PAPER

Inverse random source scattering for the Helmholtz equation in inhomogeneous media

To cite this article: Ming Li *et al* 2018 *Inverse Problems* **34** 015003

View the article online for updates and enhancements.

Inverse random source scattering for the Helmholtz equation in inhomogeneous media

Ming Li[1,4]**, Chuchu Chen**[2,5] **and Peijun Li**[3,6,7]

[1] College of Data Science, Taiyuan University of Technology, Taiyuan, Shanxi 030024, People's Republic of China
[2] LSEC, ICMSEC, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, People's Republic of China
[3] Department of Mathematics, Purdue University, West Lafayette, Indiana 47907, United States of America

E-mail: liming01@tyut.edu.cn, chenchuchu@lsec.cc.ac.cn and lipeijun@math.purdue.edu

Abstract

This paper is concerned with an inverse random source scattering problem in an inhomogeneous background medium. The wave propagation is modeled by the stochastic Helmholtz equation with the source driven by additive white noise. The goal is to reconstruct the statistical properties of the random source such as the mean and variance from the boundary measurement of the radiated random wave field at multiple frequencies. Both the direct and inverse problems are considered. We show that the direct problem has a unique mild solution by a constructive proof. For the inverse problem, we derive Fredholm integral equations, which connect the boundary measurement of the radiated wave field with the unknown source function. A regularized block Kaczmarz method is developed to solve the ill-posed integral equations. Numerical experiments are included to demonstrate the effectiveness of the proposed method.

Keywords: inverse source scattering problem, the Helmholtz equation, stochastic partial differential equation

(Some figures may appear in colour only in the online journal)

1. Introduction

The inverse source scattering problem, as an important research area in inverse scattering theory, is to determine the unknown source that generates prescribed radiated wave patten. The problem is largely motivated by medical applications where it is desirable to use electric or magnetic measurements on the surface of the human body, such as head, to infer the source currents inside the body, such as the brain, that produced these measured data [17, 21].

Due to the significant applications, the inverse source problem has been widely studied by many researchers [1, 3–5, 15, 26, 27, 31]. It is known that the problem does not have a unique solution at a fixed frequency due to the existence of nonradiating sources [16, 19]. In addition, it is ill-posed as small variations in the measured data can lead to huge errors in the reconstructions. To overcome these obstacles, one may either seek the minimum energy solution [26], which represents the pseudo-inverse of the problem, or use multi-frequency data to ensure uniqueness and gain increased stability of the solution [10–12, 24].

In this paper, we consider the random source scattering problem for acoustic waves in an inhomogeneous background medium. The wave propagation is modeled by the two-dimensional stochastic Helmholtz equation:

$$\Delta u(x, \kappa) + \kappa^2 (1 + q(x)) u(x, \kappa) = f(x), \quad x \in \mathbb{R}^2, \tag{1.1}$$

where $\kappa > 0$ is the wavenumber, $q > -1$ is the known scatterer and describes the relative electric permittivity of the inhomogeneous medium, and the electric current density f is assumed to be a random function driven by an additive white noise. Specifically, we assume

$$f(x) = g(x) + \sigma(x)\dot{W}_x, \tag{1.2}$$

where g and $\sigma \geqslant 0$ are deterministic real functions, W_x is a Brownian sheet or a two-parameter Brownian motion, and $\dot{W}_x$ denotes the white noise which can be thought as the derivative of the Brownian sheet W_x. A brief introduction can be found in the appendix of [7] on the Brownian sheets. In this random source model, g, σ, and σ^2 can be viewed as the mean, standard deviation, and variance of f, respectively. Moreover, we assume that q, g, and σ have compact supports which are contained in the rectangular domain $D \subset \mathbb{R}^2$. Let $B_R = \{x : |x| < R\}$ be the disk with center 0 and radius R, which is large enough such that $\bar{D} \subset B_R$. As usual, the wave field u is required to satisfy the Sommerfeld radiation condition, i.e.

$$\lim_{r \to \infty} r^{1/2}(\partial_r u - i\kappa u) = 0, \quad r = |x|. \tag{1.3}$$

Given the random source f, the direct problem is to determine the radiated random wave field u. Our focus is on the inverse source scattering problem, which is to determine the source f from the measurement of the radiated wave field u on $\partial B_R = \{x : |x| = R\}$. More precisely, the goal is to determine the statistical properties of the source f, i.e. the mean g and variance σ^2, from the wave field u measured on ∂B_R at a discrete set of wavenumbers $\kappa_j, j = 1, 2, \ldots, N$.

