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INCREASING STABILITY FOR THE INVERSE SOURCE SCATTERING PROBLEM WITH MULTI-FREQUENCIES

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Abstract. Consider the scattering of the two- or three-dimensional Helmholtz equation where the source of the electric current density is assumed to be compactly supported in a ball. This paper concerns the stability analysis of the inverse source scattering problem which is to reconstruct the source function. Our results show that increasing stability can be obtained for the inverse problem by using only the Dirichlet boundary data with multi-frequencies.

1. **Introduction and problem statement.** In this paper, we consider the following Helmholtz equation:

$$\Delta u(x) + \kappa^2 u(x) = f(x), \quad x \in \mathbb{R}^d, \tag{1}$$

where $d = 2$ or 3, the wavenumber $\kappa > 0$ is a constant, u is the radiated wave field, and f is the source of the electric current density which is assumed to have a compact support contained in $B_r = \{x \in \mathbb{R}^d : |x| < r\}$, where $r > 0$ is a constant. Let $R > r$ be a constant such that $\mathrm{supp} f \subset B_r \subset B_R$. Let ∂B_R be the boundary of B_R. The problem geometry is shown in Figure 1. The usual Sommerfeld radiation condition is imposed to ensure the uniqueness of the wave field:

$$\lim_{r\to\infty} r^{\frac{d-1}{2}}(\partial_r u - \mathrm{i}\kappa u) = 0, \quad r = |x|, \tag{2}$$

uniformly in all directions $\hat{x} = x/|x|$.

It is known that the scattering problem (1)–(2) has a unique solution which is given by

$$u(x) = \int_{\mathbb{R}^d} G(x, y) f(y)\mathrm{d}y, \quad x \in \mathbb{R}^d, \tag{3}$$

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where $G(x, y)$ is the Green function of the Helmholtz equation. Explicitly, we have

$$G(x,y) = \begin{cases} -\frac{\mathrm{i}}{4} H_0^{(1)}(\kappa|x-y|), & d=2, \\ -\frac{1}{4\pi}\frac{e^{\mathrm{i}\kappa|x-y|}}{|x-y|}, & d=3, \end{cases}$$

where $H_0^{(1)}$ is the Hankel function of the first kind with order 0.

For a given function u on ∂B_R in two dimensions, it has the Fourier series expansion:

$$u(R,\theta) = \sum_{n\in\mathbb{Z}} \hat{u}_n(R) e^{\mathrm{i}n\theta}, \quad \hat{u}_n(R) = \frac{1}{2\pi}\int_0^{2\pi} u(R,\theta) e^{-\mathrm{i}n\theta} \mathrm{d}\theta.$$

We introduce the Dirichlet-to-Neumann (DtN) operator $T : H^{\frac{1}{2}}(\partial B_R) \to H^{-\frac{1}{2}}(\partial B_R)$ given by

$$(Tu)(R,\theta) = \kappa \sum_{n\in\mathbb{Z}} \frac{H_n^{(1)'}(\kappa R)}{H_n^{(1)}(\kappa R)} \hat{u}_n(R) e^{\mathrm{i}n\theta}.$$

For a given function u on ∂B_R in three dimensions, it has the Fourier series expansion:

$$u(R,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \hat{u}_n^m(R) Y_n^m(\theta,\varphi), \quad \hat{u}_n^m(R) = \int_{\partial B_R} u(R,\theta,\varphi) \bar{Y}_n^m(\theta,\varphi) \mathrm{d}\gamma.$$

Similarly, we introduce the DtN operator $T : H^{\frac{1}{2}}(\partial B_R) \to H^{-\frac{1}{2}}(\partial B_R)$ as follows:

$$(Tu)(R,\theta,\varphi) = \kappa \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{h_n^{(1)'}(\kappa R)}{h_n^{(1)}(\kappa R)} \hat{u}_n^m(R) Y_n^m(\theta,\varphi).$$

Here $H_n^{(1)}$ is the Hankel function of the first kind with order n, $h_n^{(1)}$ is the spherical Hankel function of the first kind with order n, Y_n^m is the spherical harmonics of order n, and the bar denotes the complex conjugate. Using the DtN operator, we can reformulate the Sommerfeld radiation condition into a transparent boundary condition (TBC):

$$\partial_\nu u = Tu \quad \text{on } \partial B_R,$$

where ν is the unit outward normal vector on ∂B_R. Hence the the Neumann data $\partial_\nu u$ on ∂B_R can be obtained once the Dirichlet date u is available on ∂B_R.

Remark 1. Consider the following well-posed exterior problem:

$$\begin{cases} \Delta u + \kappa^2 u = 0 & \text{in } \mathbb{R}^d \setminus \bar{B}_R, \\ u = u & \text{on } \partial B_R, \\ \partial_r u - \mathrm{i}\kappa u = o(r^{-\frac{d-1}{2}}) & \text{as } r \to \infty. \end{cases}$$

The DtN operator is based on solving analytically the above problem in the polar ($d = 2$) or spherical ($d = 3$) coordinates and then taking the normal derivative of the solution.

Now we are in the position to discuss our inverse source problem:

IP. *Let f be a complex function with a compact support contained in B_R. The inverse problem is to determine f by using the boundary data $u(x,\kappa)|_{\partial B_R}$ with $\kappa \in (0, K)$ where $K > 1$ is a positive constant.*

The inverse source problem has significant applications in many aspects of scientific areas, such as antenna synthesis [2], medical and biomedical imaging [11], and various tomography problems [1, 16]. Another important example of the inverse

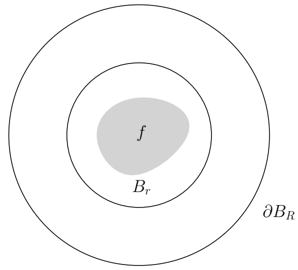

FIGURE 1. Problem geometry of the inverse source scattering problem.

problem is the recovery of acoustic sources from boundary measurements of the pressure. In this paper, we study the stability of the above inverse problem. As is known, the inverse source problem does not have a unique solution at a single frequency [8, 10, 12]. Our goal is to establish increasing stability of the inverse problem with multi-frequencies. We refer to [4, 7, 15] for increasing stability analysis of the inverse source scattering problem. In [7], the authors discussed increasing stability of the inverse source problem for the three-dimensional Helmholtz equation in a general domain Ω by using the Huygens principle. The observation data were $u(x,\kappa)|_{\partial\Omega}$ and $\nabla u(x,\kappa)|_{\partial\Omega}, \kappa \in (0,K)$. In [4], the authors studied the stability of the two- and three-dimensional Helmholtz equations via Green's functions. We point out that the stabilities in [4] are different from the stability in this paper where only the Dirichlet data is required. An initial attempt was made in [15] to study the stablity of an inverse random source scattering problem for the one-dimensional Helmholtz equation. Related results can be found in [13, 14] on increasing stability of determining potentials and in the continuation for the Helmholtz equation. We refer to [9, 5, 6] for a uniqueness result and numerical study for the inverse source scattering problem. A topic review can be found in [3] for some general inverse scattering problems with multi-frequencies.

