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Near-field imaging of biperiodic surfaces for elastic waves

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ABSTRACT

This paper is concerned with the direct and inverse scattering of elastic waves by biperiodic surfaces in three dimensions. The surface is assumed to be a small and smooth perturbation of a rigid plane. Given a time-harmonic plane incident wave, the direct problem is to determine the displacement field of the elastic wave for a given surface; the inverse problem is to reconstruct the surface from the measured displacement field. The direct problem is shown to have a unique weak solution by studying its variational formulation. Moreover, an analytic solution is deduced by using the transformed field expansion method and the convergence is established for the power series solution. A local uniqueness is proved for the inverse problem. An explicit reconstruction formula is obtained and implemented by using the fast Fourier transform. The error estimate is derived for the reconstructed surface function, and it provides an insight on the trade-off among resolution, accuracy, and stability of the solution for the inverse problem. Numerical results show that the method is effective to reconstruct biperiodic scattering surfaces with subwavelength resolution.

1. Introduction

The scattering problems for acoustic and electromagnetic waves have been undergoing extensive studies over the years [18]. Recently, the elastic wave scattering problems have received much attention from both engineering and mathematical communities due to their significant applications in diverse scientific areas [2–5,16,20–25,30]. The purpose of this paper is to study the inverse scattering problem for elastic wave scattering by biperiodic surfaces in three dimensions. It is indispensable to analyze the corresponding direct problem in order to serve this goal. Specifically, we consider a biperiodic scattering surface which is assumed to be a small and smooth perturbation of a plane. The space above the surface is filled with a linear, homogeneous, and isotropic elastic medium, while the space below the surface is elastically rigid. A time-harmonic elastic plane wave is incident on the surface from above. We consider the resonance regime where the wavelength of the incident wave is comparable with the period of the surface. Given the incident field, the direct scattering problem is to determine the displacement field of the total wave for the known surface; the inverse problem is to reconstruct the surface from the measured displacement field of the total wave at a horizontal plane over the surface. The well-posedness was studied for the direct scattering problem in [2,4,5,20,23,24] for the two-dimensional case and in [22] for the three-dimensional case. The inverse scattering problem was also investigated theoretically for its uniqueness in [1] for the two-dimensional case, and numerically by using nonlinear optimization in [21] and the factorization method in [29] for the two-dimensional problem.

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In this work, we propose an effective method for solving quantitatively the three-dimensional inverse elastic scattering problem and seek to achieve super resolved resolution. Compared with the acoustic or electromagnetic counterparts, the elastic scattering problems are more challenging due to the coexistence of compressional and shear waves that travel at different speeds. In view of this fact, we utilize the Helmholtz decomposition to split the displacement field of the elastic wave into a superposition of its compressional part and shear part by introducing two potential functions. Using the Rayleigh expansion, we derive the transparent boundary conditions for each potential function and recast the problem into a coupled boundary value problem. Based on the assumption that the surface is a small and smooth perturbation of a rigid plane, we apply the transformed field expansion to convert the three-dimensional boundary value problem into a successive sequence of two-point boundary value problems in the frequency domain. The method begins with the change of variable to flatten the curved surface into a planar surface; then it resorts to the power series expansion and the Fourier series expansion to find an analytic solution for the direct problem. Using the closed form of the analytic solution, we deduce simple expressions for the leading and linear terms of the power series solution. Dropping all higher order terms, we linearize the inverse problem and obtain an explicit and elegant identity which links the Fourier coefficients of the scattering surface and the measured displacement field. The scattering surface is then reconstructed from the truncated Fourier series expansion. The method requires only a single incident field and is efficiently implemented by the fast Fourier transform. Numerical examples show the method is effective and robust to reconstruct the scattering surfaces with subwavelength resolution. We refer to [17,31,38–40] for transformed field expansion and related boundary perturbation methods for solving various direct scattering problems.

Moreover, we provide theoretical analysis to validate the proposed method. Using a variational formulation, we establish the well-posedness and obtain an energy estimate of the solution for the direct scattering problem. Using a similar variational approach, we obtain the well-posedness of the solution for the recursive boundary value problems and prove the convergence of the power series solution. For the inverse problem, we show a local uniqueness result of the solution for a sufficiently small perturbation. We derive an error estimate for the reconstruction formula, which demonstrates a dependence of the reconstructed surface on all the physical parameters of the model problem and provides an insight on the trade-off among resolution, accuracy, and stability of the solution for the inverse problem. This paper is a non-trivial extension of our previous work on the two-dimensional elastic scattering problems [36,37] due to the obvious difference and increased challenge of the model problem. This work adds a significant contribution to our recent development of designing novel computational methods for solving a class of acoustic and electromagnetic inverse scattering problems [6,10–12,19,32–35]. We refer to [7–9,13–15,28] for other related inverse scattering problems for acoustic and electromagnetic waves.

The outline of the paper is as follows. In section 2 we introduce the model problem and the transparent boundary condition. Section 3 is devoted to the direct scattering problem where we prove the well-posedness of the solution by studying the variational formulation. In section 4 we present the transformed field expansion, derive the recursive boundary value problems, and prove the convergence of the power series expansion. The reconstruction formula, error estimate, and numerical examples are provided in section 5. We conclude the paper with comments and directions for future research in section 6.

2. Problem formulation

In this section we introduce a mathematical model for the elastic scattering by a biperiodic surface and derive a transparent boundary condition for the truncated problem.

2.1. Elastic wave equation

Let $\rho = (x, y) \in \mathbb{R}^2$ and $\boldsymbol{x} = (\rho, z) \in \mathbb{R}^3$. Let $\Lambda = (\Lambda_1, \Lambda_2)$, where $\Lambda_j > 0$ are constants. Denote a rectangular domain $R = (0, \Lambda_1) \times (0, \Lambda_2)$. Consider the part of a biperiodic surface in one periodic cell R:

$$\Gamma_f = \{\boldsymbol{x} \in \mathbb{R}^3 : z = f(\rho),\ \rho \in R\},$$

where $f \in C^k(R)$, $k \geq 2$ is a biperiodic function with period Λ. We assume that

$$f(\rho) = \varepsilon g(\rho), \tag{2.1}$$

where $\varepsilon > 0$ is a small constant and is called the surface deformation parameter, $g \in C^k(R)$ is a biperiodic function with period Λ and represents the normalized profile of f. Let

$$K = \max_{|s| \leq k} \sup_{\rho \in R} \left| D^s g(\rho) \right|, \tag{2.2}$$

which indicates the smoothness of the scattering surface and plays an important role in the subsequent convergence and error analysis. Denote

$$\Omega_f = \{\boldsymbol{x} \in \mathbb{R}^3 : z > f(\rho),\ \rho \in R\},$$

and

$$\Gamma_h = \{\boldsymbol{x} \in \mathbb{R}^3 : z = h,\ \rho \in R\},$$

where $h > \sup_{\rho\in R} f(\rho)$ is another key constant used as the measurement distance for the inverse problem. Denote by

$$\Omega = \{\boldsymbol{x} \in \mathbb{R}^3 :\ f(\rho) < z < h,\ \rho \in R\},$$

the bounded domain between Γ_f and Γ_h.

Suppose that the space above Γ_f is filled with a homogeneous, linear, and isotropic elastic medium with unit mass density, while the substrate below Γ_f is elastically rigid. Let a time-harmonic elastic plane wave $\boldsymbol{u}_{\rm in}$ be incident on Γ_f from above. For simplicity, we consider a compressional incident field with normal incidence:

$$\boldsymbol{u}_{\rm in}(\boldsymbol{x}) = -e^{-{\rm i}\kappa_1 z}\boldsymbol{e}_3, \tag{2.3}$$

where $\kappa_1 = \omega/\sqrt{\lambda + 2\mu}$ is the compressional wavenumber and $\boldsymbol{e}_3 = (0, 0, 1)$. Here $\omega > 0$ is the angular frequency, and λ, μ are the Lamé constants satisfying $\mu > 0$, $\lambda + \mu > 0$.

The displacement field of the total wave $\boldsymbol{u}$ satisfies the Navier equation:

$$\mu\Delta\boldsymbol{u} + (\lambda + \mu)\nabla(\nabla\cdot\boldsymbol{u}) + \omega^2\boldsymbol{u} = 0 \quad \text{in } \Omega_f. \tag{2.4}$$

Due to the rigid substrate assumption, $\boldsymbol{u}$ satisfies the Dirichlet boundary condition:

$$\boldsymbol{u} = 0 \quad \text{on } \Gamma_f. \tag{2.5}$$

Given the scattering surface Γ_f and the incident field $\boldsymbol{u}_{\rm in}$, the direct scattering problem is to determine the total field $\boldsymbol{u}$. Given the incident field $\boldsymbol{u}_{\rm in}$ and the measurement of the total field $\boldsymbol{u}$ at Γ_h, the inverse scattering problem is to determine the scattering surface Γ_f.

2.2. Transparent boundary condition

In order to reduce the problem from the unbounded domain Ω_f into the bounded domain Ω, it is necessary to derive a suitable boundary condition on Γ_h.

Denote by $\boldsymbol{v}$ the displacement field of the scattered wave. Consider the Helmholtz decomposition to split the wave field into its compressional part and shear part:

$$\boldsymbol{v} = \nabla\varphi + \nabla\times\boldsymbol{\psi}, \quad \nabla\cdot\boldsymbol{\psi} = 0, \tag{2.6}$$

where φ is a scalar potential function and $\boldsymbol{\psi}$ is a vector potential function. Substituting (2.6) into (2.4), we may obtain two Helmholtz equations:

$$\left(\Delta + \kappa_1^2\right)\varphi = 0, \quad \left(\Delta + \kappa_2^2\right)\boldsymbol{\psi} = 0 \quad \text{in } \Omega_f, \tag{2.7}$$

where $\kappa_2 = \omega/\sqrt{\mu}$ is the shear wavenumber. It is clear to note that $\kappa_1 < \kappa_2$.

Due to the special problem geometry, the solutions of (2.7) are periodic and admit the Fourier series expansions:

$$\varphi(\rho, z) = \sum_{n\in\mathbb{Z}^2} \varphi^{(n)}(z)e^{{\rm i}\alpha_n\cdot\rho}, \quad \boldsymbol{\psi}(\rho, z) = \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \boldsymbol{\psi}^{(n)}(z)e^{{\rm i}\alpha_n\cdot\rho}, \quad z \geq h, \tag{2.8}$$

where $n = (n_1, n_2)$, $\alpha_n = (\alpha_{1,n}, \alpha_{2,n})$, $\alpha_{j,n} = 2\pi n_j/\Lambda_j$, $\varphi^{(n)}$ and $\boldsymbol{\psi}^{(n)}$ are the n-th Fourier coefficients of φ and $\boldsymbol{\psi}$, respectively, i.e.,

$$\varphi^{(n)}(z) = (\Lambda_1\Lambda_2)^{-1}\int_R \varphi(\rho, z)e^{-{\rm i}\alpha_n\cdot\rho}{\rm d}\rho, \quad \boldsymbol{\psi}^{(n)}(z) = (\Lambda_1\Lambda_2)^{-1}\int_R \boldsymbol{\psi}(\rho, z)e^{-{\rm i}\alpha_n\cdot\rho}{\rm d}\rho.$$

Substituting (2.8) into (2.7) and using the bounded outgoing wave condition, we obtain the Rayleigh expansions:

$$\varphi(\rho, z) = \sum_{n\in\mathbb{Z}^2} \varphi^{(n)}(h)e^{{\rm i}\left(\alpha_n\cdot\rho + \beta_{1,n}(z-h)\right)}, \quad \boldsymbol{\psi}(\rho, z) = \sum_{n\in\mathbb{Z}^2} \boldsymbol{\psi}^{(n)}(h)e^{{\rm i}\left(\alpha_n\cdot\rho + \beta_{2,n}(z-h)\right)}, \quad z > h, \tag{2.9}$$

where

$$\beta_{j,n} = \begin{cases} \left(\kappa_j^2 - |\alpha_n|^2\right)^{1/2}, & |\alpha_n| < \kappa_j, \\ {\rm i}\left(|\alpha_n|^2 - \kappa_j^2\right)^{1/2}, & |\alpha_n| > \kappa_j. \end{cases} \tag{2.10}$$

Here we assume that $|\alpha_n| \neq \kappa_j$ for all $n \in \mathbb{Z}^2$ to exclude the resonance. Substituting (2.9) into (2.6)–(2.7) and evaluating at $z = h$ yields

$$\begin{bmatrix} \boldsymbol{v}^{(n)} \\ 0 \end{bmatrix} = \mathrm{i} \begin{bmatrix} \alpha_{1,n} & 0 & -\beta_{2,n} & \alpha_{2,n} \\ \alpha_{2,n} & \beta_{2,n} & 0 & -\alpha_{1,n} \\ \beta_{1,n} & -\alpha_{2,n} & \alpha_{1,n} & 0 \\ 0 & \mathrm{i}\alpha_{1,n} & \alpha_{2,n} & \beta_{2,n} \end{bmatrix} \begin{bmatrix} \varphi^{(n)} \\ \boldsymbol{\psi}^{(n)} \end{bmatrix}, \tag{2.11}$$

where $\boldsymbol{v}^{(n)}(h)$ is the n-th Fourier coefficient of $\boldsymbol{v}(\rho, h)$.