Although the deterministic counterpart has been well studied, little is known for the stochastic inverse problem due to the presence of randomness in the model [14, 22]. Recently, one-dimensional stochastic inverse source problems were considered in [9, 13, 23, 25], where the governing equations were stochastic ordinary differential equations. Utilizing the Green functions, the authors presented the first approach in [7, 8] for solving the inverse random source scattering problem in higher dimensions, where the stochastic partial differential equations were considered. Unfortunately, the technique is not applicable to the Helmholtz equation (1.1) since the explicit Green function is not available for the inhomogeneous background medium.

In this work, we propose a new approach for solving the stochastic inverse source scattering problem in inhomogeneous media. We study both the direct and inverse source scattering problems for the stochastic Helmholtz equation in an inhomogeneous medium. By constructing a sequence of regular processes approximating the white noise, we show that there exists a unique mild solution to the stochastic direct scattering problem. Motivated by [1, 2], we consider an eigenvalue problem for the inhomogeneous Helmholtz equation and deduce integral equations, which connect the scattering data u and unknown source functions g and σ^2. Computationally, we present a regularized block Kaczmarz method to solve the ill-posed Fredholm integral equations by using multi-frequency data. Numerical experiments show that the method is effective to solve the random source scattering problem for both homogeneous and inhomogeneous media.

Let the triple $(\Omega, \mathcal{F}, P)$ be a complete probability space on which the Brownian sheet W_x is defined. Here Ω is a sample space, $\mathcal{F}$ is a σ-algebra on Ω, and P is a probability measure on the measurable space $(\Omega, \mathcal{F})$. If u is a random variable, $\mathbf{E}(u)$ and $\mathbf{V}(u) = \mathbf{E}(u - \mathbf{E}(u))^2 = \mathbf{E}(u^2) - (\mathbf{E}(u))^2$ are the expectation and variance of u, respectively. If u, v are two random variables, $\mathbf{C}(u, v) = \mathbf{E}[(u - \mathbf{E}u)(v - \mathbf{E}v)]$ denotes the covariance of u and v. P-a.s. means almost surely with respect to the probability measure P. We refer to [18, 30] for more details on notations and general theory of stochastic differential equations.

The paper is organized as follows. In section 2, we introduce the model equation and discuss the solutions for the deterministic and stochastic direct problems. Section 3 is devoted to the inverse problem where integral equations are deduced and the regularized block Kaczmarz method is developed to reconstruct the mean and the variance of the random source. Numerical experiments are presented in section 4 to illustrate the performance of the proposed method. The paper is concluded with some general remarks section 5.

2. Direct scattering problem

In this section, we introduce the Helmholtz equation and discuss the solutions of the deterministic and stochastic direct source scattering problems.

2.1. Deterministic direct problem

Letting $\sigma = 0$ in (1.2), we consider the scattering problem of the two-dimensional deterministic Helmholtz equation in an inhomogeneous background medium:

$$\begin{cases} \Delta u(x, \kappa) + \kappa^2(1 + q(x))u(x, \kappa) = g(x), & x \in \mathbb{R}^2, \\ \partial_r u - \mathrm{i}\kappa u = o(r^{-1/2}), & r \to \infty, \end{cases} \quad (2.1)$$

where $g \in L^2(B_R)$ and $q \in L^\infty(B_R)$ are deterministic functions with compact supports contained in $B_R \subset \mathbb{R}^2$. Equivalently, the scattering problem (2.1) can be formulated as the Lippmann–Schwinger integral equation

$$u(x, \kappa) = -\kappa^2 \int_{B_R} G(x, y, \kappa)q(y)u(y, \kappa)\mathrm{d}y + \int_{B_R} G(x, y, \kappa)g(y)\mathrm{d}y, \quad (2.2)$$

where G is the Green function of the homogeneous Helmholtz equation. Explicitly, we have

$$G(x, y, \kappa) = -\frac{\mathrm{i}}{4}H_0^{(1)}(\kappa|x - y|).$$

Here $H_1^{(0)}$ is the Hankel function of the first kind with order zero. Define the integral operator:

$$\mathscr{G}[f](x) = \int_{B_R} G(x, y, \kappa) f(y) \mathrm{d}y, \quad \forall f \in L^2(B_R).$$

The Lippmann–Schwinger integral equation (2.2) can be rewritten as

$$u = -\kappa^2 \mathscr{G}[qu] + \mathscr{G}[g].$$

The following properties on the Green function play an important role in the subsequent analysis. The proofs can be found in [7, lemmas 2.1 and 2.2].