We point out that the approach can be used to deal with other geometries than the circular domain. For example, a DtN map can also be obtained via the boundary integral equation relating the Neumann data to the Dirichlet data on any smooth curve which encloses the compact support of the source. The rest of the paper is organized as follows. The main result is presented in section 2. Section 3 is devoted to the proof of the result. The paper is concluded in section 4 with general remarks and possible future work.

2. **Main result.**

Define a complex-valued functional space:

$$\mathcal{C}_M = \{f \in H^{n+1}(B_R) : \|f\|_{H^{n+1}(B_R)} \le M,\ \mathrm{supp} f \subset B_r \subset B_R,\ f : B_R \to \mathbb{C}\},$$

where $n \ge d$ is an integer and $M > 1$ is a constant. For any $v \in H^{\frac{1}{2}}(\partial B_R)$, we set

$$\|u(x,\kappa)\|^2_{\partial B_R} = \int_{\partial B_R} \left(|Tu(x,\kappa)|^2 + \kappa^2 |u(x,\kappa)|^2\right) \mathrm{d}\gamma.$$

Throughout the paper, $a \lesssim b$ stands for $a \le Cb$, where $C > 0$ is a constant independent of n, κ, K, M. Now we introduce the main stability result of this paper.

Theorem 2.1. *Let $f \in \mathcal{C}_M$ and u be the solution of the scattering problem (1)–(2) corresponding to f. Then*

$$\|f\|_{L^2(B_R)}^2 \lesssim \epsilon^2 + \frac{M^2}{\left(\frac{K^{\frac{2}{3}}|\ln\epsilon|^{\frac{1}{4}}}{(R+1)(6n-6d+3)^3}\right)^{2n-2d+1}}, \tag{4}$$

where

$$\epsilon = \left(\int_0^K \kappa^{d-1}\|u(x,\kappa)\|_{\partial B_R}^2 \mathrm{d}\kappa\right)^{\frac{1}{2}}. \tag{5}$$

Remark 2. First, it is clear to note that the stability estimate (4) implies the uniqueness of the inverse problem, i.e., $f = 0$ if $\epsilon = 0$. Second, there are two parts in the stability estimate: the first part is the data discrepancy and the second part is the high frequency tail of the source function. Obviously, the stability increases as K increases, i.e., the problem is more stable as the data with more frequencies are used. We can also see that when $n < \left[\frac{K^{\frac{2}{9}}|\ln\epsilon|^{\frac{1}{12}}}{(R+1)^{\frac{1}{3}}} + d - \frac{1}{2}\right]$, the stability increases as n increases, i.e., the problem is more stable as the source function has suitably higher regularity.

Remark 3. The idea was firstly proposed in [7] by separating the stability into the data discrepancy and high frequency tail where the latter was estimated by the unique continuation for the three-dimensional inverse source scattering problem. Our stability result in this work is consistent with the one in [7] for both the two- and three-dimensional inverse scattering problems.

3. Proof of Theorem 2.1.

First we present several useful lemmas.

Lemma 3.1. *Let $f \in L^2(B_R), \operatorname{supp} f \subset B_R$. Then*

$$\|f\|_{L^2(B_R)}^2 \lesssim \int_0^\infty \kappa^{d-1}\int_{\partial B_R}\left(|\partial_\nu u(x,\kappa)|^2 + \kappa^2|u(x,\kappa)|^2\right)\mathrm{d}\gamma\mathrm{d}\kappa.$$

Proof. Let $\xi \in \mathbb{R}^d$ with $|\xi| = \kappa \in (0,\infty)$. Multiplying $e^{-\mathrm{i}\xi\cdot x}$ on both sides of (1) and integrating over B_R, we obtain

$$\int_{B_R} e^{-\mathrm{i}\xi\cdot x} f(x)\mathrm{d}x = \int_{\partial B_R} e^{-\mathrm{i}\xi\cdot x}(\partial_\nu u(x,\kappa) + \mathrm{i}(\nu\cdot\xi)u(x,\kappa))\mathrm{d}\gamma.$$

Since $\operatorname{supp} f \subset B_R$, we have

$$\int_{\mathbb{R}^d} e^{-\mathrm{i}\xi\cdot x} f(x)\mathrm{d}x = \int_{\partial B_R} e^{-\mathrm{i}\xi\cdot x}(\partial_\nu u(x,\kappa) + \mathrm{i}(\nu\cdot\xi)u(x,\kappa))\mathrm{d}\gamma,$$

which gives

$$\left|\int_{\mathbb{R}^d} e^{-\mathrm{i}\xi\cdot x} f(x)\mathrm{d}x\right|^2 \le \left|\int_{\partial B_R}(|\partial_\nu u(x,\kappa)| + \kappa|u(x,\kappa)|)\mathrm{d}\gamma\right|^2.$$

Hence,

$$\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d} e^{-\mathrm{i}\xi\cdot x} f(x)\mathrm{d}x\right|^2 \mathrm{d}\xi \le \int_{\mathbb{R}^d}\left|\int_{\partial B_R}(|\partial_\nu u(x,\kappa)| + \kappa|u(x,\kappa)|)\mathrm{d}\gamma\right|^2 \mathrm{d}\xi.$$

When $d = 2$, we obtain by using the polar coordinates that

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\left|\int_{\mathbb{R}^2} e^{-\mathrm{i}\xi\cdot x} f(x)\mathrm{d}x\right|^2 \mathrm{d}\xi \\
&\leq \int_0^{2\pi}\mathrm{d}\theta\int_0^\infty \kappa\left|\int_{\partial B_R}(|\partial_\nu u(x,\kappa)|+\kappa|u(x,\kappa)|)\mathrm{d}\gamma\right|^2\mathrm{d}\kappa \\
&\lesssim \int_0^\infty \kappa\left|\int_{\partial B_R}(|\partial_\nu u(x,\kappa)|+\kappa|u(x,\kappa)|)\mathrm{d}\gamma\right|^2\mathrm{d}\kappa \\
&\lesssim \int_0^\infty \kappa\int_{\partial B_R}\left(|\partial_\nu u(x,\kappa)|^2+\kappa^2|u(x,\kappa)|^2\right)\mathrm{d}\gamma\mathrm{d}\kappa.
\end{aligned}
$$

It follows from the Plancherel theorem that

$$
\begin{aligned}
\|f\|^2_{L^2(B_R)} = \|f\|^2_{L^2(\mathbb{R}^2)} &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}|\hat{f}(\xi)|^2\mathrm{d}\xi \\
&\lesssim \int_0^\infty \kappa\int_{\partial B_R}\left(|\partial_\nu u(x,\kappa)|^2+\kappa^2|u(x,\kappa)|^2\right)\mathrm{d}\gamma\mathrm{d}\xi.
\end{aligned}
$$

