$\Delta_\Gamma Y_n^m + n(n+1) Y_n^m = 0, \tag{B.1}$$

where

$$\Delta_\Gamma = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

is the Laplace–Beltrami operator on Γ. Explicitly, the spherical harmonics of order n is written as

$$Y_n^m(\theta,\varphi) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}}\; P_n^{|m|}(\cos\theta) e^{\mathrm{i}m\varphi},$$

where the associated Legendre functions are

$$P_n^m(t) := \sqrt{(1-t^2)^m}\; \frac{\mathrm{d}^m P_n(t)}{\mathrm{d}t^m}, \quad m = 0, 1, \dots, n.$$

Here P_n is the Legendre polynomial of degree n, which is an even function if n is even and an odd function if n is odd.

Define a sequence of rescaled spherical harmonics of order n:

$$X_n^m(\theta,\varphi) = \frac{\sqrt{2}}{R}\, Y_n^m(\theta,\varphi), \tag{B.2}$$

which forms a complete orthonormal system in $L^2(\Gamma_R^+)$ for $|m| \leq n$, $m+n = \text{odd}$, $n \in \mathbb{N}$ (cf. [32, Lemma 3.1]).

Denote by $L^2(\Gamma_R^+)$ the complex square integrable functions on the hemisphere Γ_R^+. For convenience, we take the following notation for double summations:

$$\sum_{|m|\le n} w_n^m := \sum_{n=1}^{\infty} \sum_{m=-n}^{n} w_n^m,$$
$$\sum_{|m|\le n}^{\text{odd}} w_n^m := \sum_{n=1}^{\infty} \sum_{\substack{m=-n \\ m+n=\text{odd}}}^{n} w_n^m,$$
$$\sum_{|m|\le n}^{\text{even}} w_n^m := \sum_{n=1}^{\infty} \sum_{\substack{m=-n \\ m+n=\text{even}}}^{n} w_n^m.$$

To describe vector wave functions on the hemisphere, we introduce some boundary differential operators. For a smooth scalar function w defined on Γ_R^+, let

$$\nabla_\Gamma w = \frac{\partial w}{\partial \theta}\, \boldsymbol{e}_\theta + \frac{1}{\sin\theta}\frac{\partial w}{\partial \varphi}\, \boldsymbol{e}_\varphi$$

be the tangential gradient on Γ_R^+. The surface vector curl is defined by

$$\mathbf{curl}_\Gamma w = \nabla_\Gamma w \times \boldsymbol{e}_\rho.$$

Denote by div_Γ and curl_Γ the surface divergence and the surface scalar curl, respectively. For a smooth vector function $\boldsymbol{w}$ tangential to Γ_R^+, it can be represented by its coordinates in the local orthonormal basis:

$$\boldsymbol{w} = w_\theta\, \boldsymbol{e}_\theta + w_\varphi\, \boldsymbol{e}_\varphi,$$

where

$$w_\theta = \boldsymbol{w}\cdot\boldsymbol{e}_\theta \quad \text{and} \quad w_\varphi = \boldsymbol{w}\cdot\boldsymbol{e}_\varphi.$$

The surface divergence and the surface scalar curl can be defined as

$$\mathrm{div}_\Gamma \boldsymbol{w} = \frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}(w_\theta \sin\theta) + \frac{\partial w_\varphi}{\partial\varphi}\right],$$
$$\mathrm{curl}_\Gamma \boldsymbol{w} = \frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}(w_\varphi \sin\theta) - \frac{\partial w_\theta}{\partial\varphi}\right].$$

It is known (cf. [35]) that these boundary differential operators satisfy

$$\Delta_\Gamma = \mathrm{div}_\Gamma \nabla_\Gamma = -\mathrm{curl}_\Gamma\, \mathbf{curl}_\Gamma \quad \text{and} \quad \mathrm{curl}_\Gamma \nabla_\Gamma = \mathrm{div}_\Gamma \mathbf{curl}_\Gamma = 0. \tag{B.3}$$