Consider the boundary operator $\mathscr{B}$ defined by

$$\mathscr{B}\boldsymbol{v} := \mu\partial_z \boldsymbol{v} + (\lambda + \mu)(\nabla \cdot \boldsymbol{v})\boldsymbol{e}_3 \quad \text{on } \Gamma_h. \tag{2.12}$$

Substituting the Helmholtz decomposition (2.6) into (2.12), and using (2.7), we get

$$\mathscr{B}\boldsymbol{v} = \mu\partial_z (\nabla\varphi + \nabla \times \boldsymbol{\psi}) - (\lambda + \mu)\kappa_1^2 \varphi \boldsymbol{e}_3.$$

Plugging the Rayleigh expansion (2.9) into the above equation gives

$$(\mathscr{B}\boldsymbol{v})^{(n)} = \begin{bmatrix} -\mu\alpha_{1,n}\beta_{1,n} & 0 & \mu\beta_{2,n}^2 & -\mu\alpha_{2,n}\beta_{2,n} \\ -\mu\alpha_{2,n}\beta_{1,n} & -\mu\beta_{2,n}^2 & 0 & \mu\alpha_{1,n}\beta_{2,n} \\ \mu|\alpha_n|^2 - \omega^2 & \mu\alpha_{2,n}\beta_{2,n} & -\mu\alpha_{1,n}\beta_{2,n} & 0 \end{bmatrix} \begin{bmatrix} \varphi^{(n)} \\ \boldsymbol{\psi}^{(n)} \end{bmatrix}. \tag{2.13}$$

Solving (2.11) for $\varphi^{(n)}$ and $\boldsymbol{\psi}^{(n)}$, and substituting them into (2.13), we obtain the transparent boundary condition for the scattered field:

$$\mathscr{B}\boldsymbol{v} = \mathscr{T}\boldsymbol{v} := \sum_{n \in \mathbb{Z}^2} M_n \boldsymbol{v}^{(n)}(h) e^{\mathrm{i}\alpha_n \cdot \rho} \quad \text{on } \Gamma_h, \tag{2.14}$$

where

$$M_n = \frac{\mathrm{i}\mu}{\gamma_n} \begin{bmatrix} \alpha_{1,n}^2 (\beta_{1,n} - \beta_{2,n}) + \beta_{2,n}\gamma_n & \alpha_{1,n}\alpha_{2,n} (\beta_{1,n} - \beta_{2,n}) & \alpha_{1,n}\beta_{2,n} (\beta_{1,n} - \beta_{2,n}) \\ \alpha_{1,n}\alpha_{2,n} (\beta_{1,n} - \beta_{2,n}) & \alpha_{2,n}^2 (\beta_{1,n} - \beta_{2,n}) + \beta_{2,n}\gamma_n & \alpha_{2,n}\beta_{2,n} (\beta_{1,n} - \beta_{2,n}) \\ -\alpha_{1,n}\beta_{2,n} (\beta_{1,n} - \beta_{2,n}) & -\alpha_{2,n}\beta_{2,n} (\beta_{1,n} - \beta_{2,n}) & \kappa_2^2 \beta_{2,n} \end{bmatrix}. \tag{2.15}$$

Here $\gamma_n = |\alpha_n|^2 + \beta_{1,n}\beta_{2,n} \neq 0$ for all $n \in \mathbb{Z}^2$.

A simple calculation yields

$$\mathscr{B}\boldsymbol{u}_{\mathrm{in}} = \mathscr{T}\boldsymbol{u}_{\mathrm{in}} + \boldsymbol{p} \quad \text{on } \Gamma_h, \tag{2.16}$$

where

$$\boldsymbol{p} = 2\mathrm{i}(\lambda + 2\mu)\kappa_1 e^{-\mathrm{i}\kappa_1 h} \boldsymbol{e}_3.$$

Adding (2.14) and (2.16), we obtain the transparent boundary condition for the total field $\boldsymbol{u}$:

$$\mathscr{B}\boldsymbol{u} = \mathscr{T}\boldsymbol{u} + \boldsymbol{p} \quad \text{on } \Gamma_h, \tag{2.17}$$

which helps to reduce the problem from the unbounded domain Ω_f into the bounded domain Ω.

3. Direct scattering problem

In this section, we consider the direct scattering problem and establish the well-posedness by studying its variational formulation.

Define the periodic Sobolev space:

$$H^1_{\Gamma_f,\mathrm{p}}(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_f,\ u(\rho, z) = u(\rho + \Lambda, z)\},$$

which is equipped with the usual H^1-norm:

$$\|u\|_{1,\Omega} = \left(\sum_{|s| \le 1} \int_\Omega \left| D^s u \right|^2 \, \mathrm{d}\boldsymbol{x} \right)^{1/2}.$$

Let $H^1_{\Gamma_f,\mathrm{p}}(\Omega)^3$ be the triple Cartesian product space of $H^1_{\Gamma_f,\mathrm{p}}(\Omega)$ and is equipped with the norm

$$\|\boldsymbol{u}\|_{1,\Omega} = \left(\sum_{j=1}^{3} \|u_j\|_{1,\Omega}^2 \right)^{1/2}.$$

Denote by $H^{-1}_{\Gamma_f,\mathrm{p}}(\Omega)^3$ the dual space of $H^1_{\Gamma_f,\mathrm{p}}(\Omega)^3$, which consists of bounded linear functionals on $H^1_{\Gamma_f,\mathrm{p}}(\Omega)^3$, and is equipped with the norm

$$\|\boldsymbol{u}\|_{-1,\Omega} = \sup_{\boldsymbol{v}\in H^1_{\Gamma_f,\mathrm{p}}(\Omega)^3} \frac{|(\boldsymbol{u},\boldsymbol{v})_\Omega|}{\|\boldsymbol{v}\|_{1,\Omega}},$$

where the inner product $(\cdot,\cdot)_\Omega$ is defined as

$$(\boldsymbol{u},\boldsymbol{v})_\Omega = \int_\Omega \boldsymbol{u}\cdot\bar{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}.$$

For any $s\in\mathbb{R}$, denote by $H^s(\Gamma_h)$ the trace Sobolev space of periodic functions on Γ_h, which is equipped with the norm

$$\|u\|_{s,\Gamma_h} = \left[\Lambda_1\Lambda_2 \sum_{n\in\mathbb{Z}^2} \left(1+|\alpha_n|^2\right)^s \left|u^{(n)}(h)\right|^2 \right]^{1/2}.$$

Denote by $H^s(\Gamma_h)^3$ the triple Cartesian product space of $H^s(\Gamma_h)$. It is equipped with the norm

$$\|\boldsymbol{u}\|_{s,\Gamma_h} = \left(\sum_{j=1}^{3} \|u_j\|_{s,\Gamma_h}^2 \right)^{1/2}.$$

It is well known that $H^{-s}(\Gamma_h)^3$ is the dual space of $H^s(\Gamma_h)^3$ for any $s\in\mathbb{R}$ with respect to the inner product

$$\langle \boldsymbol{u},\boldsymbol{v}\rangle_{\Gamma_h} = \int_{\Gamma_h} \boldsymbol{u}\cdot\bar{\boldsymbol{v}}\,\mathrm{d}\rho.$$

The variational formulation of the direct scattering problem is to find $\boldsymbol{u}\in H^1_{\Gamma_f,\mathrm{p}}(\Omega)^3$ such that

$$a_\Omega(\boldsymbol{u},\boldsymbol{v}) = \langle \boldsymbol{p},\boldsymbol{v}\rangle_{\Gamma_h}, \quad \forall\, \boldsymbol{v}\in H^1_{\Gamma_f,\mathrm{p}}(\Omega)^3, \tag{3.1}$$

where the sesquilinear form

$$a_\Omega(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \mu\,(J\boldsymbol{u} : J\bar{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x} + (\lambda+\mu)(\nabla\cdot\boldsymbol{u},\nabla\cdot\boldsymbol{v})_\Omega - \omega^2(\boldsymbol{u},\boldsymbol{v})_\Omega - \langle \mathscr{T}\boldsymbol{u},\boldsymbol{v}\rangle_{\Gamma_h}, \tag{3.2}$$

and $J\boldsymbol{u}$ denotes the Jacobian matrix of $\boldsymbol{u}$, and $A:B=\mathrm{tr}(AB^*)$ is the Frobenius inner product for matrices A and B. For any $\boldsymbol{u}\in H^1_{\Gamma_f,\mathrm{p}}(\Omega)^3$, denote the Frobenius norm by

$$\|J\boldsymbol{u}\|_{0,\Omega} = \left(\sum_{j=1}^{3} \int_\Omega \left|\nabla u_j\right|^2 \mathrm{d}\boldsymbol{x} \right)^{1/2}.$$

Lemma 3.1. *It holds the estimate*

$$\|\boldsymbol{u}\|_{0,\Omega} \le h\|J\boldsymbol{u}\|_{0,\Omega}, \quad \forall\, \boldsymbol{u}\in H^1_{\Gamma_f,\mathrm{p}}(\Omega)^3.$$

Proof. Denote by

$$B = \{\boldsymbol{x}\in\mathbb{R}^3 : \rho\in R,\ 0<z<h\}$$

the rectangular box which contains Ω. For any $\boldsymbol{u}\in H^1_{\Gamma_f,\mathrm{p}}(\Omega)^3$, consider the zero extension to B:

$$\tilde{\boldsymbol{u}}(\boldsymbol{x}) = \begin{cases} \boldsymbol{u}(\boldsymbol{x}), & \boldsymbol{x}\in\Omega, \\ \boldsymbol{0}, & \boldsymbol{x}\in B\setminus\bar{\Omega}. \end{cases} \tag{3.3}$$

It follows from the Cauchy–Schwarz inequality that

$$|\tilde{\boldsymbol{u}}|^2 = \left|\int_0^z \partial_z \tilde{\boldsymbol{u}}\,\mathrm{d}z\right|^2 \le h \int_0^h |\partial_z \tilde{\boldsymbol{u}}|^2\,\mathrm{d}z.$$

Hence

$$\|\tilde{\boldsymbol{u}}\|_{0,B}^2 = \int_0^h \int_R |\tilde{\boldsymbol{u}}|^2\,\mathrm{d}\boldsymbol{x} \le h \int_0^h \int_R \int_0^h |\partial_z \tilde{\boldsymbol{u}}|^2\,\mathrm{d}z\mathrm{d}\boldsymbol{x} \le h^2 \|J\tilde{\boldsymbol{u}}\|_{0,B}^2,$$

which completes the proof by noting that

$$\|\boldsymbol{u}\|_{0,\Omega} = \|\tilde{\boldsymbol{u}}\|_{0,B} \quad \text{and} \quad \|J\boldsymbol{u}\|_{0,\Omega} = \|J\tilde{\boldsymbol{u}}\|_{0,B}. \qquad \square$$

Lemma 3.2. *It holds the estimate*

$$\|\boldsymbol{u}\|_{1/2,\Gamma_h} \le \|\boldsymbol{u}\|_{1,\Omega}, \quad \forall\, \boldsymbol{u} \in H^1_{\Gamma_f,\mathrm{p}}(\Omega)^3.$$

Proof. Let $\tilde{\boldsymbol{u}}$ be the zero extension of $\boldsymbol{u}$ defined in (3.3). It follows from Young's inequality that

$$\begin{aligned}
\left|\tilde{\boldsymbol{u}}^{(n)}(h)\right|^2 &= \int_0^h \frac{\mathrm{d}}{\mathrm{d}z}\left|\tilde{\boldsymbol{u}}^{(n)}(z)\right|^2\mathrm{d}z \\
&\le \int_0^h 2\left|\tilde{\boldsymbol{u}}^{(n)}(z)\right|\left|\frac{\mathrm{d}}{\mathrm{d}z}\tilde{\boldsymbol{u}}^{(n)}(z)\right|\mathrm{d}z \\
&\le (1+|\alpha_n|^2)^{1/2}\int_0^h \left|\tilde{\boldsymbol{u}}^{(n)}(z)\right|^2\mathrm{d}z + (1+|\alpha_n|^2)^{-1/2}\int_0^h \left|\frac{\mathrm{d}}{\mathrm{d}z}\tilde{\boldsymbol{u}}^{(n)}(z)\right|^2\mathrm{d}z,
\end{aligned}$$

which gives

$$(1+|\alpha_n|^2)^{1/2}\left|\tilde{\boldsymbol{u}}^{(n)}(h)\right|^2 \le (1+|\alpha_n|^2)\int_0^h \left|\tilde{\boldsymbol{u}}^{(n)}(z)\right|^2\mathrm{d}z + \int_0^h \left|\frac{\mathrm{d}}{\mathrm{d}z}\tilde{\boldsymbol{u}}^{(n)}(z)\right|^2\mathrm{d}z.$$

Using the Fourier series expansion of $\tilde{\boldsymbol{u}}$, we can verify that

$$\|\tilde{\boldsymbol{u}}\|_{1,B}^2 = \Lambda_1\Lambda_2 \sum_{n\in\mathbb{Z}^2}\left[(1+|\alpha_n|^2)\int_0^h \left|\tilde{\boldsymbol{u}}^{(n)}(z)\right|^2\mathrm{d}z + \int_0^h \left|\frac{\mathrm{d}}{\mathrm{d}z}\tilde{\boldsymbol{u}}^{(n)}(z)\right|^2\mathrm{d}z\right].$$

Hence

$$\|\tilde{\boldsymbol{u}}\|_{1/2,\Gamma_h}^2 = \Lambda_1\Lambda_2 \sum_{n\in\mathbb{Z}^2}(1+|\alpha_n|^2)^{1/2}|\tilde{\boldsymbol{u}}^{(n)}(h)|^2 \le \|\tilde{\boldsymbol{u}}\|_{1,B}^2,$$

which completes the proof by noting that

$$\|\boldsymbol{u}\|_{1/2,\Gamma_h} = \|\tilde{\boldsymbol{u}}\|_{1/2,\Gamma_h}, \quad \|\boldsymbol{u}\|_{1,\Omega} = \|\tilde{\boldsymbol{u}}\|_{1,B}. \qquad \square$$

Lemma 3.3. *For any $\eta > 0$, it holds the estimate*

$$\|\boldsymbol{u}\|_{-1/2,\Gamma_h}^2 \le \eta^{-1}\|\boldsymbol{u}\|_{0,\Omega}^2 + \eta\,\|\partial_z\boldsymbol{u}\|_{0,\Omega}^2, \quad \forall\, \boldsymbol{u} \in H^1_{\Gamma_f,\mathrm{p}}(\Omega)^3.$$

Proof. Let $\tilde{\boldsymbol{u}}$ be the zero extension of $\boldsymbol{u}$ defined in (3.3). It follows from Young's inequality that

$$\begin{aligned}
\left|\tilde{\boldsymbol{u}}^{(n)}(h)\right|^2 &= \int_0^h \frac{\mathrm{d}}{\mathrm{d}z}|\tilde{\boldsymbol{u}}^{(n)}(z)|^2\,\mathrm{d}z \le \int_0^h 2|\tilde{\boldsymbol{u}}^{(n)}(z)|\left|\frac{\mathrm{d}}{\mathrm{d}z}\tilde{\boldsymbol{u}}^{(n)}(z)\right|\mathrm{d}z \\
&\le \eta^{-1}(1+|\alpha_n|^2)^{1/2}\int_0^h |\tilde{\boldsymbol{u}}^{(n)}(z)|^2\,\mathrm{d}z + \eta(1+|\alpha_n|^2)^{-1/2}\int_0^h \left|\frac{\mathrm{d}}{\mathrm{d}z}\tilde{\boldsymbol{u}}^{(n)}(z)\right|^2\mathrm{d}z,
\end{aligned}$$

which gives

$$(1+|\alpha_n|^2)^{-1/2}\left|\tilde{\boldsymbol{u}}^{(n)}(h)\right|^2 \le \eta^{-1}\int_0^h \left|\tilde{\boldsymbol{u}}^{(n)}(z)\right|^2 \mathrm{d}z + \eta(1+|\alpha_n|^2)^{-1}\int_0^h \left|\frac{\mathrm{d}}{\mathrm{d}z}\tilde{\boldsymbol{u}}^{(n)}(z)\right|^2 \mathrm{d}z.$$