When $d=3$, we obtain by using the spherical coordinates that

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\left|\int_{\mathbb{R}^3} e^{-\mathrm{i}\xi\cdot x} f(x)\mathrm{d}x\right|^2 \mathrm{d}\xi \\
&\leq \int_0^{2\pi}\mathrm{d}\theta\int_0^\pi \sin\varphi\mathrm{d}\varphi\int_0^\infty \kappa^2\left|\int_{\partial B_R}(|\partial_\nu u(x,\kappa)|+\kappa|u(x,\kappa)|)\mathrm{d}\gamma\right|^2\mathrm{d}\kappa \\
&\lesssim \int_0^\infty \kappa^2\left|\int_{\partial B_R}(|\partial_\nu u(x,\kappa)|+\kappa|u(x,\kappa)|)\mathrm{d}\gamma\right|^2\mathrm{d}\kappa \\
&\lesssim \int_0^\infty \kappa^2\int_{\partial B_R}\left(|\partial_\nu u(x,\kappa)|^2+\kappa^2|u(x,\kappa)|^2\right)\mathrm{d}\gamma\mathrm{d}\kappa.
\end{aligned}
$$

It follows from the Plancherel theorem again that

$$
\begin{aligned}
\|f\|^2_{L^2(B_R)} = \|f\|^2_{L^2(\mathbb{R}^3)} &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}|\hat{f}(\xi)|^2\mathrm{d}\xi \\
&\lesssim \int_0^\infty \kappa^2\int_{\partial B_R}\left(|\partial_\nu u(x,\kappa)|^2+\kappa^2|u(x,\kappa)|^2\right)\mathrm{d}\gamma\mathrm{d}\xi,
\end{aligned}
$$

which completes the proof. □

For $d=2$, let

$$
I_1(s) = \int_0^s \kappa^3\int_{\partial B_R}\left|\int_{B_R}\frac{\mathrm{i}}{4}H_0^{(1)}(\kappa|x-y|)f(y)\mathrm{d}y\right|^2\mathrm{d}\gamma(x)\mathrm{d}\kappa, \tag{6}
$$

$$
I_2(s) = \int_0^s \kappa\int_{\partial B_R}\left|\int_{B_R}\frac{\mathrm{i}}{4}\partial_{\nu_x}H_0^{(1)}(\kappa|x-y|)f(y)\mathrm{d}y\right|^2\mathrm{d}\gamma(x)\mathrm{d}\kappa. \tag{7}
$$

For $d=3$, let

$$
I_1(s) = \int_0^s \kappa^4\int_{\partial B_R}\left|\int_{B_R}\frac{e^{\mathrm{i}\kappa|x-y|}}{4\pi|x-y|}f(y)\mathrm{d}y\right|^2\mathrm{d}\gamma(x)\mathrm{d}\kappa, \tag{8}
$$

$$
I_2(s) = \int_0^s \kappa^2\int_{\partial B_R}\left|\int_{B_R}\partial_{\nu_x}\frac{e^{\mathrm{i}\kappa|x-y|}}{4\pi|x-y|}f(y)\mathrm{d}y\right|^2\mathrm{d}\gamma(x)\mathrm{d}\kappa. \tag{9}
$$

Denote

$$
S = \{z = x+\mathrm{i}y\in\mathbb{C} : -\frac{\pi}{4} < \arg z < \frac{\pi}{4}\}.
$$

The integrands in (6)–(9) are analytic functions of κ in S. The integrals with respect to κ can be taken over any path joining points 0 and s in S. Thus $I_1(s)$ and $I_2(s)$ are analytic functions of $s = s_1 + {\rm i}s_2 \in S, s_1, s_2 \in \mathbb{R}$.

Lemma 3.2. *Let $f \in H^1(B_R), {\rm supp} f \subset B_R$. For any $s = s_1 + {\rm i}s_2 \in S$, we have:*

1. *for $d = 2$,*

$$|I_1(s)| \lesssim |s|^3 e^{4R|s_2|} \|f\|^2_{L^2(B_R)}, \tag{10}$$

$$|I_2(s)| \lesssim |s| e^{4R|s_2|} \|f\|^2_{H^1(B_R)}; \tag{11}$$

2. *for $d = 3$,*

$$|I_1(s)| \lesssim |s|^5 e^{4R|s_2|} \|f\|^2_{L^2(B_R)}, \tag{12}$$

$$|I_2(s)| \lesssim |s|^3 e^{4R|s_2|} \|f\|^2_{H^1(B_R)}. \tag{13}$$

Proof. First we consider $d = 3$. Let $\kappa = st, t \in (0, 1)$. It follows from the change of variables that

$$I_1(s) = \int_0^1 s^5 t^4 \int_{\partial B_R} \left| \int_{B_R} \frac{e^{{\rm i}st|x-y|}}{4\pi|x-y|} f(y){\rm d}y \right|^2 {\rm d}\gamma(x){\rm d}t.$$

Noting

$$|e^{{\rm i}st|x-y|}| \leq e^{2R|s_2|} \quad \text{for all}\, x \in \partial B_R, y \in B_R,$$

we have from the Cauchy–Schwarz inequality that

$$\begin{aligned}
|I_1(s)| &\leq \int_0^1 |s|^5 t^4 \int_{\partial B_R} \left| \int_{B_R} \frac{e^{2R|s_2|}}{4\pi|x-y|} |f(y)|{\rm d}y \right|^2 {\rm d}\gamma(x){\rm d}t \\
&\lesssim \int_0^1 |s|^5 t^4 \int_{\partial B_R} \left(\int_{B_R} |f(y)|^2 {\rm d}y \right) \left(\int_{B_R} \frac{e^{4R|s_2|}}{|x-y|^2} {\rm d}y \right) {\rm d}\gamma(x){\rm d}t \\
&\lesssim \int_0^1 |s|^5 t^4 \int_{\partial B_R} \left(\int_{B_R} |f(y)|^2 {\rm d}y \right) \left(\int_{B_{2R}(x)} \frac{e^{4R|s_2|}}{|x-y|^2} {\rm d}y \right) {\rm d}\gamma(x){\rm d}t,
\end{aligned}$$

where $B_{2R}(x)$ is the ball with a radius $2R$ and center at x. Using the spherical coordinates (ρ, θ, φ) with respect to y where $\rho = |x - y|$, we get

$$\begin{aligned}
&|I_1(s)| \\
&\lesssim \int_0^1 |s|^5 t^4 \left(\int_{B_R} |f(y)|^2 {\rm d}y \right) \int_{\partial B_R} \left(\int_0^{2\pi} {\rm d}\theta \int_0^{\pi} \sin(\varphi){\rm d}\varphi \int_0^{2R} e^{4R|s_2|} {\rm d}\rho \right) {\rm d}\gamma(x){\rm d}t \\
&\lesssim \int_0^1 |s|^5 \left(\int_{B_R} |f(y)|^2 {\rm d}y \right) \int_{\partial B_R} \left(\int_0^{2R} e^{4R|s_2|} {\rm d}\rho \right) {\rm d}\gamma(x){\rm d}t \\
&\lesssim |s|^5 e^{4R|s_2|} \|f\|^2_{L^2(B_R)},
\end{aligned}$$

which shows (12).