It is also know (cf. [16, Theorem 6.23]) that an orthonormal basis for $\boldsymbol{L}_\mathrm{t}^2(\Gamma_R) = \{\boldsymbol{w} \in \boldsymbol{L}^2(\Gamma_R) :\ \boldsymbol{e}_\rho \cdot \boldsymbol{w} = 0\}$, the tangential fields on Γ_R, consists of functions of the form

$$\boldsymbol{U}_n^m(\theta,\varphi) = \frac{1}{R\sqrt{n(n+1)}}\nabla_\Gamma Y_n^m(\theta,\varphi)$$

and

$$\boldsymbol{V}_n^m(\theta,\varphi) = \boldsymbol{e}_\rho \times \boldsymbol{U}_n^m(\theta,\varphi) = -\frac{1}{R\sqrt{n(n+1)}}\mathbf{curl}_\Gamma Y_n^m$$

for $|m| \le n$, $n \in \mathbb{N}$. It follows from (B.1) and (B.3) that

$$\mathrm{div}_\Gamma \boldsymbol{U}_n^m = -\frac{\sqrt{n(n+1)}}{R} Y_n^m, \quad \mathrm{curl}_\Gamma \boldsymbol{V}_n^m = -\frac{\sqrt{n(n+1)}}{R} Y_n^m,$$

and

$$\mathrm{curl}_\Gamma \boldsymbol{U}_n^m = \mathrm{div}_\Gamma \boldsymbol{V}_n^m = 0.$$

Define two sequences of tangential fields

$$\boldsymbol{X}_n^m(\theta,\varphi) = \frac{1}{\sqrt{n(n+1)}} \nabla_\Gamma X_n^m(\theta,\varphi) = \sqrt{2}\, \boldsymbol{U}_n^m(\theta,\varphi) \tag{B.4}$$

and

$$\boldsymbol{Y}_n^m(\theta,\varphi) = \boldsymbol{e}_\rho \times \boldsymbol{X}_n^m(\theta,\varphi) = \sqrt{2}\, \boldsymbol{V}_n^m(\theta,\varphi).$$

Using the definition of the tangential gradient, and noticing that $\boldsymbol{e}_\theta \times \boldsymbol{e}_\varphi = \boldsymbol{e}_\rho$, $\boldsymbol{e}_\varphi \times \boldsymbol{e}_\rho = \boldsymbol{e}_\theta$, $\boldsymbol{e}_\rho \times \boldsymbol{e}_\theta = \boldsymbol{e}_\varphi$, we get

$$\boldsymbol{e}_\theta \times \boldsymbol{X}_n^m \left(\frac{\pi}{2}, \varphi\right) = 0 \quad \text{for } |m| \le n,\ m+n = \text{odd},\ n \in \mathbb{N},$$

and

$$\boldsymbol{e}_\theta \times \boldsymbol{Y}_n^m (\frac{\pi}{2}, \varphi) = 0 \quad \text{for } |m| \le n,\ m+n = \text{even},\ n \in \mathbb{N}.$$

Define a subspace of complex square integrable tangential fields functions on the hemisphere Γ_R^+:

$$\boldsymbol{L}_{\mathrm{t}}^2(\Gamma_R^+) = \{\boldsymbol{w} \in \boldsymbol{L}^2(\Gamma_R^+) :\ \boldsymbol{e}_\rho \cdot \boldsymbol{w} = 0\}.$$

It is shown (cf. [32, Lemma 3.2]) that the vector spherical harmonics $\{\boldsymbol{X}_n^m : m+n = \text{odd}\}$ and $\{\boldsymbol{Y}_n^m : m+n = \text{even}\}$ for $|m| \le n$, $n \in \mathbb{N}$ form a complete orthonormal system in $\boldsymbol{L}_{\mathrm{t}}^2(\Gamma_R^+)$.

Appendix C. Hankel functions. For $\nu \in \mathbb{R}$, the two Hankel functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$, where $z \in \mathbb{C}$, are two fundamental solutions of the Bessel equation of order ν:

$$z^2 \frac{\mathrm{d}^2 u}{\mathrm{d}z^2} + z \frac{\mathrm{d}u}{\mathrm{d}z} + (z^2 - \nu^2) u = 0.$$

Recall the Bessel functions of imaginary argument $K_\nu(z)$, also called the modified Bessel functions, which is the solution of the differential equation

$$z^2 \frac{\mathrm{d}^2 u}{\mathrm{d}z^2} + z \frac{\mathrm{d}u}{\mathrm{d}z} - (z^2 + \nu^2) u = 0.$$

It is connected with $H_\nu^{(1)}(z)$ through the relation

$$K_\nu(z) = \frac{1}{2}\pi \mathrm{i} e^{\frac{1}{2}\nu\pi \mathrm{i}} H_\nu^{(1)}(\mathrm{i}z). \tag{C.1}$$

It is known (cf. [41, p. 511]) that $K_\nu(z)$ has no zeros if $|\arg z| \le \frac{1}{2}\pi$, which implies from (C.1) that $H_\nu^{(1)}(z)$ has no zeros when $\mathrm{Im} z \le 0$.