Using the Fourier series expansion of $\tilde{\boldsymbol{u}}$, we can verify that

$$\|\tilde{\boldsymbol{u}}\|_{0,B}^2 = \Lambda_1\Lambda_2 \sum_{n\in\mathbb{Z}^2}\int_0^h \left|\tilde{\boldsymbol{u}}^{(n)}(z)\right|^2 \mathrm{d}z,$$

and

$$\left\|\partial_z\tilde{\boldsymbol{u}}\right\|_{0,B}^2 = \Lambda_1\Lambda_2 \sum_{n\in\mathbb{Z}^2}\int_0^h \left|\frac{\mathrm{d}}{\mathrm{d}z}\tilde{\boldsymbol{u}}^{(n)}(z)\right|^2 \mathrm{d}z.$$

Hence we have

$$\begin{aligned}
\|\tilde{\boldsymbol{u}}\|_{-1/2,\Gamma_h}^2 &= \Lambda_1\Lambda_2 \sum_{n\in\mathbb{Z}^2}(1+|\alpha_n|^2)^{-1/2}|\tilde{\boldsymbol{u}}^{(n)}(h)|^2 \\
&\le \Lambda_1\Lambda_2 \sum_{n\in\mathbb{Z}^2}\eta^{-1}\int_0^h \left|\tilde{\boldsymbol{u}}^{(n)}(z)\right|^2 \mathrm{d}z + \eta(1+|\alpha_n|^2)^{-1}\int_0^h \left|\frac{\mathrm{d}}{\mathrm{d}z}\tilde{\boldsymbol{u}}^{(n)}(z)\right|^2 \mathrm{d}z \\
&\le \Lambda_1\Lambda_2 \sum_{n\in\mathbb{Z}^2}\eta^{-1}\int_0^h \left|\tilde{\boldsymbol{u}}^{(n)}(z)\right|^2 \mathrm{d}z + \eta\int_0^h \left|\frac{\mathrm{d}}{\mathrm{d}z}\tilde{\boldsymbol{u}}^{(n)}(z)\right|^2 \mathrm{d}z \\
&= \eta^{-1}\|\tilde{\boldsymbol{u}}\|_{0,B}^2 + \eta\left\|\partial_z\tilde{\boldsymbol{u}}\right\|_{0,B}^2,
\end{aligned}$$

which completes the proof by noting that

$$\|\boldsymbol{u}\|_{-1/2,\Gamma_h} = \left\|\tilde{\boldsymbol{u}}\right\|_{-1/2,\Gamma_h}, \quad \|\boldsymbol{u}\|_{0,\Omega} = \left\|\tilde{\boldsymbol{u}}\right\|_{0,B}, \quad \|\partial_z\boldsymbol{u}\|_{0,\Omega} = \left\|\partial_z\tilde{\boldsymbol{u}}\right\|_{0,B}. \qquad \square$$

Lemma 3.4. *The boundary operator* $\mathscr{T}: H^{1/2}(\Gamma_h)^3 \to H^{-1/2}(\Gamma_h)^3$ *is continuous, i.e.,*

$$\|\mathscr{T}\boldsymbol{u}\|_{-1/2,\Gamma_h} \lesssim \|\boldsymbol{u}\|_{1/2,\Gamma_h}, \quad \forall\, \boldsymbol{u}\in H^{1/2}(\Gamma_h)^3.$$

Proof. It follows from the definition of $\beta_{j,n}$ in (2.10) that we get

$$\begin{aligned}
\beta_{j,n} &= \mathrm{i}(|\alpha_n|^2-\kappa_j^2)^{1/2} \sim |n|, \\
|\alpha_n|^2+\beta_{1,n}\beta_{2,n} &= |\alpha_n|^2\left[1-\left(1-\frac{\kappa_1^2}{|\alpha_n|^2}\right)^{1/2}\left(1-\frac{\kappa_s^2}{|\alpha_n|^2}\right)^{1/2}\right] \sim 1, \\
\beta_{1,n}-\beta_{2,n} &= \mathrm{i}(|\alpha_n|^2-\kappa_1^2)^{1/2}-\mathrm{i}(|\alpha_n|^2-\kappa_2^2)^{1/2} \sim |n|^{-1},
\end{aligned}$$

as $|n|\to\infty$. Denote by $\|M_n\|_2$ the Euclidean norm of matrix M_n. It follows from (2.15) that

$$\|M_n\|_2 \sim |n| \quad \text{as } |n|\to\infty.$$

Hence we have

$$\begin{aligned}
\|\mathscr{T}\boldsymbol{u}\|_{-1/2,\Gamma_h}^2 &= \sum_{n\in\mathbb{Z}^2}(1+|\alpha_n|^2)^{-1/2}\left|M_n\boldsymbol{u}^{(n)}(h)\right|^2 \\
&\lesssim \sum_{n\in\mathbb{Z}^2}(1+|\alpha_n|^2)^{1/2}\left|\boldsymbol{u}^{(n)}(h)\right|^2 = \|\boldsymbol{u}\|_{1/2,\Gamma_h}^2,
\end{aligned}$$

which completes the proof. $\square$

Lemma 3.5. *Let $\hat{M}_n = -(M_n + M_n^*)/2$. It holds that*

(1) $\|\hat{M}_n\|_2 \lesssim 1$ *if* $|\alpha_n| < \kappa_2$;
(2) $\hat{M}_n$ *is positive definite if* $|\alpha_n| > \kappa_2$.

Proof. For any $n \in \mathbb{Z}^2$, denote $\zeta_{j,n} = \left|\kappa_j^2 - |\alpha_n|^2\right|^{1/2}$. If $|\alpha_n| < \kappa_2$, then it is easy to verify $\|\hat{M}_n\| \lesssim 1$. If $|\alpha_n| > \kappa_2$, then $\zeta_{j,n} > 0$, $\beta_{j,n} = \mathrm{i}\zeta_{j,n}$, and $\gamma_n = |\alpha_n|^2 - \zeta_{1,n}\zeta_{2,n} > 0$. It follows from (2.15) that

$$\hat{M}_n = \frac{\mu}{\gamma_n}\begin{bmatrix} \alpha_{1,n}^2\left(\zeta_{1,n} - \zeta_{2,n}\right) + \zeta_{2,n}\gamma_n & \alpha_{1,n}\alpha_{2,n}\left(\zeta_{1,n} - \zeta_{2,n}\right) & \mathrm{i}\alpha_{1,n}\zeta_{2,n}\left(\zeta_{1,n} - \zeta_{2,n}\right) \\ \alpha_{1,n}\alpha_{2,n}\left(\zeta_{1,n} - \zeta_{2,n}\right) & \alpha_{2,n}^2\left(\zeta_{1,n} - \zeta_{2,n}\right) + \zeta_{2,n}\gamma_n & \mathrm{i}\alpha_{2,n}\zeta_{2,n}\left(\zeta_{1,n} - \zeta_{2,n}\right) \\ -\mathrm{i}\alpha_{1,n}\zeta_{2,n}\left(\zeta_{1,n} - \zeta_{2,n}\right) & -\mathrm{i}\alpha_{2,n}\zeta_{2,n}\left(\zeta_{1,n} - \zeta_{1,n}\right) & \kappa_2^2\zeta_{2,n} \end{bmatrix}.$$

Since $\kappa_1 < \kappa_2$, we have $\zeta_{1,n} > \zeta_{2,n}$. The first leading principle minor of $\hat{M}_n$ is

$$\alpha_{1,n}^2\left(\zeta_{1,n} - \zeta_{2,n}\right) + \zeta_{2,n}\gamma_n > 0.$$

The second leading principle minor of $\hat{M}_n$ is

$$|\alpha_n|^2(\zeta_{1,n} - \zeta_{2,n})\zeta_{2,n}\gamma_n + \zeta_{2,n}^2\gamma_n^2 > 0.$$

Using the determinant for block matrices, we can compute

$$\begin{aligned}\det(\hat{M}_n) &= \gamma_n\zeta_{2,n}^2\left[|\alpha_n|^2(\zeta_{1,n} - \zeta_{2,n})\left(\kappa_2^2 - \zeta_{2,n}(\zeta_{1,n} - \zeta_{2,n})\right) + \kappa_2^2\gamma_n\zeta_{2,n}\right] \\ &= \gamma_n\zeta_{2,n}^2\left[|\alpha_n|^2(\zeta_{1,n} - \zeta_{2,n})\left(|\alpha_n|^2 - \zeta_{1,n}\zeta_{2,n}\right) + \kappa_2^2\gamma_n\zeta_{2,n}\right] > 0.\end{aligned}$$

It follows from Sylvester's rule that the matrix $\hat{M}_n$ is positive definite. □

Theorem 3.6. *The variational problem (3.1) has a unique solution $\boldsymbol{u} \in H^1_{\Gamma_f,\mathrm{p}}(\Omega)^3$ for sufficiently small h. The solution satisfies the estimate*

$$\|\boldsymbol{u}\|_{1,\Omega} \lesssim \|\boldsymbol{p}\|_{-1/2,\Gamma_h}.$$

Proof. Following the Cauchy–Schwarz inequality, we have

$$\begin{aligned}|a_\Omega(\boldsymbol{u}, \boldsymbol{v})| &\le \mu\|J\boldsymbol{u}\|_{0,\Omega}\|J\boldsymbol{v}\|_{0,\Omega} + (\lambda + \mu)\|\nabla\cdot\boldsymbol{u}\|_{0,\Omega}\|\nabla\cdot\boldsymbol{v}\|_{0,\Omega} + \omega^2\|\boldsymbol{u}\|_{0,\Omega}\|\boldsymbol{v}\|_{0,\Omega} \\ &\quad + \|\mathscr{T}\boldsymbol{u}\|_{-1/2,\Gamma_h}\|\boldsymbol{v}\|_{1/2,\Gamma_h} \\ &\lesssim \|\boldsymbol{u}\|_{1,\Omega}\|\boldsymbol{v}\|_{1,\Omega} + \|\mathscr{T}\boldsymbol{u}\|_{-1/2,\Gamma_h}\|\boldsymbol{v}\|_{1/2,\Gamma_h}.\end{aligned}$$

Applying Lemma 3.2 and Lemma 3.4 yields

$$|a_\Omega(\boldsymbol{u}, \boldsymbol{v})| \lesssim \|\boldsymbol{u}\|_{1,\Omega}\|\boldsymbol{v}\|_{1,\Omega},$$

which shows the sesquilinear form is bounded.

It follows from the definition (2.14) that

$$-\mathrm{Re}\,\langle\mathscr{T}\boldsymbol{u}, \boldsymbol{u}\rangle_{\Gamma_h} = \Lambda_1\Lambda_2\sum_{n\in\mathbb{Z}^2}\hat{M}_n\boldsymbol{u}^{(n)}(h)\overline{\boldsymbol{u}^{(n)}(h)}.$$

By Lemma 3.5, we have

$$\begin{aligned}\sum_{|\alpha_n|<\kappa_2}\hat{M}_n\boldsymbol{u}^{(n)}(h)\overline{\boldsymbol{u}^{(n)}(h)} &\lesssim \sum_{|\alpha_n|<\kappa_2}\left|\boldsymbol{u}^{(n)}(h)\right|^2 \\ &\lesssim \sum_{|\alpha_n|<\kappa_2}(1 + |\alpha_n|^2)^{-1/2}\left|\boldsymbol{u}^{(n)}(h)\right|^2 \lesssim \|\boldsymbol{u}\|_{-1/2,\Gamma_h},\end{aligned}$$

and

$$\sum_{|\alpha_n|>\kappa_2}\hat{M}_n\boldsymbol{u}^{(n)}(h)\overline{\boldsymbol{u}^{(n)}(h)} \ge 0,$$

which shows that

$$-\mathrm{Re}\langle \mathscr{T}\boldsymbol{u},\boldsymbol{u}\rangle_{\Gamma_h} \geq -c\|\boldsymbol{u}\|_{-1/2,\Gamma_h},$$

where $c > 0$ is a constant.

Using Lemma 3.1 and Lemma 3.3, we obtain

$$\begin{aligned}
\mathrm{Re}a_\Omega(\boldsymbol{u},\boldsymbol{u}) &= \mu\|J\boldsymbol{u}\|^2_{0,\Omega} + (\lambda+\mu)\|\nabla\cdot\boldsymbol{u}\|^2_{0,\Omega} - \omega^2\|\boldsymbol{u}\|^2_{0,\Omega} - \mathrm{Re}\,\langle \mathscr{T}\boldsymbol{u},\boldsymbol{u}\rangle_{\Gamma_h} \\
&\geq \mu\|J\boldsymbol{u}\|^2_{0,\Omega} - \omega^2\|\boldsymbol{u}\|^2_{0,\Omega} - c\|\boldsymbol{u}\|_{-1/2,\Gamma_h} \\
&\geq \mu\|J\boldsymbol{u}\|^2_{0,\Omega} - \omega^2\|\boldsymbol{u}\|^2_{0,\Omega} - c\eta^{-1}\|\boldsymbol{u}\|^2_{0,\Omega} - c\eta\|\partial_z\boldsymbol{u}\|^2_{0,\Omega} \\
&\geq (\mu - c\eta)\|J\boldsymbol{u}\|^2_{0,\Omega} - \left(\omega^2 + c\eta^{-1}\right)\|\boldsymbol{u}\|^2_{0,\Omega} \\
&\geq \left[(\mu - c\eta) - \left(\omega^2 + c\eta^{-1}\right)h^2\right]\|J\boldsymbol{u}\|^2_{0,\Omega} \\
&\geq \left[(\mu - c\eta) - (\omega^2 + c\eta^{-1})h^2\right]\left(1+h^2\right)^{-1}\|\boldsymbol{u}\|^2_{1,\Omega},
\end{aligned}$$

where $\eta > 0$ is a constant. Taking $\eta = \mu/(2c)$, we have $|a_\Omega(\boldsymbol{u},\boldsymbol{u})| \gtrsim \|\boldsymbol{u}\|^2_{1,\Omega}$ for sufficiently small h. Hence the sesquilinear form is coercive and the proof is completed by using the Lax–Milgram lemma. □

4. Transformed field expansion

In this section, we introduce the transformed field expansion to obtain an analytic solution for the direct scattering problem.