Next we prove (13). Let $\kappa = st, t \in (0, 1)$. It follows from the change of variables again that

$$I_2(s) = \int_0^1 s^3 t^2 \int_{\partial B_R} \left| \int_{B_R} \partial_{\nu_x} \frac{e^{{\rm i}st|x-y|}}{4\pi|x-y|} f(y){\rm d}y \right|^2 {\rm d}\gamma(x){\rm d}t,$$

which gives

$$|I_2(s)| \lesssim \int_0^1 |s|^3 t^2 \int_{\partial B_R} \left| \int_{B_R} \nu_x \cdot \nabla_x \left(\frac{e^{\mathrm{i}st|x-y|}}{|x-y|} \right) f(y)\mathrm{d}y \right|^2 \mathrm{d}\gamma(x)\mathrm{d}t.$$

Noting

$$\nabla_x \left(\frac{e^{\mathrm{i}st|x-y|}}{|x-y|} \right) = -\nabla_y \left(\frac{e^{\mathrm{i}st|x-y|}}{|x-y|} \right),$$

we have from the integration by parts that

$$\begin{aligned} |I_2(s)| &\lesssim \int_0^1 |s|^3 t^2 \int_{\partial B_R} \left| \int_{B_R} \frac{e^{\mathrm{i}st|x-y|}}{|x-y|} \nu \cdot \nabla f(y)\mathrm{d}y \right|^2 \mathrm{d}\gamma(x)\mathrm{d}t \\ &\lesssim \int_0^1 |s|^3 t^2 \int_{\partial B_R} \left(\int_{B_R} |\nabla f|^2 \mathrm{d}y \right) \left(\int_{B_R} \frac{e^{4R|s_2|}}{|x-y|^2} \mathrm{d}y \right) \mathrm{d}\gamma(x)\mathrm{d}t \\ &\lesssim |s|^3 e^{4R|s_2|} \|f\|^2_{H^1(B_R)}, \end{aligned}$$

where we have used the Cauchy–Schwarz inequality, the fact that $e^{\mathrm{i}st|x-y|} \leq e^{2R|s_2|}$ for all $x \in \partial B_R, y \in B_R$, and the change of the Cartesian coordinates to the spherical coordinates.

Second we consider $d = 2$. Letting $\kappa = st, t \in (0, 1)$, we have from the change of variables that

$$I_1(s) = \int_0^1 s^4 t^3 \int_{\partial B_R} \left| \int_{B_R} \frac{\mathrm{i}}{4} H_0^{(1)}(st|x-y|) f(y)\mathrm{d}y \right|^2 \mathrm{d}\gamma(x)\mathrm{d}t.$$

The Hankel function can be expressed by the following integral when $\mathrm{Re}z > 0$ (e.g.,[17], Chapter VI):

$$H_0^{(1)}(z) = \frac{1}{\mathrm{i}\pi} \int_{1+\infty\mathrm{i}}^1 e^{\mathrm{i}z\tau} (\tau^2 - 1)^{-\frac{1}{2}} \mathrm{d}\tau.$$

Consequently,

$$\begin{aligned} |H_0^{(1)}(z)| &= \left| \frac{1}{\pi} \int_{+\infty}^0 e^{\mathrm{i}(\mathrm{Re}z + \mathrm{i}\mathrm{Im}z)(1+t\mathrm{i})} ((1+t\mathrm{i})^2 - 1)^{-\frac{1}{2}} \mathrm{d}t \right| \\ &\leq \left| \frac{1}{\pi} e^{\mathrm{i}\mathrm{Re}z - \mathrm{Im}z} \int_{+\infty}^0 e^{-t\mathrm{Re}z - \mathrm{i}t\mathrm{Im}z} (2t\mathrm{i} - t^2)^{-\frac{1}{2}} \mathrm{d}t \right| \\ &\leq \frac{1}{\pi} e^{|\mathrm{Im}z|} \int_0^{+\infty} \frac{e^{-t\mathrm{Re}z}}{\left| t^{\frac{1}{2}} (2\mathrm{i} - t^{\frac{1}{2}}) \right|} \mathrm{d}t \\ &\leq \frac{1}{\pi} e^{|\mathrm{Im}z|} \int_0^{+\infty} \frac{e^{-t\mathrm{Re}z}}{t^{\frac{1}{2}} (t^2 + 4)^{\frac{1}{4}}} \mathrm{d}t \\ &\leq \frac{1}{\pi} e^{|\mathrm{Im}z|} \int_0^{+\infty} \frac{e^{-t\mathrm{Re}z}}{t^{\frac{1}{2}} 2^{\frac{1}{2}}} \mathrm{d}t \\ &= \frac{1}{\pi} e^{|\mathrm{Im}z|} \left(\int_0^{\frac{1}{\mathrm{Re}z}} \frac{e^{-t\mathrm{Re}z}}{t^{\frac{1}{2}} 2^{\frac{1}{2}}} \mathrm{d}t + \int_{\frac{1}{\mathrm{Re}z}}^{+\infty} \frac{e^{-t\mathrm{Re}z}}{t^{\frac{1}{2}} 2^{\frac{1}{2}}} \mathrm{d}t \right) \end{aligned}$$

$$\leq \frac{1}{\pi} e^{|\mathrm{Im} z|} \left(\int_0^{\frac{1}{\mathrm{Re} z}} \frac{1}{t^{\frac{1}{2}}} \mathrm{d}t + \frac{(\mathrm{Re} z)^{\frac{1}{2}}}{2^{\frac{1}{2}}} \int_{\frac{1}{\mathrm{Re} z}}^{+\infty} e^{-t \mathrm{Re} z} \mathrm{d}t \right)$$
$$\leq \frac{3}{\pi} \frac{e^{|\mathrm{Im} z|}}{(\mathrm{Re} z)^{\frac{1}{2}}}.$$

Hence we have from the Cauchy–Schwarz inequality that

$$|I_1(s)| \lesssim \int_0^1 |s|^4 t^3 \int_{\partial B_R} \left(\int_{B_R} |f(y)|^2 \mathrm{d}y \right) \left(\int_{B_{2R}(x)} \frac{e^{4R|s_2|}}{s_1 t |x-y|} \mathrm{d}y \right) \mathrm{d}\gamma(x) \mathrm{d}t.$$

Using the polar coordinates (ρ, θ) with respect to y with $\rho = |x-y|$ and the fact that

$$\frac{|s|}{s_1} \leq 2^{\frac{1}{2}} \quad \text{for all}\, s \in S,$$

we obtain

$$|I_1(s)| \lesssim \int_0^1 |s|^3 t^2 \left(\int_{B_R} |f(y)|^2 \mathrm{d}y \right) \int_{\partial B_R} \left(\int_0^{2\pi} \mathrm{d}\theta \int_0^{2R} e^{4R|s_2|} \mathrm{d}\rho \right) \mathrm{d}\gamma(x) \mathrm{d}t$$
$$\lesssim |s|^3 e^{4R|s_2|} \|f\|^2_{L^2(B_R)},$$

which shows (10).