The spherical Hankel function $h_n^{(1)}(z)$ can also be defined by the Hankel function of half integer order:

$$h_n^{(1)}(z) = \sqrt{\frac{\pi}{2z}} H^{(1)}_{n+\frac{1}{2}}(z). \tag{C.2}$$

Combining (C.1) and (C.2) yields

$$h_n^{(1)}(z) = -\sqrt{\frac{2\pi}{z}} \mathrm{i} e^{-\frac{1}{2}(n+\frac{1}{2})\pi \mathrm{i}} K_{n+\frac{1}{2}}(-\mathrm{i}z),$$

which implies that $h_n^{(1)}(z)$ has no zeros when $\mathrm{Im} z \ge 0$.

The following two lemmas on the spherical Hankel functions for the complex number are proved in [13].

Lemma C.1. *Let $R > 0, n \in \mathbb{Z}, s = s_1 + \mathrm{i}s_2$ with $s_1 > 0$. It holds that*

$$\mathrm{Re}\big\{1 + r_n^{(1)}(\mathrm{i}sR)\big\} < 0,$$

where $r_n^{(1)}(z) = z(h_n^{(1)})'(z)/h_n^{(1)}(z)$.

Lemma C.2. *Let $R > 0, n \in \mathbb{Z}, s = s_1 + \mathrm{i}s_2$ with $s_1 > 0$. It holds that*

$$\mathrm{Im}\big\{s_2(1 + r_n^{(1)}(\mathrm{i}sR))\big\} \le 0.$$

Combining the result in Lemma C.1 for small value of n and the proof in [16] (cf. [28, Lemma 3.1]) for large value of n, we may obtain the following estimate.

Lemma C.3. *Let $R > 0, n \in \mathbb{Z}, s = s_1 + \mathrm{i}s_2$ with $s_1 > 0$. There exist two positive constants C_1 and C_2 such that*

$$C_1 n \le |1 + r_n^{(1)}(\mathrm{i}sR)| \le C_2 n.$$

Appendix D. Functional spaces. Denote by $H^s(\Gamma_R^+)$ the Sobolev space, the completion of $C_0^\infty(\Gamma_R^+)$ in the norm $\|\cdot\|_{H^s(\Gamma_R^+)}$ characterized by

$$\|w\|^2_{H^s(\Gamma_R^+)} = \sum_{|m|\le n}^{\mathrm{odd}} \left(1 + n(n+1)\right)^s \, |w_n^m|^2,$$

where

$$w(\theta,\varphi) = \sum_{|m|\le n}^{\mathrm{odd}} w_n^m \, X_n^m(\theta,\varphi).$$

Introduce three tangential trace spaces:

$$\begin{aligned}
\boldsymbol{H}_{\mathrm{t}}^s(\Gamma_R^+) &= \big\{\boldsymbol{w} \in (H^s(\Gamma_R^+))^3,\ \boldsymbol{e}_\rho \cdot \boldsymbol{w} = 0,\ \boldsymbol{e}_\theta \times \boldsymbol{w}(\pi/2,\varphi) = 0\big\},\\
\boldsymbol{H}^{-1/2}(\mathrm{curl}, \Gamma_R^+) &= \big\{\boldsymbol{w} \in \boldsymbol{H}_{\mathrm{t}}^{-1/2}(\Gamma_R^+),\ \mathrm{curl}_\Gamma \boldsymbol{w} \in H^{-1/2}(\Gamma_R^+)\big\},\\
\boldsymbol{H}^{-1/2}(\mathrm{div}, \Gamma_R^+) &= \big\{\boldsymbol{w} \in \boldsymbol{H}_{\mathrm{t}}^{-1/2}(\Gamma_R^+),\ \mathrm{div}_\Gamma \boldsymbol{w} \in H^{-1/2}(\Gamma_R^+)\big\}.
\end{aligned}$$

For any tangential field $\boldsymbol{w} \in \boldsymbol{H}_{\mathrm{t}}^{s}(\Gamma_R^+)$, it can be represented in the series expansion

$$\boldsymbol{w} = \sum_{|m|\le n}^{\text{odd}} w_{1n}^m \, \boldsymbol{X}_n^m(\theta,\varphi) + \sum_{|m|\le n}^{\text{even}} w_{2n}^m \, \boldsymbol{Y}_n^m(\theta,\varphi).$$