4.1. Change of variables

Consider the change of variables:

$$\tilde{x} = x, \quad \tilde{y} = y, \quad \tilde{z} = h\left(\frac{z-f}{h-f}\right), \tag{4.1}$$

which transforms the domain Ω into the rectangular box B. In particular, it maps Γ_f to $\Gamma_0 = \{\tilde{\boldsymbol{x}} \in \mathbb{R}^3 : \tilde{z} = 0,\ \tilde{\rho} \in R\}$ and maps Γ_h still to Γ_h. Using the chain rule, we have the differential rules:

$$\begin{aligned}
\partial_x &= \partial_{\tilde{x}} - f_{\tilde{x}}\left(\frac{h-\tilde{z}}{h-f}\right)\partial_{\tilde{z}}, \\
\partial_y &= \partial_{\tilde{y}} - f_{\tilde{y}}\left(\frac{h-\tilde{z}}{h-f}\right)\partial_{\tilde{z}}, \\
\partial_z &= \left(\frac{h}{h-f}\right)\partial_{\tilde{z}}.
\end{aligned}$$

Let $\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}}) = \boldsymbol{u}(\boldsymbol{x})$. It can be verified that the new function $\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}})$, after dropping the tilde for simplicity of notation, satisfies

$$\sum_{j=1}^{3} \mathscr{C}_{ij}u_j = 0, \tag{4.2}$$

where $\mathscr{C}_{ij} = \mathscr{C}_{ji}$ and

$$\begin{aligned}
\mathscr{C}_{11} = \mu\Big\{&(h-f)^2(\partial_{xx}+\partial_{yy}) + \left[h^2 + (h-z)^2|\nabla f|^2\right]\partial_{zz} \\
&- 2(h-z)(h-f)(f_x\partial_{xz} + f_y\partial_{yz}) - (h-z)\left[(h-f)\Delta f + 2|\nabla f|^2\right]\partial_z\Big\} \\
&+ (\lambda+\mu)\Big\{(h-f)^2\partial_{xx} + (h-z)^2 f_x^2\partial_{zz} - 2(h-z)(h-f)f_x\partial_{xz} \\
&- (h-z)\left[(h-f)f_{xx} + 2f_x^2\right]\partial_z\Big\} + \omega^2(h-f)^2, \\
\mathscr{C}_{12} = (\lambda+\mu)\Big\{&(h-f)^2\partial_{xy} - (h-z)\big[(h-f)(f_y\partial_{xz} + f_x\partial_{yz}) - f_x f_y\partial_{zz} \\
&+ ((h-f)f_{xy} + 2f_x f_y)\partial_z\big]\Big\},
\end{aligned}$$

$$\mathscr{C}_{13} = (\lambda+\mu)\big[h(h-f)\partial_{xz} - h(h-z)f_x\partial_{zz} + hf_x\partial_z\big],$$
$$\mathscr{C}_{22} = \mu\Big\{(h-f)^2(\partial_{xx}+\partial_{yy}) + \Big[h^2+(h-z)^2|\nabla f|^2\Big]\partial_{zz}$$
$$- 2(h-z)(h-f)(f_x\partial_{xz}+f_y\partial_{yz}) - (h-z)\Big[(h-f)\Delta f + 2|\nabla f|^2\Big]\partial_z\Big\}$$
$$+ (\lambda+\mu)\Big\{(h-f)^2\partial_{yy} + (h-z)^2 f_y^2\partial_{zz} - 2(h-z)(h-f)f_y\partial_{yz}$$
$$- (h-z)\Big[(h-f)f_{yy} + 2f_y^2\Big]\partial_z\Big\} + \omega^2(h-f)^2,$$
$$\mathscr{C}_{23} = (\lambda+\mu)\big[h(h-f)\partial_{yz} - h(h-z)f_y\partial_{zz} + hf_y\partial_z\big],$$
$$\mathscr{C}_{33} = \mu\Big\{(h-f)^2(\partial_{xx}+\partial_{yy}) + \Big[h^2+(h-z)^2|\nabla f|^2\Big]\partial_{zz}$$
$$- 2(h-z)(h-f)(f_x\partial_{xz}+f_y\partial_{yz}) - (h-z)\Big[(h-f)\Delta f + 2|\nabla f|^2\Big]\partial_z\Big\}$$
$$+ (\lambda+\mu)h^2\partial_{zz} + \omega^2(h-f)^2.$$

In the new coordinates, the Dirichlet boundary condition (2.5) becomes

$$\boldsymbol{u} = 0 \quad \text{on } \Gamma_0.$$

Note that $\mathscr{T}\boldsymbol{u} = \mathscr{T}\tilde{\boldsymbol{u}}$ on Γ_h. The transparent boundary condition (2.17) becomes

$$\mu h\partial_z\boldsymbol{u} + (\lambda+\mu)\big[(h-f)\partial_x u_1 + (h-f)\partial_y u_2 + h\partial_z u_3\big]\boldsymbol{e}_3 = (h-f)\,(\mathscr{T}\boldsymbol{u} + \boldsymbol{p}) \quad \text{on } \Gamma_h. \tag{4.3}$$

4.2. Power series expansion

Consider the power series expansion of ε:

$$\boldsymbol{u}(\boldsymbol{r},\varepsilon) = \sum_{m=0}^{\infty} \varepsilon^m \boldsymbol{u}_m(\boldsymbol{r}). \tag{4.4}$$

Substituting (4.4) and (2.1) into (4.2)–(4.3) yields successive sequence of boundary value problems:

$$\begin{cases} \mu\Delta\boldsymbol{u}_m + (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u}_m) + \omega^2\boldsymbol{u}_m = \boldsymbol{q}_m & \text{in } B, \\ \boldsymbol{u}_m = \boldsymbol{0} & \text{on } \Gamma_0, \\ \mathscr{B}\boldsymbol{u}_m = \mathscr{T}\boldsymbol{u}_m + \boldsymbol{p}_m & \text{on } \Gamma_h, \end{cases} \tag{4.5}$$

where $\boldsymbol{q}_m = (q_{1,m}, q_{2,m}, q_{3,m})$ with

$$q_{i,m} = h^{-1}\sum_{j=1}^{3}\mathscr{D}_{ij}^{(1)}u_{j,m-1} + h^{-2}\sum_{j=1}^{3}\mathscr{D}_{ij}^{(2)}u_{j,m-2}, \quad i=1,2,3, \tag{4.6}$$

$\mathscr{D}_{ij}^{(k)} = \mathscr{D}_{ji}^{(k)}$ for $k=1,2$ with

$$\mathscr{D}_{11}^{(1)} = \mu\big[2g(\partial_{xx}+\partial_{yy}) + 2(h-z)(g_x\partial_{xz}+g_y\partial_{yz}) + (h-z)\Delta g\partial_z\big]$$
$$+ (\lambda+\mu)\big[2g\partial_{xx} + 2(h-z)g_x\partial_{xz} + (h-z)g_{xx}\partial_z\big] + 2\omega^2 g,$$
$$\mathscr{D}_{12}^{(1)} = (\lambda+\mu)\big[2g\partial_{xy} + (h-z)(g_y\partial_{xz}+g_x\partial_{yz}+g_{xy}\partial_z)\big],$$
$$\mathscr{D}_{13}^{(1)} = (\lambda+\mu)\big[g\partial_{xz} + (h-z)g_x\partial_{zz} - g_x\partial_z\big],$$
$$\mathscr{D}_{22}^{(1)} = \mu\big[2g(\partial_{xx}+\partial_{yy}) + 2(h-z)(g_x\partial_{xz}+g_y\partial_{yz}) + (h-z)\Delta g\partial_z\big]$$
$$+ (\lambda+\mu)\big[2g\partial_{yy} + 2(h-z)g_y\partial_{yz} + (h-z)g_{yy}\partial_z\big] + 2\omega^2 g,$$
$$\mathscr{D}_{23}^{(1)} = (\lambda+\mu)\big[g\partial_{yz} + (h-z)g_y\partial_{zz} - g_y\partial_z\big],$$
$$\mathscr{D}_{33}^{(1)} = \mu\big[2g(\partial_{xx}+\partial_{yy}) + 2(h-z)(g_x\partial_{xz}+g_y\partial_{yz}) + (h-z)\Delta g\partial_z\big] + 2\omega^2 g,$$

and

$$\begin{aligned}
\mathscr{D}_{11}^{(2)} =& -\mu\Big[g^2(\partial_{xx}+\partial_{yy})+(h-z)^2|\nabla g|^2\partial_{zz}\\
&+2(h-z)g(g_x\partial_{xz}+g_y\partial_{yz})-(h-z)(2|\nabla g|^2-g\Delta g)\partial_z\Big]\\
&-(\lambda+\mu)\Big[g^2\partial_{xx}+(h-z)^2g_x^2\partial_{zz}+2(h-z)gg_x\partial_{xz}-(h-z)(2g_x^2-gg_{xx})\partial_z\Big]-\omega^2g^2,\\
\mathscr{D}_{12}^{(2)} =& -(\lambda+\mu)\Big[g^2\partial_{xy}+(h-z)(gg_y\partial_{xz}+gg_x\partial_{yz}+g_xg_y\partial_{zz}+(gg_{xy}-2g_xg_y)\partial_z\Big],\\
\mathscr{D}_{13}^{(2)} =& 0,\\
\mathscr{D}_{22}^{(2)} =& -\mu\Big[g^2(\partial_{xx}+\partial_{yy})+(h-z)^2|\nabla g|^2\partial_{zz}\\
&+2(h-z)g(g_x\partial_{xz}+g_y\partial_{yz})-(h-z)(2|\nabla g|^2-g\Delta g)\partial_z\Big]\\
&-(\lambda+\mu)\Big[g^2\partial_{yy}+(h-z)^2g_y^2\partial_{zz}+2(h-z)gg_y\partial_{yz}-(h-z)(2g_y^2-gg_{yy})\partial_z\Big]-\omega^2g^2,\\
\mathscr{D}_{23}^{(2)} =& 0,\\
\mathscr{D}_{33}^{(2)} =& -\mu\Big[g^2(\partial_{xx}+\partial_{yy})+(h-z)^2|\nabla g|^2\partial_{zz}\\
&+2(h-z)g(g_x\partial_{xz}+g_y\partial_{yz})-(h-z)(2|\nabla g|^2-g\Delta g)\partial_z\Big]-\omega^2g^2,
\end{aligned}$$

and

$$\begin{cases}\boldsymbol{p}_0=\boldsymbol{p}=2\mathrm{i}(\lambda+2\mu)\kappa_1 e^{-\mathrm{i}\kappa_1 h}\boldsymbol{e}_3,\\ \boldsymbol{p}_1=h^{-1}\left[(\lambda+\mu)(\partial_x u_{1,0}+\partial_y u_{2,0})\boldsymbol{e}_3-(\mathscr{T}\boldsymbol{u}_0+\boldsymbol{p})\right]g,\\ \boldsymbol{p}_m=h^{-1}\left[(\lambda+\mu)(\partial_x u_{1,m-1}+\partial_y u_{2,m-1})\boldsymbol{e}_3-\mathscr{T}\boldsymbol{u}_{m-1}\right]g,\quad m\geq 2.\end{cases}\tag{4.7}$$

The variational problem of (4.5) is to find $\boldsymbol{u}\in H^1_{\Gamma_0,\mathrm{p}}(\Omega)^3$ such that

$$a_B(\boldsymbol{u}_m,\boldsymbol{v})=\langle\boldsymbol{p}_m,\boldsymbol{v}\rangle_{\Gamma_h}-(\boldsymbol{q}_m,\boldsymbol{v})_B,\quad\forall\,\boldsymbol{v}\in H^1_{\Gamma_0,\mathrm{p}}(\Omega)^3,\tag{4.8}$$

where the bilinear form $a_B(\cdot,\cdot)$ is given in (3.2) except changing the domain Ω by the domain B.

Following the same proof in Theorem 3.6, we can show the well-posedness of the variational problem (4.8).

Theorem 4.1. *The variational problem (4.8) has a unique solution $\boldsymbol{u}_m\in H^1_{\Gamma_0,\mathrm{p}}(\Omega)^3$ for sufficiently small h. The solution $\boldsymbol{u}_m$ satisfies the estimate*

$$\|\boldsymbol{u}_m\|_{1,B}\lesssim\|\boldsymbol{p}_m\|_{-1/2,\Gamma_h}+\|\boldsymbol{q}_m\|_{-1,B}.$$

Lemma 4.2. *It holds the estimate*

$$\|g\boldsymbol{v}\|_{1/2,\Gamma_h}\lesssim K\|\boldsymbol{v}\|_{1/2,\Gamma_h},\quad\forall\,\boldsymbol{v}\in H^{1/2}(\Gamma_h)^3.$$

Proof. Let $v\in H^{1/2}(\Gamma_h)$. Using an equivalent norm of $H^{1/2}(\Gamma_h)$, we have

$$\|gv\|^2_{1/2,\Gamma_h}=\|gv\|^2_{0,\Gamma_h}+\int_{\Gamma_h}\int_{\Gamma_h}\frac{|g(t)v(t)-g(s)v(s)|^2}{|t-s|^2}\,\mathrm{d}t\mathrm{d}s.$$

Applying the mean value theorem gives

$$\|gv\|^2_{0,\Gamma_h}\lesssim K^2\|v\|^2_{0,\Gamma_h}\leq K^2\|v\|^2_{1/2,\Gamma_h}$$

and

$$\begin{aligned}&\int_{\Gamma_h}\int_{\Gamma_h}\frac{|g(t)v(t)-g(s)v(s)|^2}{|t-s|^2}\,\mathrm{d}t\mathrm{d}s\\ &\lesssim\int_{\Gamma_h}\int_{\Gamma_h}\frac{|g(t)-g(s)|^2}{|t-s|^2}|u(t)|^2\,\mathrm{d}t\mathrm{d}s+\int_{\Gamma_h}\int_{\Gamma_h}\frac{|v(t)-v(s)|^2}{|t-s|^2}|g(s)|^2\,\mathrm{d}t\mathrm{d}s\end{aligned}$$

$$\lesssim K^2 \int_{\Gamma_h}\int_{\Gamma_h} |v(t)|^2 \,\mathrm{d}t\mathrm{d}s + K^2 \int_{\Gamma_h}\int_{\Gamma_h} \frac{|v(t)-v(s)|^2}{|t-s|^2}\,\mathrm{d}t\mathrm{d}s$$

$$\lesssim K^2\|v\|^2_{0,\Gamma_h} + K^2\|v\|^2_{1/2,\Gamma_h} \lesssim K^2\|v\|^2_{1/2,\Gamma_h}.$$

The lemma follows from the above estimates and the definition of norm in $H^{1/2}(\Gamma_h)^3$. □