Noting that $\partial_{\nu_x} H_0^{(1)}(\kappa|x-y|) = \nu_x \cdot \nabla_x H_0^{(1)}(\kappa|x-y|)$ and $\nabla_x H_0^{(1)}(\kappa|x-y|) = -\nabla_y H_0^{(1)}(\kappa|x-y|)$, we can prove (11) in a similar way by taking the integration by parts, which completes the proof. $\square$

Lemma 3.3. *Let $f \in \mathcal{C}_M$. For any $s \geq 1$, we have*

(14)
$$\int_s^{+\infty} \int_{\partial B_R} \kappa^{d-1} \big(|\partial_\nu u(x,\kappa)|^2 + \kappa^2 |u(x,\kappa)|^2 \big) \mathrm{d}\gamma \mathrm{d}\kappa \lesssim s^{-(2n-2d+1)} \|f\|^2_{H^{n+1}(B_R)}.$$

Proof. It is easy to note that

$$\int_s^{+\infty} \int_{\partial B_R} \kappa^{d-1} \big(|\partial_\nu u(x,\kappa)|^2 + \kappa^2 |u(x,\kappa)|^2 \big) \mathrm{d}\gamma \mathrm{d}\kappa$$
$$= \int_s^{+\infty} \int_{\partial B_R} \kappa^{d+1} |u(x,\kappa)|^2 \mathrm{d}\gamma \mathrm{d}\kappa + \int_s^{+\infty} \int_{\partial B_R} \kappa^{d-1} |\partial_\nu u(x,\kappa)|^2 \mathrm{d}\gamma \mathrm{d}\kappa$$
$$= L_1 + L_2.$$

We estimate L_1 and L_2, respectively.

First we consider $d = 3$. Using (3) yields

$$L_1 = \int_s^{+\infty} \int_{\partial B_R} \kappa^4 |u(x,\kappa)|^2 \mathrm{d}\gamma \mathrm{d}\kappa$$
$$= \int_s^{+\infty} \int_{\partial B_R} \kappa^4 \left| \int_{\mathbb{R}^3} \frac{e^{\mathrm{i}\kappa|x-y|}}{4\pi|x-y|} f(y) \mathrm{d}y \right|^2 \mathrm{d}\gamma(x) \mathrm{d}\kappa.$$

Using the spherical coordinates $\rho = |x-y|$ originated at x with respect to y, we have

$$L_1 = \frac{1}{4\pi} \int_s^{+\infty} \int_{\partial B_R} \kappa^4 \left| \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi} \sin\varphi \mathrm{d}\varphi \int_0^{+\infty} e^{\mathrm{i}\kappa\rho} f \rho \mathrm{d}\rho \right|^2 \mathrm{d}\gamma(x) \mathrm{d}\kappa.$$

Using the integration by parts and noting $\mathrm{supp} f \subset B_r \subset B_R$, we obtain

$$L_1 = \frac{1}{4\pi}\int_s^{+\infty}\int_{\partial B_R}\kappa^4\left|\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\sin\varphi\mathrm{d}\varphi\int_{(R-r)}^{2R}\frac{e^{\mathrm{i}\kappa\rho}}{(\mathrm{i}\kappa)^n}\frac{\partial^n(f\rho)}{\partial\rho^n}\mathrm{d}\rho\right|^2\mathrm{d}\gamma(x)\mathrm{d}\kappa.$$

Consequently,

$$\begin{aligned}
L_1 \lesssim & \int_s^{+\infty}\int_{\partial B_R}\kappa^4\left|\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\sin\varphi\mathrm{d}\varphi\int_{(R-r)}^{2R}\frac{1}{\kappa^n}\right.\\
&\left.\left(\left|\sum_{|\alpha|=n}\partial_y^\alpha f\right|\rho\ +n\left|\sum_{|\alpha|=n-1}\partial_y^\alpha f\right|\right)\mathrm{d}\rho\right|^2\mathrm{d}\gamma(x)\mathrm{d}\kappa\\
= & \int_s^{+\infty}\int_{\partial B_R}\kappa^4\left|\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\sin\varphi\mathrm{d}\varphi\int_{(R-r)}^{2R}\frac{1}{\kappa^n}\right.\\
&\left.\left(\left|\sum_{|\alpha|=n}\partial_y^\alpha f\right|\frac{1}{\rho}\ +\left|\sum_{|\alpha|=n-1}\partial_y^\alpha f\right|\frac{n}{\rho^2}\right)\rho^2\mathrm{d}\rho\right|^2\mathrm{d}\gamma(x)\mathrm{d}\kappa\\
\leq & \int_s^{+\infty}\int_{\partial B_R}\kappa^4\left|\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\sin\varphi\mathrm{d}\varphi\int_{(R-r)}^{2R}\frac{1}{\kappa^n}\right.\\
&\left.\left(\left|\sum_{|\alpha|=n}\partial_y^\alpha f\right|\frac{1}{(R-r)}\ +\left|\sum_{|\alpha|=n-1}\partial_y^\alpha f\right|\frac{n}{(R-r)^2}\right)\rho^2\mathrm{d}\rho\right|^2\mathrm{d}\gamma(x)\mathrm{d}\kappa\\
\leq & \int_s^{+\infty}\int_{\partial B_R}\kappa^4\left|\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\sin\varphi\mathrm{d}\varphi\int_0^{+\infty}\frac{1}{\kappa^n}\right.\\
&\left.\left(\left|\sum_{|\alpha|=n}\partial_y^\alpha f\right|\frac{1}{(R-r)}\ +\left|\sum_{|\alpha|=n-1}\partial_y^\alpha f\right|\frac{n}{(R-r)^2}\right)\rho^2\mathrm{d}\rho\right|^2\mathrm{d}\gamma\mathrm{d}\kappa.
\end{aligned}$$

Changing back to the Cartesian coordinates with respect to y, we have

$$\begin{aligned}
L_1 \lesssim & \int_s^{+\infty}\int_{\partial B_R}\kappa^4\left|\int_{\mathbb{R}^3}\frac{1}{\kappa^n}\right.\\
&\left.\left(\left|\sum_{|\alpha|=n}\partial_y^\alpha f\right|\frac{1}{(R-r)}\ +\left|\sum_{|\alpha|=n-1}\partial_y^\alpha f\right|\frac{n}{(R-r)^2}\right)dy\right|^2\mathrm{d}\gamma(x)\mathrm{d}\kappa\\
\lesssim & \, n\|f\|_{H^n(B_R)}\int_s^{+\infty}\kappa^{4-2n}\mathrm{d}\kappa\\
\lesssim & \left(\frac{n}{2n-5}\right)\frac{1}{s^{2n-5}}\|f\|_{H^n(B_R)}\\
\lesssim & \frac{1}{s^{2n-5}}\|f\|^2_{H^n(B_R)},
\end{aligned}\tag{15}$$

where we have used the fact that $n \geq d = 3$.