Using the series coefficients, the norm of the space $\boldsymbol{H}_{\mathrm{t}}^{s}(\Gamma_R^+)$ can be characterized by

$$\|\boldsymbol{w}\|^2_{\boldsymbol{H}_{\mathrm{t}}^{s}(\Gamma_R^+)} = \sum_{|m|\le n}^{\text{odd}} (1+n(n+1))^s \, |w_{1n}^m|^2 + \sum_{|m|\le n}^{\text{even}} (1+n(n+1))^s \, |w_{2n}^m|^2;$$

the norm of the space $\boldsymbol{H}^{-1/2}(\mathrm{curl},\, \Gamma_R^+)$ can be characterized by

$$\|\boldsymbol{w}\|^2_{\boldsymbol{H}^{-1/2}(\mathrm{curl},\, \Gamma_R^+)} = \sum_{|m|\le n}^{\text{odd}} \frac{1}{\sqrt{1+n(n+1)}} \, |w_{1n}^m|^2 + \sum_{|m|\le n}^{\text{even}} \sqrt{1+n(n+1)} \, |w_{2n}^m|^2;$$

and the norm of the space $\boldsymbol{H}^{-1/2}(\mathrm{div},\, \Gamma_R^+)$ can be characterized by

$$\|\boldsymbol{w}\|^2_{\boldsymbol{H}^{-1/2}(\mathrm{div},\, \Gamma_R^+)} = \sum_{|m|\le n}^{\text{odd}} \sqrt{1+n(n+1)} \, |w_{1n}^m|^2 + \sum_{|m|\le n}^{\text{even}} \frac{1}{\sqrt{1+n(n+1)}} \, |w_{2n}^m|^2.$$

Define a dual pairing by

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \int_{\Gamma_R^+} \boldsymbol{u} \cdot \bar{\boldsymbol{v}} \, \mathrm{d}\gamma = \sum_{|m|\le n}^{\text{odd}} u_{1n}^m \, \bar{v}_{1n}^m + \sum_{|m|\le n}^{\text{even}} u_{2n}^m \, \bar{v}_{2n}^m,$$

where

$$\boldsymbol{u} = \sum_{|m|\le n}^{\text{odd}} u_{1n}^m \, \boldsymbol{X}_n^m + \sum_{|m|\le n}^{\text{even}} u_{2n}^m \, \boldsymbol{Y}_n^m \quad \text{and} \quad \boldsymbol{v} = \sum_{|m|\le n}^{\text{odd}} v_{1n}^m \, \boldsymbol{X}_n^m + \sum_{|m|\le n}^{\text{even}} v_{2n}^m \, \boldsymbol{Y}_n^m.$$

Introduce three functional spaces

$$\begin{aligned} \boldsymbol{H}(\mathrm{curl},\, \Omega) &= \big\{ \boldsymbol{u} \in \boldsymbol{L}^2(\Omega),\ \nabla \times \boldsymbol{E} \in \boldsymbol{L}^2(\Omega) \big\}, \\ \boldsymbol{H}_S(\mathrm{curl},\, \Omega) &= \big\{ \boldsymbol{u} \in \boldsymbol{H}(\mathrm{curl},\, \Omega),\ \boldsymbol{n} \times \boldsymbol{u} = 0 \text{ on } S \big\}, \\ \boldsymbol{H}_0(\mathrm{curl},\, \Omega) &= \big\{ \boldsymbol{u} \in \boldsymbol{H}(\mathrm{curl},\, \Omega),\ \boldsymbol{n} \times \boldsymbol{u} = 0 \text{ on } S \cup \Gamma_R^+ \big\}, \end{aligned}$$

which are Sobolev spaces with the norm

$$\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathrm{curl},\, \Omega)} = \Big(\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla \times \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} \Big)^{1/2}.$$

Given a vector field $\boldsymbol{u}$ on Γ_R^+, denote by

$$\boldsymbol{u}_{\Gamma_R^+} = -\boldsymbol{e}_\rho \times (\boldsymbol{e}_\rho \times \boldsymbol{u})$$

the tangential component of $\boldsymbol{u}$ on Γ_R^+, which satisfies the following trace estimate (cf. [32, Lemma 3.3]).

LEMMA D.1. *For any* $\boldsymbol{u} \in \boldsymbol{H}_S(\mathrm{curl},\, \Omega)$*, it holds that*

$$\|\boldsymbol{u}_{\Gamma_R^+}\|_{\boldsymbol{H}^{-1/2}(\mathrm{curl},\, \Gamma_R^+)} \lesssim \|\boldsymbol{u}\|_{\boldsymbol{H}(\mathrm{curl},\, \Omega)}.$$

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