Lemma 4.3. *For any $s\in\mathbb{R}$, it holds the estimate*

$$\|\partial_x u\|_{s,\Gamma_h} \le \|u\|_{s+1,\Gamma_h}, \quad \|\partial_y u\|_{s,\Gamma_h} \le \|u\|_{s+1,\Gamma_h}, \quad \forall\, u\in H^{s+1}(\Gamma_h).$$

Proof. It is easy to note that

$$\begin{aligned}\|\partial_x u\|^2_{s,\Gamma_h} &= \Lambda_1\Lambda_2 \sum_{n\in\mathbb{Z}^2} (1+|\alpha_n|^2)^s \left|(\partial_x u)^{(n)}(h)\right|^2 \\ &= \Lambda_1\Lambda_2 \sum_{n\in\mathbb{Z}^2} (1+|\alpha_n|^2)^s |\alpha_{1,n}|^2 \left|u^{(n)}(h)\right|^2 \\ &\le \Lambda_1\Lambda_2 \sum_{n\in\mathbb{Z}^2} (1+|\alpha_n|^2)^{s+1} \left|u^{(n)}(h)\right|^2 = \|u\|^2_{s+1,\Gamma_h}.\end{aligned}$$

The proof for $\partial_y u$ is similar and is omitted here. □

Lemma 4.4. *It holds the estimate*

$$\|\boldsymbol{p}_m\|_{-1/2,\Gamma_h} \lesssim Kh^{-1}\|\boldsymbol{u}_{m-1}\|_{1,B}, \quad m\ge 2.$$

Proof. It follows from Lemma 3.2, 3.4, 4.2, and 4.3 that

$$\begin{aligned}\left|\langle \boldsymbol{p}_m, \boldsymbol{v}\rangle_{\Gamma_h}\right| &\lesssim h^{-1}\left|\left\langle(\partial_x u_{1,m-1}+\partial_y u_{2,m-1})\boldsymbol{e}_3, g\boldsymbol{v}\right\rangle_{\Gamma_h}\right| + h^{-1}\left|\langle \mathscr{T}\boldsymbol{u}_{m-1}, g\boldsymbol{v}\rangle_{\Gamma_h}\right| \\ &\lesssim h^{-1}\left(\|\partial_x u_{1,m-1}+\partial_y u_{2,m-1}\|_{-1/2,\Gamma_h} + \|\mathscr{T}\boldsymbol{u}_{m-1}\|_{-1/2,\Gamma_h}\right)\|g\boldsymbol{v}\|_{1/2,\Gamma_h} \\ &\lesssim Kh^{-1}\|\boldsymbol{u}_{m-1}\|_{1/2,\Gamma_h}\|\boldsymbol{v}\|_{1/2,\Gamma_h} \le Kh^{-1}\|\boldsymbol{u}_{m-1}\|_{1,B}\|\boldsymbol{v}\|_{1/2,\Gamma_h}.\end{aligned}$$

Hence we have

$$\|\boldsymbol{p}_m\|_{-1/2,\Gamma_h} = \sup_{\boldsymbol{v}\in H^{1/2}(\Gamma_h)^3} \frac{\left|\langle \boldsymbol{p}_m, \boldsymbol{v}\rangle_{\Gamma_h}\right|}{\|\boldsymbol{v}\|_{1/2,\Gamma_h}} \lesssim Kh^{-1}\|\boldsymbol{u}_{m-1}\|_{1,B},$$

which completes the proof. □

Lemma 4.5. *It holds the estimate*

$$\|\boldsymbol{q}_m\|_{-1,B} \lesssim \left(Kh^{-1}\right)\|\boldsymbol{u}_{m-1}\|_{1,B} + \left(Kh^{-1}\right)^2\|\boldsymbol{u}_{m-2}\|_{1,B}, \quad m\ge 2.$$

Proof. Using the integration by parts and the periodic boundary condition, we have for any $u, v\in H^1_{\Gamma_0,\mathrm{p}}(B)$ that

$$\begin{aligned}\left(\mathscr{D}^{(1)}_{11}u, v\right)_B = \mu\big[&-2\,(\partial_x u, \partial_x(gv))_\Omega - 2\left(\partial_y u, \partial_y(gv)\right)_B - 2\left(\partial_z u, (h-z)(\partial_x(g_x v)+\partial_y(g_y v))\right)_B \\ &+ (\partial_z u, (h-z)\Delta g v)_B\,\big] \\ &+ (\lambda+\mu)\big[-2\,(\partial_x u, \partial_x(gv))_B - 2\,(\partial_z u, (h-z)\partial_x(g_x v))_B + (\partial_z u, (h-z)g_{xx}v)_B\,\big] \\ &+ 2\omega^2(u, gv)_B.\end{aligned}$$

It is easy to verify that

$$\left|\left(\mathscr{D}^{(1)}_{11}u, v\right)_B\right| \lesssim K\|u\|_{1,B}\|v\|_{1,B}.$$

Similarly we can show that

$$\left|\left(\mathscr{D}^{(1)}_{ij}u, v\right)_B\right| \lesssim K\|u\|_{1,B}\|v\|_{1,B}, \quad \left|\left(\mathscr{D}^{(2)}_{ij}u, v\right)_B\right| \lesssim K^2\|u\|_{1,B}\|v\|_{1,B}, \quad i,j=1,2,3.$$

It follows from (4.6) and the above estimates that

$$|(\boldsymbol{q}_m, \boldsymbol{v})_B| \lesssim \left[\left(Kh^{-1}\right) \|\boldsymbol{u}_{m-1}\|_{1,B} + \left(Kh^{-1}\right)^2 \|\boldsymbol{u}_{m-2}\|_{1,B} \right] \|\boldsymbol{v}\|_{1,B}, \quad \forall\, \boldsymbol{v} \in H^1_{\Gamma_0,\mathrm{p}}(B)^3.$$

The proof is completed by using the definition of the norm $\|\cdot\|_{-1,B}$. □

Theorem 4.6. *Let $\boldsymbol{u}_m$ be the solution of the variational problem (4.8). It satisfies*

$$\|\boldsymbol{u}_m\|_{1,B} \le \left(cKh^{-1}\right)^m, \quad m \ge 0.$$

Proof. It follows from Theorem 4.1, Lemma 4.4, and Lemma 4.5 that

$$\|\boldsymbol{u}_m\|_{1,B} \le \tilde{c} \left[\left(Kh^{-1}\right) \|\boldsymbol{u}_{m-1}\|_{1,B} + \left(Kh^{-1}\right)^2 \|\boldsymbol{u}_{m-2}\|_{1,B} \right],$$

where $\tilde{c}$ is a constant. Consider the recurrence relation

$$a_m \le \tilde{c} \left(t a_{m-1} + t^2 a_{m-2}\right), \quad m \ge 2,$$

where $\tilde{c}$, t, a_0, and a_1 are nonnegative numbers. It suffices to show that the above recurrence relation implies $a_m \le (ct)^m$ for some constant $c > 0$ depending only on $\tilde{c}$. By mathematical induction, it requires to find $c > 0$ such that

$$\tilde{c} \left[t(ct)^{m-1} + t^2 (ct)^{m-2} \right] \le (ct)^m, \quad m \ge 2,$$

which leads to the condition $\tilde{c}(1+c) \le c^2$. The proof is completed by taking any c such that

$$c \ge \frac{1}{2}(\tilde{c} + (\tilde{c}^2 + 4\tilde{c})^{1/2}). \qquad \square$$

Theorem 4.7. *The power series expansion (4.4) converges strongly if $cK\varepsilon h^{-1} < 1$.*

Proof. It follows from Theorem 4.6 that we have

$$\left\|\varepsilon^m \boldsymbol{u}_m\right\|_{1,B} \le \left(cK\varepsilon h^{-1}\right)^m, \quad m \ge 2.$$

The proof is completed by applying the dominated convergence theorem. □

5. Inverse problem

First we show a uniqueness result for the inverse problem if the deformation parameter ε is sufficiently small.

Lemma 5.1. *Let $G \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary ∂G. The boundary value problem*

$$\begin{cases} \mu \Delta \boldsymbol{u} + (\lambda + \mu)\nabla\nabla \cdot \boldsymbol{u} + \omega^2 \boldsymbol{u} = 0 & \text{in } G, \\ \boldsymbol{u} = 0 & \text{on } \partial G \end{cases}$$

has only the trivial solution if $\omega\, \mathrm{dep}(G) < \sqrt{\mu}$, where

$$\mathrm{dep}(G) = \sup\{|z_1 - z_2| : \boldsymbol{r}_1, \boldsymbol{r}_2 \in G\}$$

denotes the depth of G along the z-axis.

Proof. Let $\boldsymbol{u} \ne 0$ be a solution to the boundary value problem. Multiplying the equation by $\bar{\boldsymbol{u}}$ and applying integration by parts, we obtain

$$\omega^2 \|\boldsymbol{u}\|^2_{0,G} = \mu \|\nabla \boldsymbol{u}\|^2_{0,G} + (\lambda + \mu)\|\mathrm{div}\, \boldsymbol{u}\|^2_{0,G} \ge \mu \|\nabla \boldsymbol{u}\|^2_{0,G}.$$

Following the proof of Lemma 3.1, we can show

$$\|\boldsymbol{u}\|_{0,G} \le \mathrm{dep}(G) \|\nabla \boldsymbol{u}\|_{0,G}.$$

Combining the above inequalities yields $\omega \mathrm{dep}(G) \ge \sqrt{\mu}$, which is a contradiction. □

Theorem 5.2. *Let $f_j = \varepsilon g_j$, $g_j \in C^k(\mathbb{R}^2)$, $j = 1, 2$ be two periodic functions with the same period Λ and $\Omega_j = D \times (f_j, h)$. Let $\boldsymbol{u}_j$ be the unique solution of the variational problem (3.1) in Ω_j. If ε is sufficiently small and $\boldsymbol{u}_1 = \boldsymbol{u}_2$ on Γ_h, then $f_1 = f_2$.*

Proof. Let $S_j = \{\boldsymbol{r} \in \mathbb{R}^3 : z = f_j(\rho),\ \ \rho \in D\}$, $j = 1, 2$ and $\Omega = \Omega_1 \cap \Omega_2$. If $f_1 \neq f_2$, then $\Omega_1 \setminus \Omega$ or $\Omega_2 \setminus \Omega$ is nonempty. Without loss of generality, let $G = \Omega_1 \setminus \Omega$ be nonempty and let $\partial G = C_1 \cup C_2$ where $C_j \subset S_j$, $j = 1, 2$.

Let $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ and let φ_1, $\boldsymbol{\psi}_1$, φ_2, $\boldsymbol{\psi}_2$, φ, $\boldsymbol{\psi}$ be the scalar and vector potential functions for $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}$ respectively. Since $\boldsymbol{u}$ consists of bounded and outgoing waves and $\boldsymbol{u} = 0$ on Γ_h it follows from (2.11) that $\varphi = \boldsymbol{\psi} = 0$ on Γ_h. Using the Rayleigh expansions (2.9) yields $\partial_z \varphi = \partial_z \boldsymbol{\psi} = 0$ on Γ_h and $\varphi = \boldsymbol{\psi} = 0$ in the domain above Γ_h. It follows from unique continuation that $\varphi = \boldsymbol{\psi} = 0$ in $\bar{\Omega}$. Hence $\boldsymbol{u} = 0$ in $\bar{\Omega}$, and in particular $\boldsymbol{u} = 0$ on C_2. Since $\boldsymbol{u}_2 = 0$ on C_2 we have $\boldsymbol{u}_1 = \boldsymbol{u} - \boldsymbol{u}_2 = 0$ on C_2. Since it also holds $\boldsymbol{u}_1 = 0$ on C_1, we have $\boldsymbol{u}_1$ satisfy the boundary value problem in Lemma 5.1. If ε is sufficiently small, then we have $\operatorname{dep}(G) \le \varepsilon \max\{\|g_1\|_\infty, \|g_2\|_\infty\} < \omega \operatorname{dep}(G)\sqrt{\mu}$, which implies $\boldsymbol{u}_1 = 0$ in G by Lemma 5.1. It follows from the Helmholtz decomposition and unique continuation that $\boldsymbol{u}_1 = 0$ in Ω_1, which is a contradiction with the non-homogeneous transparent boundary condition (2.17) for $\boldsymbol{u}_1$ on Γ_h. $\square$

5.1. Analytical solutions

In this section we derive analytical solutions for $\boldsymbol{u}_m$ in the power series solution (4.4), particularly for the leading term $\boldsymbol{u}_0$ and the linear term $\boldsymbol{u}_1$, which serve as the basis for the reconstruction formula.