Next we estimate L_2 for $d = 3$. It follows from (3) again that

$$L_2 = \int_s^{+\infty} \int_{\partial B_R} \kappa^2 |\partial_\nu u(x,\kappa)|^2 \mathrm{d}\gamma \mathrm{d}\kappa$$
$$= \int_s^{+\infty} \int_{\partial B_R} \kappa^4 \left| \int_{\mathbb{R}^3} \left(\nu_x \cdot \nabla_x \frac{e^{\mathrm{i}\kappa|x-y|}}{4\pi|x-y|} \right) f(y) \mathrm{d}y \right|^2 \mathrm{d}\gamma(x) \mathrm{d}\kappa.$$

Noting that $\nabla_y \left(\frac{e^{\mathrm{i}\kappa|x-y|}}{|x-y|} \right) = -\nabla_x \left(\frac{e^{\mathrm{i}\kappa|x-y|}}{|x-y|} \right)$ and $\mathrm{supp} f \subset B_R$, we have

$$L_2 = \int_s^{+\infty} \int_{\partial B_R} \kappa^2 |\partial_\nu u(x,\kappa)|^2 \mathrm{d}\gamma(x) \mathrm{d}\kappa$$
$$= \int_s^{+\infty} \int_{\partial B_R} \kappa^2 \left| \int_{\mathbb{R}^3} \left(\nu_x \cdot \nabla_y \frac{e^{\mathrm{i}\kappa|x-y|}}{4\pi|x-y|} \right) f(y) \mathrm{d}y \right|^2 \mathrm{d}\gamma(x) \mathrm{d}\kappa$$
$$= \int_s^{+\infty} \int_{\partial B_R} \kappa^2 \left| \int_{\mathbb{R}^3} \frac{e^{\mathrm{i}\kappa|x-y|}}{4\pi|x-y|} \left(\nu_x \cdot \nabla_y f(y) \right) \mathrm{d}y \right|^2 \mathrm{d}\gamma(x) \mathrm{d}\kappa.$$

Following a similar argument as that for the proof of (15), we obtain

$$L_2 \lesssim n \|f\|_{H^{n+1}(B_R)} \int_s^{+\infty} \kappa^{2-2n} \mathrm{d}\kappa = \left(\frac{n}{2n-3} \right) \|f\|_{H^{n+1}(B_R)} \frac{1}{s^{2n-3}}$$
$$\lesssim \frac{1}{s^{2n-3}} \|f\|^2_{H^{n+1}(B_R)}. \tag{16}$$

Combining (15)–(16) and noting $s > 1$, we obtain (14) for $d = 3$.

Second we consider $d = 2$. Similarly we have

$$L_1 = \int_s^{+\infty} \int_{\partial B_R} \kappa^3 |u(x,\kappa)|^2 \mathrm{d}\gamma \mathrm{d}\kappa$$
$$= \int_s^{+\infty} \int_{\partial B_R} \kappa^3 \left| \int_{\mathbb{R}^2} \frac{\mathrm{i}}{4} H_0^{(1)}(\kappa|x-y|) f(y) \mathrm{d}y \right|^2 \mathrm{d}\gamma(x) \mathrm{d}\kappa.$$

The Hankel function can also be expressed by the following integral when $t > 0$ (e.g., [17], Chapter VI):

$$H_0^{(1)}(t) = \frac{2}{\mathrm{i}\pi} \int_1^{+\infty} e^{\mathrm{i}t\tau} (\tau^2 - 1)^{-\frac{1}{2}} \mathrm{d}\tau.$$

Using the polar coordinates $\rho = |y - x|$ originated at x with respect to y, we have

$$L_1 = \int_s^{+\infty} \int_{\partial B_R} \kappa^3 \left| \int_0^{2\pi} \mathrm{d}\theta \int_0^{+\infty} \frac{1}{4} H_0^{(1)}(\kappa\rho) f \rho \mathrm{d}\rho \right|^2 \mathrm{d}\gamma(x) \mathrm{d}\kappa.$$

Let

$$H_k(t) = \frac{2}{\mathrm{i}\pi} \int_1^{+\infty} \frac{e^{\mathrm{i}t\tau}}{(\mathrm{i}\tau)^k (\tau^2 - 1)^{1/2}} \mathrm{d}\tau, \quad k = 1, 2, \cdots. \tag{17}$$

It is clear to note that

$$H_0(t) = H_0^{(1)}(t) \quad \text{and} \quad \frac{\mathrm{d}H_k(t)}{\mathrm{d}t} = H_{k-1}(t), \ t > 0, \ k \in \mathbb{N}.$$

Using the integration by parts and noting $\mathrm{supp} f \subset B_r \subset B_R$, we obtain

$$\begin{aligned} L_1 =& \frac{1}{4}\int_s^{+\infty}\int_{\partial B_R}\kappa^3\left|\int_0^{2\pi}\mathrm{d}\theta\int_{R-r}^{2R}\frac{H_1(\kappa\rho)}{\kappa^2}\frac{\partial(f\rho)}{\partial\rho}\mathrm{d}\rho\right|^2\mathrm{d}\gamma(x)\mathrm{d}\kappa \\ =& \frac{1}{4}\int_s^{+\infty}\int_{\partial B_R}\kappa^3\left|\int_0^{2\pi}\mathrm{d}\theta\int_{(R-r)}^{2R}\frac{H_n(\kappa\rho)}{\kappa^{n+1}}\frac{\partial^n(f\rho)}{\partial\rho^n}\mathrm{d}\rho\right|^2\mathrm{d}\gamma(x)\mathrm{d}\kappa. \end{aligned}$$