Combining the Dirichlet boundary condition (2.5) and the Helmholtz decomposition (2.6) yields the coupled boundary condition

$$\partial_z \begin{bmatrix} \varphi \\ \boldsymbol{\psi} \end{bmatrix} = \begin{bmatrix} 0 & \partial_y & -\partial_x & 0 \\ -\partial_y & 0 & 0 & \partial_x \\ \partial_x & 0 & 0 & \partial_y \\ 0 & -\partial_x & -\partial_y & 0 \end{bmatrix} \begin{bmatrix} \varphi \\ \boldsymbol{\psi} \end{bmatrix} \quad \text{on } \Gamma_f. \tag{5.1}$$

Denote by φ_{in}, $\boldsymbol{\psi}_{\text{in}}$ the potential functions for the incident field $\boldsymbol{u}_{\text{in}}$ defined in (2.3). It is easy to verify from the Helmholtz decomposition (2.6) that

$$\varphi_{\text{in}} = (\mathrm{i}\kappa_1)^{-1} e^{-\mathrm{i}\kappa_1 z}, \quad \boldsymbol{\psi}_{\text{in}} = \boldsymbol{0}.$$

Using the Rayleigh expansions (2.9), we obtain the transparent boundary conditions

$$\partial_z \varphi = \mathscr{T}_1 \varphi + p, \quad \partial_z \boldsymbol{\psi} = \mathscr{T}_2 \boldsymbol{\psi} \quad \text{on } \Gamma_h, \tag{5.2}$$

where $p = -2e^{-\mathrm{i}\kappa_1 h}$ and

$$\mathscr{T}_j v = \sum_{n \in \mathbb{Z}^2} \mathrm{i}\beta_{j,n} v^{(n)} e^{\mathrm{i}\alpha_n \cdot \rho}.$$

Under the change of variables (4.1), the Helmholtz equations (2.7) become

$$\mathscr{C}_1 \varphi = 0, \quad \mathscr{C}_2 \boldsymbol{\psi} = 0 \quad \text{in } B, \tag{5.3}$$

where

$$\begin{aligned} \mathscr{C}_j = {} & (h - f)^2 (\partial_{xx} + \partial_{yy}) + \left[h^2 + (h - z)^2 |\nabla f|^2\right] \partial_{zz} \\ & - 2(h - z)(h - f)(f_x \partial_{xz} + f_y \partial_{yz}) - (h - z)\left[(h - f)\Delta f + 2|\nabla f|^2\right] \partial_z + \kappa_j^2 (h - f)^2. \end{aligned}$$

The boundary condition (5.1) becomes

$$\begin{cases} h\partial_z \varphi = -\left[(h - f)\partial_x - h f_x \partial_z\right]\psi_2 + \left[(h - f)\partial_y - h f_y \partial_z\right]\psi_1, \\ h\partial_z \psi_1 = \left[(h - f)\partial_x - h f_x \partial_z\right]\psi_3 - \left[(h - f)\partial_y - h f_y \partial_z\right]\varphi, \\ h\partial_z \psi_2 = \left[(h - f)\partial_x - h f_x \partial_z\right]\varphi + \left[(h - f)\partial_y - h f_y \partial_z\right]\psi_3, \\ h\partial_z \psi_3 = -\left[(h - f)\partial_x - h f_x \partial_z\right]\psi_1 - \left[(h - f)\partial_y - h f_y \partial_z\right]\psi_2, \end{cases} \quad \text{on } \Gamma_0. \tag{5.4}$$

The boundary condition (5.2) becomes

$$h\partial_z \varphi = (h - f)\left(\mathscr{T}_1 \varphi + p\right), \quad h\partial_z \boldsymbol{\psi} = (h - f)\mathscr{T}_2 \boldsymbol{\psi} \quad \text{on } \Gamma_h. \tag{5.5}$$

It can be verified that the power series expansion (4.4) is equivalent to the following two power series expansions:

$$\varphi(\boldsymbol{x},\varepsilon)=\sum_{m=0}^{\infty}\varepsilon^m\varphi_m(\boldsymbol{x}),\quad \boldsymbol{\psi}(\boldsymbol{x},\varepsilon)=\sum_{m=0}^{\infty}\varepsilon^m\boldsymbol{\psi}_m(\boldsymbol{x}).$$

Substituting the above expansions and (2.1) into (5.3), we obtain the recurrence equations

$$\left(\Delta+\kappa_{\mathrm{p}}^2\right)\varphi_m=v_m,\quad \left(\Delta+\kappa_{\mathrm{s}}^2\right)\boldsymbol{\psi}_m=\boldsymbol{w}_m, \tag{5.6}$$

where

$$\begin{cases} v_m=h^{-1}\mathscr{D}_1^{(1)}\varphi_{m-1}+h^{-2}\mathscr{D}_1^{(2)}\varphi_{m-2},\\ \boldsymbol{w}_m=h^{-1}\mathscr{D}_2^{(1)}\boldsymbol{\psi}_{m-1}+h^{-2}\mathscr{D}_2^{(2)}\boldsymbol{\psi}_{m-2}, \end{cases} \tag{5.7}$$

and

$$\begin{aligned} \mathscr{D}_j^{(1)}&=2g(\partial_{xx}+\partial_{yy})+2(h-z)(g_x\partial_{xz}+g_y\partial_{yz})+(h-z)\Delta g\partial_z+2\kappa_j^2 g,\\ \mathscr{D}_j^{(2)}&=-g^2(\partial_{xx}+\partial_{yy})-(h-z)^2|\nabla g|^2\partial_{zz}+2(h-z)g(g_x\partial_{xz}+g_y\partial_{yz})\\ &\quad-(h-z)\left(g\Delta g-2|\nabla g|^2\right)\partial_z-\kappa_j^2 g^2. \end{aligned}$$

Similarly the boundary condition (5.4) yields

$$\partial_z\begin{bmatrix}\varphi_m\\ \boldsymbol{\psi}_m\end{bmatrix}=\begin{bmatrix}0&\partial_y&-\partial_x&0\\ -\partial_y&0&0&\partial_x\\ \partial_x&0&0&\partial_y\\ 0&-\partial_x&-\partial_y&0\end{bmatrix}\begin{bmatrix}\varphi_m\\ \boldsymbol{\psi}_m\end{bmatrix}+\begin{bmatrix}r_m\\ \boldsymbol{s}_m\end{bmatrix}\quad\text{on }\Gamma_0,$$

where

$$\begin{cases} r_m=\left(h^{-1}g\partial_x+g_x\partial_z\right)\psi_{2,m-1}-\left(h^{-1}g\partial_y+g_y\partial_z\right)\psi_{1,m-1},\\ s_{1,m}=-\left(h^{-1}g\partial_x+g_x\partial_z\right)\psi_{3,m-1}+\left(h^{-1}g\partial_y+g_y\partial_z\right)\varphi_{m-1},\\ s_{2,m}=-\left(h^{-1}g\partial_x+g_x\partial_z\right)\varphi_{m-1}-\left(h^{-1}g\partial_y+g_y\partial_z\right)\psi_{3,m-1},\\ s_{3,m}=\left(h^{-1}g\partial_x+g_x\partial_z\right)\psi_{1,m-1}+\left(h^{-1}g\partial_y+g_y\partial_z\right)\psi_{2,m-1}, \end{cases} \tag{5.8}$$

and the boundary condition (5.5) yields

$$\partial_z\varphi_m=\mathscr{T}_1\varphi_m+p_m,\quad \partial_z\psi_m=\mathscr{T}_2\psi_m+\boldsymbol{q}_m, \tag{5.9}$$

where

$$\begin{cases} p_0=\rho,\quad p_1=-h^{-1}g\left(\mathscr{T}_1\varphi_0+p\right),\\ p_m=-h^{-1}g\mathscr{T}_1\varphi_{m-1},\quad m\geq 2, \end{cases}\qquad \boldsymbol{q}_m=-h^{-1}g\mathscr{T}_2\boldsymbol{\psi}_{m-1},\quad m\geq 0. \tag{5.10}$$

Applying the Fourier series expansions to the boundary value problem (5.6)–(5.9) yields the two-point boundary value problem:

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\begin{bmatrix}\varphi_m^{(n)}\\ \boldsymbol{\psi}_m^{(n)}\end{bmatrix}+\mathrm{diag}\left(\beta_{1,n}^2,\beta_{2,n}^2,\beta_{2,n}^2,\beta_{2,n}^2\right)\begin{bmatrix}\varphi_m^{(n)}\\ \boldsymbol{\psi}_m^{(n)}\end{bmatrix}=\begin{bmatrix}v_m^{(n)}\\ \boldsymbol{w}_m^{(n)}\end{bmatrix},\quad 0<z<h,$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{bmatrix}\varphi_m^{(n)}\\ \boldsymbol{\psi}_m^{(n)}\end{bmatrix}=\mathrm{i}\begin{bmatrix}0&\alpha_{2,n}&-\alpha_{1,n}&0\\ -\alpha_{2,n}&0&0&\alpha_{1,n}\\ \alpha_{1,n}&0&0&\alpha_{2,n}\\ 0&-\alpha_{1,n}&-\alpha_{2,n}&0\end{bmatrix}\begin{bmatrix}\varphi_m^{(n)}\\ \boldsymbol{\psi}_m^{(n)}\end{bmatrix}+\begin{bmatrix}r_m^{(n)}\\ \boldsymbol{s}_m^{(n)}\end{bmatrix},\quad z=0,$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{bmatrix}\varphi_m^{(n)}\\ \boldsymbol{\psi}_m^{(n)}\end{bmatrix}-\mathrm{diag}\left(\beta_{1,n},\beta_{2,n},\beta_{2,n},\beta_{2,n}\right)\begin{bmatrix}\varphi_m^{(n)}\\ \boldsymbol{\psi}_m^{(n)}\end{bmatrix}=\begin{bmatrix}p_m^{(n)}\\ \boldsymbol{q}_m^{(n)}\end{bmatrix},\quad z=h.$$

Solving the above coupled system, we obtain the analytic solution

$$\begin{bmatrix} \varphi_m^{(n)}(z) \\ \boldsymbol{\psi}_m^{(n)}(z) \end{bmatrix} = \int_0^h \operatorname{diag}(K_1, K_2, K_2, K_2)(z, z') \begin{bmatrix} v_m^{(n)}(z') \\ \boldsymbol{w}_m^{(n)}(z') \end{bmatrix} \mathrm{d}z'$$
$$+ \operatorname{diag}(K_1, K_2, K_2, K_2)(z, 0) \left(\mathrm{i} \begin{bmatrix} 0 & \alpha_{2,n} & -\alpha_{1,n} & 0 \\ -\alpha_{2,n} & 0 & 0 & \alpha_{1,n} \\ \alpha_{1,n} & 0 & 0 & \alpha_{2,n} \\ 0 & -\alpha_{1,n} & -\alpha_{2,n} & 0 \end{bmatrix} \begin{bmatrix} \varphi_m^{(n)}(0) \\ \boldsymbol{\psi}_m^{(n)}(0) \end{bmatrix} + \begin{bmatrix} r_m^{(n)} \\ \boldsymbol{s}_m^{(n)} \end{bmatrix} \right)$$
$$- \operatorname{diag}(K_1, K_2, K_2, K_2)(z, h) \begin{bmatrix} p_m^{(n)} \\ \boldsymbol{q}_m^{(n)} \end{bmatrix}, \tag{5.11}$$

where the integral kernel

$$K_j(z, z') = \frac{1}{2\mathrm{i}\beta_{j,n}} \begin{cases} e^{\mathrm{i}\beta_{j,n}z}\left(e^{\mathrm{i}\beta_{j,n}z'} + e^{-\mathrm{i}\beta_{j,n}z'}\right), & z' < z, \\ e^{\mathrm{i}\beta_{j,n}z'}\left(e^{\mathrm{i}\beta_{j,n}z} + e^{-\mathrm{i}\beta_{j,n}z}\right), & z' > z, \end{cases}.$$

Evaluating (5.11) at $z = 0$ yields the linear system of algebraic equations for $\varphi_m^{(n)}(0)$ and $\boldsymbol{\psi}_m^{(n)}$:

$$\begin{bmatrix} 1 & -\alpha_{2,n}/\beta_{1,n} & \alpha_{1,n}/\beta_{1,n} & 0 \\ \alpha_{2,n}/\beta_{2,n} & 1 & 0 & -\alpha_{1,n}/\beta_{2,n} \\ -\alpha_{1,n}/\beta_{2,n} & 0 & 1 & -\alpha_{2,n}/\beta_{2,n} \\ 0 & \alpha_{1,n}/\beta_{2,n} & \alpha_{2,n}/\beta_{2,n} & 1 \end{bmatrix} \begin{bmatrix} \varphi_m^{(n)}(0) \\ \boldsymbol{\psi}_m^{(n)}(0) \end{bmatrix} = \begin{bmatrix} a_m^{(n)} \\ \boldsymbol{b}_m^{(n)}, \end{bmatrix} \tag{5.12}$$

where

$$\begin{cases} a_m^{(n)} = \left(\mathrm{i}\beta_{1,n}\right)^{-1} \left[\int_0^h e^{\mathrm{i}\beta_{1,n}z'} v_m^{(n)}(z')\,\mathrm{d}z' + r_m^{(n)} - e^{\mathrm{i}\beta_{1,n}h} p_m^{(n)}\right], \\ \boldsymbol{b}_m^{(n)} = \left(\mathrm{i}\beta_{2,n}\right)^{-1} \left[\int_0^h e^{\mathrm{i}\beta_{2,n}z'} \boldsymbol{w}_m^{(n)}(z')\,\mathrm{d}z' + \boldsymbol{s}_m^{(n)} - e^{\mathrm{i}\beta_{2,n}h} \boldsymbol{q}_m^{(n)}\right]. \end{cases} \tag{5.13}$$

For the leading term, i.e., $m = 0$, it follows from (5.7), (5.8), and (5.10) that

$$v_0 = \boldsymbol{w}_0 = r_0 = \boldsymbol{s}_0 = \boldsymbol{q}_0 = 0, \quad p_0 = p = -2e^{-\mathrm{i}\kappa_1 h},$$

and their Fourier coefficients

$$v_0^{(n)} = \boldsymbol{w}_0^{(n)} = r_0^{(n)} = \boldsymbol{s}_0^{(n)} = \boldsymbol{q}_0^{(n)} = 0, \quad p_0^{(n)} = -2e^{-\mathrm{i}\kappa_1 h}\delta_{0n}, \tag{5.14}$$

where δ is the Kronecker delta. Substituting the above results into (5.13) yields

$$a_0^{(n)} = 2(\mathrm{i}\kappa_1)^{-1}\delta_{0n}, \quad \boldsymbol{b}_0^{(n)} = \boldsymbol{0}. \tag{5.15}$$

Substituting (5.15) into (5.12) and solving for $\varphi_0^{(n)}(0)$, $\boldsymbol{\psi}_0^{(n)}(0)$, we obtain

$$\varphi_0^{(n)}(0) = 2(\mathrm{i}\kappa_1)^{-1}\delta_{0n}, \quad \boldsymbol{\psi}_0^{(n)}(0) = \boldsymbol{0}. \tag{5.16}$$

Substituting (5.14) and (5.16) into (5.11), we obtain

$$\varphi_0^{(n)}(z) = 2(\mathrm{i}\kappa_1)^{-1}\cos(\kappa_1 z)\delta_{0n}, \quad \boldsymbol{\psi}_0^{(n)} = \boldsymbol{0}.$$

Hence the leading terms are given by

$$\varphi_0 = 2(\mathrm{i}\kappa_1)^{-1}\cos(\kappa_1 z), \quad \boldsymbol{\psi}_0 = \boldsymbol{0}.$$

Using the Helmholtz decomposition (2.6), we obtain the leading term for the total field

$$\boldsymbol{u}_0 = 2\mathrm{i}\sin(\kappa_1 z)\boldsymbol{e}_3.$$

For the linear terms, i.e., $m = 1$, it follows from (5.7), (5.8) and (5.10) that

$$v_1 = h^{-1}\left[2\mathrm{i}(h-z)\sin(\kappa_1 z)(\Delta g) - 4\mathrm{i}\kappa_1\cos(\kappa_1 z)g\right],$$
$$p_1 = -h^{-1}2\mathrm{i}\sin(\kappa_1 h)g, \quad \boldsymbol{w}_1 = r_1 = \boldsymbol{s}_1 = \boldsymbol{q}_1 = 0,$$

and their Fourier coefficients

$$\begin{cases} v_1^{(n)} = -2\mathrm{i}h^{-1}\left[|\alpha_n|^2(h-z)\sin(\kappa_1 z) + 2\kappa_1\cos(\kappa_1 z)\right]g^{(\boldsymbol{n})}, \\ p_1^{(n)} = -2\mathrm{i}h^{-1}\sin(\kappa_1 h)g^{(n)}, \quad \boldsymbol{w}_1^{(n)} = r_1^{(n)} = \boldsymbol{s}_1^{(n)} = \boldsymbol{q}_1^{(n)} = 0. \end{cases} \tag{5.17}$$