Consequently, we have

$$\begin{aligned} L_1 \lesssim & \int_s^{+\infty}\int_{\partial B_R}\kappa^3\left|\int_0^{2\pi}\mathrm{d}\theta\int_{(R-r)}^{2R}\left|\frac{H_n(\kappa\rho)}{\kappa^{n+1}}\right|\left|\frac{\partial^n(f\rho)}{\partial\rho^n}\right|\mathrm{d}\rho\right|^2\mathrm{d}\gamma(x)\mathrm{d}\kappa \\ \le & \int_s^{+\infty}\int_{\partial B_R}\kappa^3\left|\int_0^{2\pi}\mathrm{d}\theta\int_{(R-r)}^{2R}\left|\frac{H_n(\kappa\rho)}{\kappa^{n+1}}\right|\right. \\ & \left.\left(\left|\sum_{|\alpha|=n}\partial_y^\alpha f\right|+\left|\sum_{|\alpha|=n-1}\partial_y^\alpha f\right|\frac{n}{\rho}\right)\rho\mathrm{d}\rho\right|^2\mathrm{d}\gamma(x)\mathrm{d}\kappa \\ \le & \int_s^{+\infty}\int_{\partial B_R}\kappa^3\left|\int_0^{2\pi}\mathrm{d}\theta\int_{(R-r)}^{2R}\left|\frac{H_n(\kappa\rho)}{\kappa^{n+1}}\right|\right. \\ & \left.\left(\left|\sum_{|\alpha|=n}\partial_y^\alpha f\right|+\left|\sum_{|\alpha|=n-1}\partial_y^\alpha f\right|\frac{n}{(R-r)}\right)\rho\mathrm{d}\rho\right|^2\mathrm{d}\gamma(x)\mathrm{d}\kappa. \end{aligned}$$

Noting (17), we see that there exists a constant $C > 0$ such that $|H_n(\kappa\rho)| \le C$ for $n \ge 1$. Hence,

$$\begin{aligned} L_1 \lesssim & \int_s^{+\infty}\int_{\partial B_R}\kappa^3\left|\int_0^{2\pi}\mathrm{d}\theta\int_{R-r}^{2R}\frac{1}{\kappa^{n+1}}\right. \\ & \left.\left(\left|\sum_{|\alpha|=n}\partial_y^\alpha f\right|+\left|\sum_{|\alpha|=n-1}\partial_y^\alpha f\right|\frac{n}{(R-r)}\right)\rho\mathrm{d}\rho\right|^2\mathrm{d}\gamma(x)\mathrm{d}\kappa. \end{aligned}$$

Changing back to the Cartesian coordinates with respect to y, we have

$$\begin{aligned} L_1 \lesssim & \int_s^{+\infty}\int_{\partial B_R}\kappa^3\left|\int_{B_R}\frac{1}{\kappa^{n+1}}\right. \\ & \left.\left(\left|\sum_{|\alpha|=n}\partial_y^\alpha f\right|+\left|\sum_{|\alpha|=n-1}\partial_y^\alpha f\right|\frac{n}{(R-r)}\right)\mathrm{d}x\right|^2\mathrm{d}\gamma\mathrm{d}\kappa \\ \lesssim & \, n\|f\|_{H^n(B_R)}\int_s^{+\infty}\kappa^{1-2n}\mathrm{d}\kappa \\ = & \left(\frac{n}{2n-2}\right)\frac{1}{s^{2n-2}}\|f\|^2_{H^n(B_R)} \lesssim \frac{1}{s^{2n-2}}\|f\|^2_{H^n(B_R)}. \end{aligned} \tag{18}$$

Next we estimate L_2 for $d=2$. A simple calculation yields

$$\begin{aligned}L_2 =&\frac{1}{4}\int_s^{+\infty}\int_{\partial B_R}\kappa^2|\partial_\nu u(x,\kappa)|^2\mathrm{d}\gamma\mathrm{d}\kappa\\ =&\int_s^{+\infty}\int_{\partial B_R}\kappa^4\left|\int_{\mathbb{R}^2}\left(\nu_x\cdot\nabla_x H_0^{(1)}(\kappa|x-y|)\right)f(y)\mathrm{d}y\right|^2\mathrm{d}\gamma(x)\mathrm{d}\kappa.\end{aligned}$$

Noting that $\nabla_y H_0^{(1)}(\kappa|x-y|)=-\nabla_x H_0^{(1)}(k|x-y|)$ and $\mathrm{supp}f\subset B_r\subset B_R$, we have from the integration by parts that

$$\begin{aligned}L_2 =&\int_s^{+\infty}\int_{\partial B_R}\kappa^2|\partial_\nu u(x,\kappa)|^2\mathrm{d}\gamma\mathrm{d}\kappa\\ =&\frac{1}{4}\int_s^{+\infty}\int_{\partial B_R}\kappa^2\left|\int_{\mathbb{R}^3}\left(\nu_x\cdot\nabla_y H_0^{(1)}(\kappa|x-y|)\right)f(y)\mathrm{d}y\right|^2\mathrm{d}\gamma(x)\mathrm{d}\kappa\\ =&\frac{1}{4}\int_s^{+\infty}\int_{\partial B_R}\kappa^2\left|\int_{\mathbb{R}^2}H_0^{(1)}(\kappa|x-y|)\left(\nu_x\cdot\nabla_y f(y)\right)\mathrm{d}y\right|^2\mathrm{d}\gamma(x)\mathrm{d}\kappa.\end{aligned}$$

Following a similar argument as the proof of (18), we can obtain

$$\begin{aligned}L_2 &\lesssim n\|f\|^2_{H^{n+1}(B_R)}\int_s^{+\infty}\kappa^{-2n}\mathrm{d}\kappa=\left(\frac{n}{2n-1}\right)\|f\|^2_{H^{n+1}(B_R)}\frac{1}{s^{2n-1}}\\ &\lesssim\frac{1}{s^{2n-1}}\|f\|^2_{H^{n+1}(B_R)}.\end{aligned}\tag{19}$$

Combining (18) and (19) completes the proof of (14) for $d=2$. □

The following lemma is proved in [7].

Lemma 3.4. *Let $J(z)$ be an analytic function in $S=\{z=x+\mathrm{i}y\in\mathbb{C}:-\frac{\pi}{4}<\arg z<\frac{\pi}{4}\}$ and continuous in $\bar{S}$ satisfying*

$$\begin{cases}|J(z)|\le\epsilon, & z\in(0,\ L],\\ |J(z)|\le V, & z\in S,\\ |J(0)|=0.\end{cases}$$

Then there exits a function $\mu(z)$ satisfying

$$\begin{cases}\mu(z)\ge\frac{1}{2}, & z\in(L,\ 2^{\frac{1}{4}}L),\\ \mu(z)\ge\frac{1}{\pi}((\frac{z}{L})^4-1)^{-\frac{1}{2}}, & z\in(2^{\frac{1}{4}}L,\ \infty)\end{cases}$$

such that

$$|J(z)|\le V\epsilon^{\mu(z)},\quad\forall\, z\in(L,\ \infty).$$

Lemma 3.5. *Let $f\in\mathcal{C}_M$. Then there exists a function $\mu(z)$ satisfying*

$$\begin{cases}\mu(s)\ge\frac{1}{2}, & s\in(K,\ 2^{\frac{1}{4}}K),\\ \mu(s)\ge\frac{1}{\pi}((\frac{s}{K})^4-1)^{-\frac{1}{2}}, & s\in(2^{\frac{1}{4}}K,\ \infty),\end{cases}\tag{20}$$

such that

$$|I_1(s)+I_2(s)|\lesssim M^2e^{(4R+1)s}\epsilon^{2\mu(s)},\quad\forall s\in(K,\ \infty).$$