Substituting (5.17) into (5.13) yields

$$a_1^{(n)} = \left(\mathrm{i}\beta_{1,n}\right)^{-1}\left[\int_0^h e^{\mathrm{i}\beta_{1,n}z'}v_1^{(n)}(z')\,\mathrm{d}z' - e^{\mathrm{i}\beta_{1,n}h}p_1^{(n)}\right]g^{(n)}, \quad \boldsymbol{b}_1^{(n)} = \boldsymbol{0}.$$

After a tedious but straightforward simplification, we find

$$a_1^{(n)} = \frac{-2\kappa_1}{\beta_{1,n}}g^{(n)}, \quad \boldsymbol{b}_1^{(n)} = \boldsymbol{0}. \tag{5.18}$$

Substituting (5.18) into (5.12) yields

$$\begin{bmatrix} \varphi_1^{(n)}(0) \\ \boldsymbol{\psi}_1^{(n)}(0) \end{bmatrix} = -2\kappa_1\gamma_n^{-1}(\beta_{2,n}, -\alpha_{2,n}, \alpha_{1,n}, 0)^\top g^{(n)}. \tag{5.19}$$

Substituting (5.17) and (5.19) into (5.11) yields

$$\begin{aligned} \begin{bmatrix} \varphi_1^{(n)}(z) \\ \boldsymbol{\psi}_1^{(n)}(z) \end{bmatrix} &= \left[\int_0^h K_1(z,z')v_1^{(n)}(z')\,\mathrm{d}z'\right]\boldsymbol{e}_1 - \frac{e^{\mathrm{i}\beta_{1,n}h}}{2\mathrm{i}\beta_{1,n}}\left(e^{\mathrm{i}\beta_{1,n}z} + e^{-\mathrm{i}\beta_{1,n}z}\right)p_1^{(n)}\boldsymbol{e}_1 \\ &\quad + \frac{2\kappa_1}{\gamma_n}\left(\frac{|\alpha_n|^2}{\beta_{1,n}}e^{\mathrm{i}\beta_{1,n}z}, \alpha_{2,n}e^{\mathrm{i}\beta_{2,n}z}, -\alpha_{1,n}e^{\mathrm{i}\beta_{2,n}z}, 0\right)^\top g^{(n)} \end{aligned} \tag{5.20}$$

Evaluating (5.20) at $z = h$ and simplifying, we obtain

$$\begin{aligned} \begin{bmatrix} \varphi_1^{(n)}(h) \\ \boldsymbol{\psi}_1^{(n)}(h) \end{bmatrix} &= \frac{e^{\mathrm{i}\beta_{1,n}h}}{2\mathrm{i}\beta_{1,n}}\left[\int_0^h \left(e^{\mathrm{i}\beta_{1,n}z} + e^{-\mathrm{i}\beta_{1,n}z}\right)v_1^{(n)}(z)\,\mathrm{d}z - \left(e^{\mathrm{i}\beta_{1,n}h} + e^{-\mathrm{i}\beta_{1,n}h}\right)p_1^{(n)}\right]\boldsymbol{e}_1 \\ &\quad + \frac{2\kappa_1}{\gamma_n}\left(\frac{|\alpha_n|^2}{\beta_{1,n}}e^{\mathrm{i}\beta_{1,n}h}, \alpha_{2,n}e^{\mathrm{i}\beta_{2,n}h}, -\alpha_{1,n}e^{\mathrm{i}\beta_{2,n}h}, 0\right)^\top g^{(n)}. \end{aligned}$$

Simplifying the integral yields

$$\begin{bmatrix} \varphi_1^{(n)}(h) \\ \boldsymbol{\psi}_1^{(n)}(h) \end{bmatrix} = \frac{2\kappa_1}{\gamma_n}(-\beta_{2,n}e^{\mathrm{i}\beta_{1,n}h}, \alpha_{2,n}e^{\mathrm{i}\beta_{2,n}h}, -\alpha_{1,n}e^{\mathrm{i}\beta_{2,n}h}, 0)^\top g^{(n)}.$$

Using (2.11), we obtain the linear term of the total field at Γ_h:

$$\boldsymbol{u}_1^{(n)}(h) = \frac{2\mathrm{i}\kappa_1}{\gamma_n}\begin{bmatrix} \alpha_{1,n}\beta_{2,n}\left(e^{\mathrm{i}\beta_{2,n}h} - e^{\mathrm{i}\beta_{1,n}h}\right) \\ \alpha_{2,n}\beta_{2,n}\left(e^{\mathrm{i}\beta_{2,n}h} - e^{\mathrm{i}\beta_{1,n}h}\right) \\ -|\alpha_n|^2e^{\mathrm{i}\beta_{2,n}h} - \beta_{1,n}\beta_{2,n}e^{\mathrm{i}\beta_{1,n}h} \end{bmatrix}g^{(n)}. \tag{5.21}$$

It is easy to note that each of the three equations in (5.21) may be used to solve for $g^{(n)}$ but they are not invertible for all $n \in \mathbb{Z}^2$. A simple observation gives

$$(\alpha_{1,n}, \alpha_{2,n}, \beta_{2,n}) \cdot \boldsymbol{u}_1^{(n)}(h) = -2\mathrm{i}\kappa_1\beta_{2,n}e^{\mathrm{i}\beta_{1,n}h}g^{(n)}, \tag{5.22}$$

which is invertible for all $n \in \mathbb{Z}^2$ and is the key result for our reconstruction formula.

5.2. Reconstruction formula and error estimate

Recalling the power series (4.4), we have

$$\boldsymbol{u} = \boldsymbol{u}_0 + \varepsilon \boldsymbol{u}_1 + \boldsymbol{r}, \tag{5.23}$$

where the remainder

$$\boldsymbol{r} = \sum_{m=2}^{\infty} \varepsilon^m \boldsymbol{u}_m.$$

Combining (5.22)–(5.23) and recalling (2.1), we obtain

$$f^{(n)} = \frac{\mathrm{i}e^{-\mathrm{i}\beta_{1,n}h}}{2\kappa_1\beta_{2,n}}(\alpha_{1,n}, \alpha_{2,n}, \beta_{2,n}) \cdot (\boldsymbol{u} - \boldsymbol{u}_0 - \boldsymbol{r}). \tag{5.24}$$

Dropping the remainder $\boldsymbol{r}$ linearizes the inverse problem and yields the reconstruction formula for the Fourier coefficient of f:

$$f_\varepsilon^{(n)} = \frac{\mathrm{i}e^{-\mathrm{i}\beta_{1,n}h}}{2\kappa_1\beta_{2,n}}(\alpha_{1,n}, \alpha_{2,n}, \beta_{2,n}) \cdot (\boldsymbol{u}^{(n)}(h) - \boldsymbol{u}_0^{(n)}(h)).$$

If the measured data contains noise with level $\delta > 0$, then the reconstruction formula for the Fourier coefficients of f is given by

$$f_{\varepsilon,\delta}^{(n)} = \frac{\mathrm{i}e^{-\mathrm{i}\beta_{1,n}h}}{2\kappa_1\beta_{2,n}}(\alpha_{1,n}, \alpha_{2,n}, \beta_{2,n}) \cdot (\boldsymbol{u}_\delta^{(n)}(h) - \boldsymbol{u}_0^{(n)}(h)), \tag{5.25}$$

where $\boldsymbol{u}_\delta$ denotes the noisy measured data such that

$$\|\boldsymbol{u}_\delta - \boldsymbol{u}\|_{0,\Gamma_h} \le \delta. \tag{5.26}$$

Finally the reconstruction formula for f is given by

$$f_{\varepsilon,\delta}(\rho) = \sum_{|\alpha_n|_\infty \le \kappa_c} f_{\varepsilon,\delta}^{(n)} e^{\mathrm{i}\alpha_n\cdot\rho}, \tag{5.27}$$

where $\kappa_c > 0$ is the spectral cut-off frequency and plays the role of the regularization parameter for the linearized inverse problem.

Lemma 5.3. *If $g \in C^k(R)$ is a periodic function with periodicity Λ and $K > 0$ is defined in (2.2), then*

$$\sum_{|\alpha_n|_\infty > \kappa_c} \left| g^{(n)} \right|^2 \lesssim K^2 \kappa_c^{-2k+2}.$$

Proof. It follows from integration by parts that we have for any $k_1 \in \mathbb{N}_0$, $k_2 \in \mathbb{N}_0$, $k_1 + k_2 \le k$ and $\boldsymbol{n} \neq \boldsymbol{0}$

$$\begin{aligned} g^{(n)} &= (\Lambda_1\Lambda_2)^{-1} \int_R g(\rho) e^{-\mathrm{i}\alpha_n\cdot\rho}\,\mathrm{d}\rho \\ &= (\Lambda_1\Lambda_2)^{-1} \left(\mathrm{i}\alpha_{1,n}\right)^{-k_1} \left(\mathrm{i}\alpha_{2,n}\right)^{-k_2} \int_0^{\Lambda_2}\int_0^{\Lambda_1} \partial_x^{k_1}\partial_y^{k_2} g(x,y) e^{-\mathrm{i}(\alpha_{1,n}x + \alpha_{2,n}y)}\,\mathrm{d}x\mathrm{d}y. \end{aligned}$$

Hence

$$\begin{aligned} \left| g^{(n)} \right| &\le K \min\{|\alpha_{1,n}|^{-k_1} |\alpha_{2,n}|^{-k_2} : k_1 \in \mathbb{N}_0, k_2 \in \mathbb{N}_0, k_1 + k_2 \le k\} \\ &= K \begin{cases} |\alpha_{1,n}|^{-k} & \text{if } |\alpha_{1,n}| \ge |\alpha_{2,n}|, \\ |\alpha_{2,n}|^{-k} & \text{if } |\alpha_{2,n}| \ge |\alpha_{1,n}|. \end{cases} \end{aligned}$$

Using the integral test, we get

$$\sum_{|\alpha_n|_\infty > \kappa_c} \left|g^{(n)}\right|^2 \le \sum_{|\alpha_n| > \kappa_c} \left|g^{(n)}\right|^2 \lesssim K^2 \left(\int_{D_1} |x|^{-2k}\, \mathrm{d}x\mathrm{d}y + \int_{D_2} |y|^{-2k}\, \mathrm{d}x\mathrm{d}y \right),$$

where

$$D_1 = \{(x,y) : (x^2+y^2)^{1/2} \ge \kappa_c,\ |x| \ge |y|\}, \quad D_2 = \{(x,y) : (x^2+y^2)^{1/2} \ge \kappa_c,\ |y| \ge |x|\}.$$

A simple calculation gives

$$\int_{D_1} |x|^{-2k}\, \mathrm{d}x\mathrm{d}y = 2 \int_{-\pi/4}^{\pi/4} \int_{\kappa_c}^{\infty} r^{-2k+1} (\cos\theta)^{-2k}\, \mathrm{d}r\mathrm{d}\theta \lesssim \int_{\kappa_c}^{\infty} r^{-2k+1}\, \mathrm{d}r \lesssim \kappa_c^{-2k+2}.$$

Similarly we have

$$\int_{D_2} |y|^{-2k}\, \mathrm{d}x\mathrm{d}y \lesssim \kappa_c^{-2k+2},$$

which completes the proof. □

Lemma 5.4. *It holds the estimate*

$$\left|\frac{\alpha_n}{\beta_{2,n}}\right| \lesssim 1, \quad \forall\, n \in \mathbb{Z}^2.$$

Proof. It follows from (2.10) that

$$\left|\frac{\alpha_n}{\beta_{2,n}}\right|^2 = \frac{|\alpha_n|^2}{\left|\kappa_2^2 - |\alpha_n|^2\right|},$$

where $\kappa_s^2 \neq |\alpha_n|$ for all $n \in \mathbb{Z}^2$ by assumption. Consider the function

$$\xi(t) = \frac{t}{\left|\kappa_2^2 - t\right|}, \quad t \ge 0,\ t \neq \kappa_2^2.$$

It is easy to see $\xi(t)$ is increasing for $0 < t < \kappa_2^2$ and decreasing for $t > \kappa_2^2$. Hence there exists an $n^* \in \mathbb{Z}^2$ such that

$$\left|\frac{\alpha_n}{\beta_{2,n}}\right|^2 \le \frac{|\alpha_{n^*}|^2}{\left|\kappa_2^2 - |\alpha_{n^*}|^2\right|},$$

which completes the proof. □

Theorem 5.5. *Let f be the exact scattering surface function and $f_{\varepsilon,\delta}$ be the reconstructed scattering function using (5.27). It holds the error estimate*

$$\|f_{\varepsilon,\delta} - f\|_{0,\Gamma_h} \lesssim \left|e^{h(\kappa_c^2-\kappa_1^2)^{1/2}}\right| \left[\delta + \left(K\varepsilon h^{-1}\right)^2\right] + K\varepsilon\kappa_c^{-k+1}. \tag{5.28}$$

Proof. It follows from (5.24), (5.25) and (5.27) that

$$\|f_{\varepsilon,\delta} - f\|^2_{0,\Gamma_h} = \Lambda_1\Lambda_2(E_1 + E_2 + E_3), \tag{5.29}$$

where

$$E_1 = \sum_{|\alpha_n|_\infty \le \kappa_c} \left| \frac{\mathrm{i}e^{-\mathrm{i}\beta_{1,n}h}}{2\kappa_1\beta_{2,n}} (\alpha_{1,n}, \alpha_{2,n}, \beta_{2,n}) \cdot (\boldsymbol{u}_\delta^{(n)}(h) - \boldsymbol{u}^{(n)}(h)) \right|^2,$$

$$E_2 = \sum_{|\alpha_n|_\infty \le \kappa_c} \left| \frac{\mathrm{i}e^{-\mathrm{i}\beta_{1,n}h}}{2\kappa_1\beta_{2,n}} (\alpha_{1,n}, \alpha_{2,n}, \beta_{2,n}) \cdot \cdot \boldsymbol{r}^{(n)}(h) \right|^2,$$

$$E_3 = \sum_{|\alpha_n|_\infty > \kappa_c} \left| f^{(n)} \right|^2.$$

It is easy to verify

$$\left|e^{-\mathrm{i}\beta_{1,n}h}\right| \le \left|e^{h\left(\kappa_c^2-\kappa_1^2\right)^{1/2}}\right| \quad \text{for } |\alpha_n|_\infty \le \kappa_c. \tag{5.30}$$

Combining (5.26), Lemma 5.4, and (5.30) yields

$$E_1^{1/2} \lesssim \left|e^{h\left(\kappa_c^2-\kappa_1^2\right)^{1/2}}\right| \delta. \tag{5.31}$$

It follows from Lemma 3.2 and Theorem 4.6 that

$$\begin{aligned}\|\boldsymbol{r}\|_{0,\Gamma_h} &= \left\|\sum_{m=2}^{\infty}\varepsilon^m \boldsymbol{u}_m\right\|_{0,\Gamma_h} \le \sum_{m=2}^{\infty}\varepsilon^m\|\boldsymbol{u}_m\|_{0,\Gamma_h} \\ &\le \sum_{m=2}^{\infty}\varepsilon^m\|\boldsymbol{u}_m\|_{1/2,\Gamma_h} \le \sum_{m=2}^{\infty}\varepsilon^m\|\boldsymbol{u}_m\|_{1,B} \lesssim \left(K\varepsilon h^{-1}\right)^2.\end{aligned} \tag{5.32}$$

Combining Lemma 5.4, (5.30) and (5.32) yields

$$E_2^{1/2} \lesssim \left|e^{h\left(\kappa_c^2-\kappa_1^2\right)^{1/2}}\right| \left(K\varepsilon h^{-1}\right)^2. \tag{5.33}$$

It follows from Lemma 5.3 that

$$E_3^{1/2} \le K\varepsilon\kappa_c^{-k+1}. \tag{5.34}$$

Combining (5.29), (5.31), (5.33), and (5.34) completes the proof. □

The error estimate (5.28) has an explicit dependence on h, ε, δ, κ_c, K and an implicit dependence on Λ_1, Λ_2, ω, λ, μ, k. Among these parameters, ε, δ, K, Λ_1, Λ_2, λ, μ, k are intrinsic and not controllable by the user, but h, κ_c are user specified parameters that can be tuned in practice. The error estimate consists of three parts: the measurement noise, the linearization error, and the regularization error due to the spectral cut-off. It provides an insight on the trade-off among resolution, accuracy, and stability of the solution for the inverse problem.

5.3. Numerical experiments

In this section, we show numerical examples to test the proposed method and investigate the effect of parameters on the reconstructed surface. For a given surface function f, the displacement field of the total wave $\boldsymbol{u}$ is obtained by solving the direct scattering problem using the finite element method with the perfectly matched layer (PML) technique, which is implemented by FreeFem++ [27]. The finite element solution is interpolated at a 513×513 uniform grid on Γ_h and a random noise of relative magnitude δ is added to the data:

$$\boldsymbol{u}_\delta(x_i, y_j, h) = \boldsymbol{u}(x_i, y_j, h)(1 + \delta r_{ij}),$$

where r_{ij} are independent random numbers drawn from the uniform distribution in $[-1, 1]$. The Fourier coefficients $\boldsymbol{u}_\delta^{(n)}(h)$ are computed via the fast Fourier transform (FFT). In all of the following experiments, we take $\omega = \pi$, $\lambda = 2.0$, $\mu = 1.0$, $\Lambda_1 = \Lambda_2 = 1.0$. Hence the pressure wavelength is 4 and the shear wavelength is 2. The resonance does not occur for this set of parameters since $\beta_{j,n} \ne 0$ for $j = 1, 2$ and for all $n \in \mathbb{Z}^2$.

In the first example, the profile function is given by $f(x, y) = \varepsilon g(x, y)$, where

$$\begin{aligned}g(x, y) &= 2\sigma\,(5(x-0.5), 5(y-0.5))\sin(\pi x)\sin(\pi y), \\ \sigma(x, y) &= 0.3(1-x)^2 e^{-x^2-(y+1)^2} - (0.2x - x^3 - y^5)e^{-x^2-y^2} - 0.03e^{-(x+1)^2-y^2}.\end{aligned}$$

The graph of g is shown in Fig. 1.

First we investigate the effect of the deformation parameter ε. Fig. 2 shows the reconstructed surface function with $h = 0.1$, $\delta = 0$ and $\varepsilon = 0.08, 0.04$ respectively. The cut-off frequency is taken as $\kappa_c = 2, 3$ respectively. Observe that the subwavelength features of the surface are well reconstructed. A better reconstruction is obtained for a smaller value of ε, which is consistent with the error estimate. Fig. 3 shows the reconstructed surface function with $h = 0.1$, $\varepsilon = 0.01$ and $\delta = 0, 10\%$ respectively. The cut-off frequency is taken as $\kappa_c = 3$. It is clear to note that our method is very robust with respect to the measurement noise. Fig. 4 shows the reconstructed profile function with $\varepsilon = 0.01$, $\delta = 10\%$ and $h = 0.1, 0.02$ respectively. The cut-off frequency is taken as $\kappa_c = 3, 5$ respectively. Clearly, a smaller measurement distance yields a better resolution.

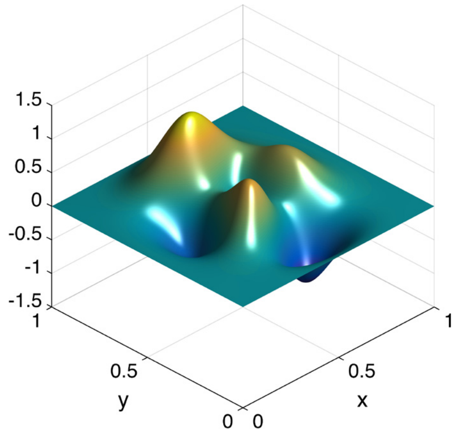

Fig. 1. (Color online.) Graph of the exact scattering surface.

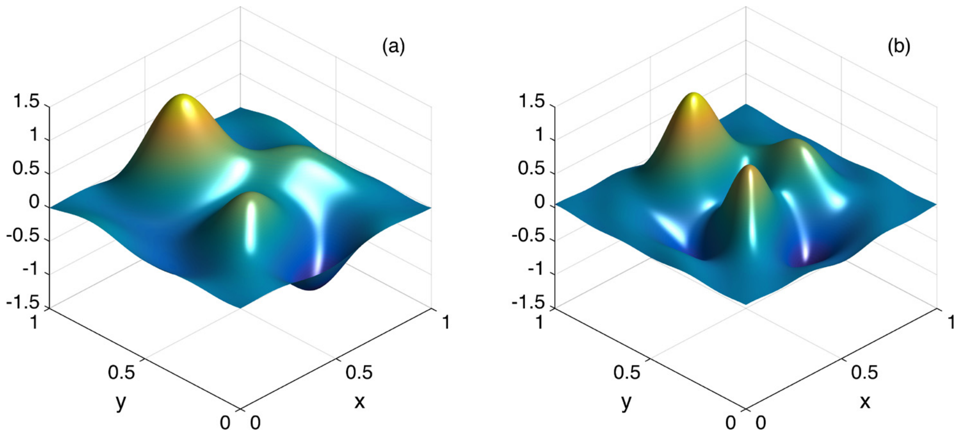

Fig. 2. (Color online.) The reconstructed surface function. (a) $\varepsilon = 0.08$; (b) $\varepsilon = 0.04$.

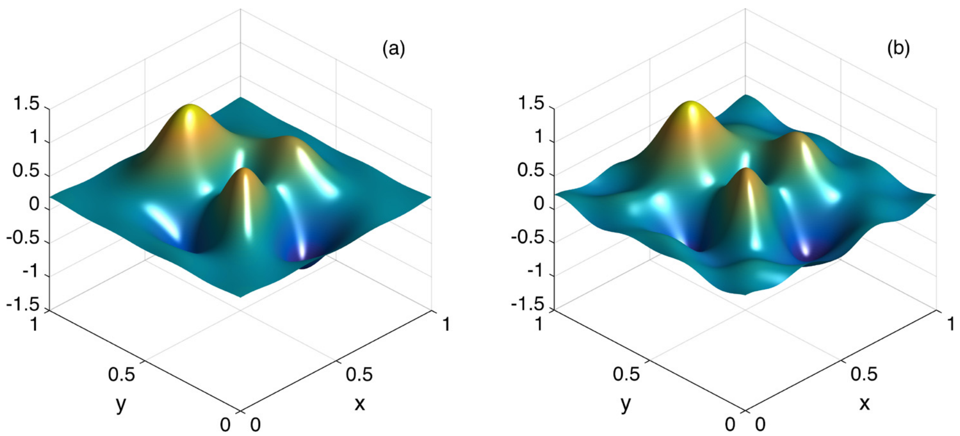

Fig. 3. (Color online.) The reconstructed surface function. (a) $\delta = 0$; (b) $\delta = 10\%$.

In the second example, we consider a non-smooth function $f(x, y) = \varepsilon g(x, y)$, where

$$g(x, y) = \chi_{[0.6,0.8]\times[0.2,0.4]} + \chi_{[0.2,0.4]\times[0.6,0.8]},$$

and the graph of g is shown in Fig. 5. This example shows that our method works well for non-smooth surface functions, although the mathematical justification requires smooth functions. Fig. 6 shows the reconstructed surface function with $\varepsilon = 0.01$, $h = 0.02$, $\delta = 10\%$ and $\kappa_c = 6, 7$ respectively. It displays a subwavelength resolution and also the well-known Gibbs phenomenon which can be greatly reduced by applying a suitable low-pass filter [26] in practice.

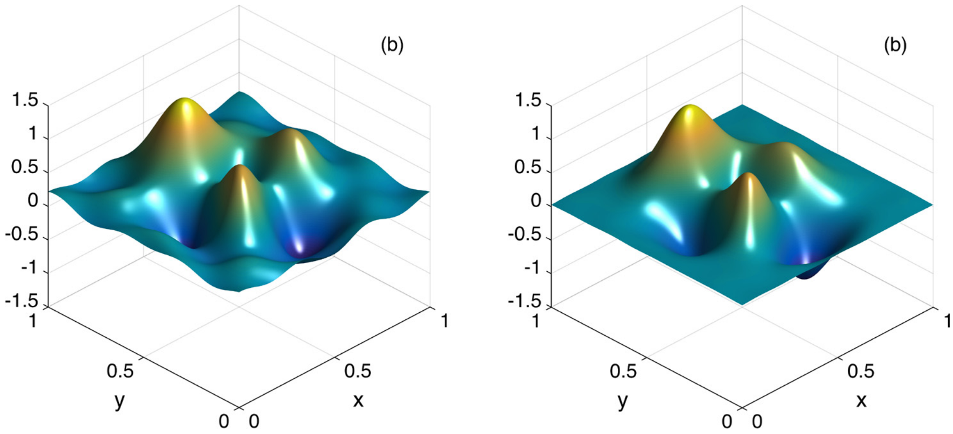

Fig. 4. (Color online.) The reconstructed surface function. (a) $h = 0.1$; (b) $h = 0.02$.

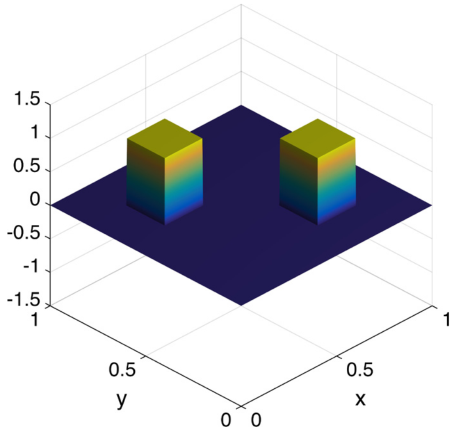

Fig. 5. (Color online.) Graph of the exact scattering surface.

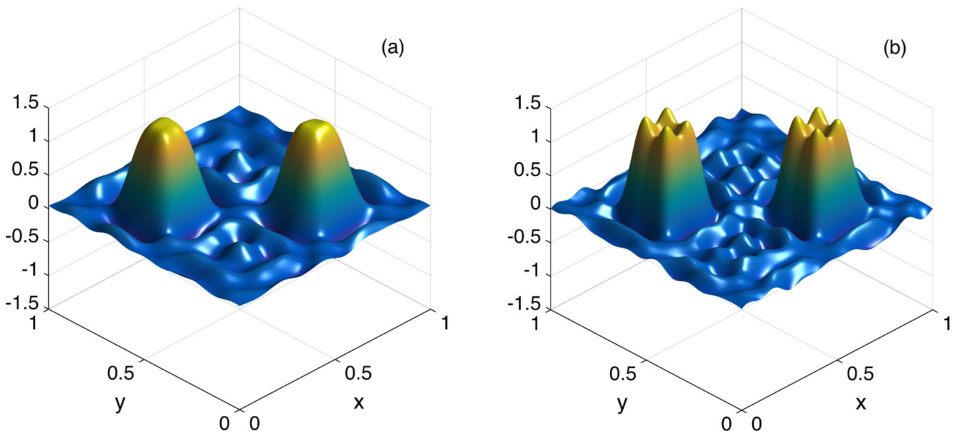

Fig. 6. (Color online.) The reconstructed surface function. (a) $\kappa_c = 6$; (b) $\kappa_c = 7$.

6. Conclusion

We presented a novel method for solving the inverse elastic scattering problem for biperiodic surfaces in three dimensions. The scattering surface was assumed to be a small and smooth perturbation of a rigid planar surface. Using the Helmholtz decomposition and the transformed field expansion, we obtained power series expansions for both the scalar and vector potential functions. The terms of the power series were shown to satisfy a recursive system of boundary value problems and were solved in closed forms. By keeping only the leading and linear terms, which essentially linearized the inverse problem, we deduced an explicit and simple relation between the Fourier coefficients of the surface function and the measured displacement field. The surface function was reconstructed by the truncated Fourier series expansion.

By deriving a transparent boundary condition and using the variational formulation, we showed the well-posedness of the solution for the direct problem. We also showed the well-posedness of the solution for the recursive boundary value problems and the convergence of the power series expansion. We proved the solution of the inverse scattering problem is unique for a sufficiently small deformation parameter. We derived an error estimate of the reconstructed surface scattering. The error estimate provided a clear dependence on all the physical parameters and shed a light on the trade-off among the accuracy, stability and resolution for the inverse problem. The method requires only a single incident field. It is simple and is efficiently implemented by using the fast Fourier transform. Numerical examples show that the method is effective and robust to reconstruct both smooth and non-smooth biperiodic surfaces with subwavelength resolution.

Although the method is derived for the Dirichlet boundary condition in this paper, it can be readily extended to solve the inverse elastic surface scattering problems for other boundary conditions. We may also consider extending the method to closed surfaces such as obstacles and cavities. More challenging problems in this direction include the cases where the scattering surface is random or contains multi-scale features, and when the measured data is phaseless or limited aperture. We hope to address some of those problems in future work.

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