Proof. It follows from Lemma 3.2 that

$$|\left(I_1(s)+I_2(s)\right)e^{-(4R+1)s}|\lesssim M^2,\quad\forall s\in S.$$

Recalling (5), (6)–(9), we have

$$|\,(I_1(s)+I_2(s))\,e^{-(4R+1)s}| \le \epsilon^2, \quad s\in[0,\ K].$$

A direct application of Lemma 3.5 shows that there exists a function $\mu(s)$ satisfying (20) such that

$$|\,(I_1(s)+I_2(s))\,e^{-(4R+1)s}| \lesssim M^2\epsilon^{2\mu}, \quad \forall s\in(K,\ \infty),$$

which completes the proof. □

Now we show the proof of Theorem 2.1.

Proof. We can assume that $\epsilon < e^{-1}$, otherwise the estimate is obvious. Let

$$s=\begin{cases}\frac{1}{((4R+3)\pi)^{\frac{1}{3}}}K^{\frac{2}{3}}|\ln\epsilon|^{\frac{1}{4}} & \text{if } 2^{\frac{1}{4}}((4R+3)\pi)^{\frac{1}{3}}K^{\frac{1}{3}} < |\ln\epsilon|^{\frac{1}{4}},\\ K & \text{if } |\ln\epsilon| \le 2^{\frac{1}{4}}((4R+3)\pi)^{\frac{1}{3}}K^{\frac{1}{3}}.\end{cases}$$

If $2^{\frac{1}{4}}(((4R+3)\pi)^{\frac{1}{3}}K^{\frac{1}{3}} < |\ln\epsilon|^{\frac{1}{4}}$, then we have

$$\begin{aligned}|I_1(s)+I_2(s)| &\lesssim M^2 e^{(4R+3)s}e^{-\frac{2|\ln\epsilon|}{\pi}((\frac{s}{K})^4-1)^{-\frac{1}{2}}}\\ &\lesssim M^2 e^{\frac{(4R+3)}{((4R+3)\pi)^{\frac{1}{3}}}K^{\frac{2}{3}}|\ln\epsilon|^{\frac{1}{4}}-\frac{2|\ln\epsilon|}{\pi}(\frac{K}{s})^2}\\ &\lesssim M^2 e^{-2\left(\frac{(4R+3)^2}{\pi}\right)^{\frac{1}{3}}K^{\frac{2}{3}}|\ln\epsilon|^{\frac{1}{2}}\left(1-\frac{1}{2}|\ln\epsilon|^{-\frac{1}{4}}\right)}.\end{aligned}$$

Noting that

$$\frac{1}{2}|\ln\epsilon|^{-\frac{1}{4}} < \frac{1}{2}, \quad \left(\frac{(4R+3)^2}{\pi}\right)^{\frac{1}{3}} > 1,$$

we have

$$|I_1(s)+I_2(s)| \lesssim M^2 e^{-K^{\frac{2}{3}}|\ln\epsilon|^{\frac{1}{2}}}.$$

Using the elementary inequality

$$e^{-x} \le \frac{(6n-6d+3)!}{x^{3(2n-2d+1)}}, \quad x>0,$$

we get

$$|I_1(s)+I_2(s)| \lesssim \frac{M^2}{\left(\frac{K^2|\ln\epsilon|^{\frac{3}{2}}}{(6n-6d+3)^3}\right)^{2n-2d+1}}. \tag{21}$$

If $|\ln\epsilon| \le 2^{\frac{1}{4}}(((4R+3)\pi)^{\frac{1}{3}}K^{\frac{1}{3}}$, then $s=K$. We have from (5), (6)-(9) that

$$|I_1(s)+I_2(s)| \le \epsilon^2,$$

Here we have noted that

$$I_1(s)+I_2(s)=\int_0^s\int_{\partial B_R}\kappa^{d-1}\big(|\partial_\nu u(x,\kappa)|^2+\kappa^2|u(x,\kappa)|^2\big)\mathrm{d}\gamma\mathrm{d}\kappa, \quad s>0.$$

Hence we obtain from Lemma 3.3 and (21) that

$$\begin{aligned}
&\int_0^\infty \int_{\partial B_R} \kappa^{d-1}\big(|\partial_\nu u(x,\kappa)|^2 + \kappa^2|u(x,\kappa)|^2\big)\mathrm{d}\gamma\mathrm{d}\kappa \\
&\le I_1(s) + I_2(s) + \int_s^\infty \int_{\partial B_R} \kappa^{d-1}\big(|\partial_\nu u(x,\kappa)|^2 + \kappa^2|u(x,\kappa)|^2\big)\mathrm{d}\gamma\mathrm{d}\kappa \\
&\lesssim \epsilon^2 + \frac{M^2}{\left(\frac{K^2|\ln\epsilon|^{\frac{3}{2}}}{(6n-6d+3)^3}\right)^{2n-2d+1}} + \frac{\|f\|^2_{H^{n+1}(B_R)}}{\left(2^{-\frac{1}{4}}((4R+3)\pi)^{-\frac{1}{3}}K^{\frac{2}{3}}|\ln\epsilon|^{\frac{1}{4}}\right)^{2n-2d+1}}.
\end{aligned}$$

By Lemma 3.1, we have

$$\|f\|^2_{L^2(B_R)} \lesssim \epsilon^2 + \frac{M^2}{\left(\frac{K^2|\ln\epsilon|^{\frac{3}{2}}}{(6n-6d+3)^3}\right)^{2n-2d+1}} + \frac{M^2}{\left(\frac{K^{\frac{2}{3}}|\ln\epsilon|^{\frac{1}{4}}}{(R+1)(6n-6d+3)^3}\right)^{2n-2d+1}}.$$

Since $K^{\frac{2}{3}}|\ln\epsilon|^{\frac{1}{4}} \le K^2|\ln\epsilon|^{\frac{3}{2}}$ when $K > 1$ and $|\ln\epsilon| > 1$, we obtain the stability estimate. □

4. **Conclusion.** In this work, we have shown that the increasing stability can be achieved for the inverse source scattering problem by using multi-frequency Dirichlet data on a sphere which encloses the compact support of the source. The stability estimate consists of the data discrepancy and the high frequency tail of the source function. Future work are to investigate the stability with partial data, and to extend the method for solving the inverse source scattering problems for elastic and electromagnetic waves.

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