Lemma 2.1.   *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. We have $G(x, y, \kappa) \in L^2(\Omega), \forall y \in \Omega$.*

Lemma 2.2.   *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. We have for any $\alpha \in (\frac{3}{2}, \infty)$ that*

$$\int_\Omega |G(x, y, \kappa) - G(x, z, \kappa)|^\alpha \mathrm{d}x \lesssim |y - z|^{\frac{3}{2}}, \quad \forall y, z \in \Omega. \tag{2.3}$$

Throughout the paper, $a \lesssim b$ stands for $a \leqslant Cb$, where $C > 0$ is a constant. The specific value of C is not required but should be clear from the context.

It is known that the scattering problem (2.1) has a unique solution for all the wavenumbers. The proof can be found in [10, theorem 2.2].

Theorem 2.3.   *For any $\kappa > 0$, the scattering problem (2.1) admits a unique weak solution $u \in H^1(B_R)$, which satisfies*

$$\|u\|_{H^1(B_R)} \lesssim \|g\|_{L^2(B_R)}.$$

2.2. Stochastic direct problem

In this section, we discuss the solution for the stochastic Helmholtz equation in an inhomogeneous medium. Consider the scattering problem for the stochastic Helmholtz equation

$$\begin{cases} \Delta u(x, \kappa) + \kappa^2(1 + q(x))u(x, \kappa) = g(x) + \sigma(x)\dot{W}_x, & x \in \mathbb{R}^2, \\ \partial_r u - \mathrm{i}\kappa u = o(r^{-1/2}), & r \to \infty, \end{cases} \tag{2.4}$$

where $q \in L^\infty(B_R)$, $g \in L^2(B_R)$, and the regularity of σ is chosen such that the stochastic integral

$$\int_{B_R} G(x, y, \kappa)\sigma(y)\mathrm{d}W_y$$

satisfies

$$\mathbf{E}\left(\left|\int_{B_R} G(x, y, \kappa)\sigma(y)\mathrm{d}W_y\right|^2\right) = \int_{B_R} |G(x, y, \kappa)|^2 \sigma^2(y)\mathrm{d}y < \infty,$$

where the Itô isometry is used in the above identity.

We consider the singular part of the Green function. It follows from the Hölder inequality that

$$\int_{B_R} \left|\log\frac{1}{|x - y|}\right|^2 \sigma^2(y)\mathrm{d}y \leqslant \left(\int_{B_R} \left|\log\frac{1}{|x - y|}\right|^{\frac{2p}{p-2}} \mathrm{d}y\right)^{\frac{p-2}{p}} \left(\int_{B_R} |\sigma(y)|^p \mathrm{d}y\right)^{\frac{2}{p}}.$$

Since the first term on the right-hand side of the above inequality is a singular integral, we should choose p such that it is well defined. Let $\rho > 0$ be sufficiently large such that $\bar{B}_R \subset B_\rho(x)$, where $B_\rho(x)$ is the disc with radius ρ and center at x. A simple calculation yields

$$\int_{B_R} \left| \log \frac{1}{|x-y|} \right|^{\frac{2p}{p-2}} \mathrm{d}y \leqslant \int_{B_\rho(x)} \left| \log \frac{1}{|x-y|} \right|^{\frac{2p}{p-2}} \mathrm{d}y \lesssim \int_0^\rho r \left| \log \frac{1}{r} \right|^{\frac{2p}{p-2}} \mathrm{d}r.$$

It is clear to note that the above integral is well defined when $p > 2$.

From now on, we assume that $\sigma \in L^p(B_R)$ where $p \in (2, \infty]$. Moreover, we require that $\sigma \in C^{0,\eta}(B_R)$, i.e. η-Hölder continuous, where $\eta \in (0, 1]$. The Hölder continuity will be used in the analysis for existence of the solution.

The following theorem is the main result for the direct problem of the stochastic Helmholtz equation.

Theorem 2.4. *For any $\kappa > 0$, the stochastic direct problem (2.4) admits a unique continuous stochastic process $u : B_R \to \mathbb{C}$ which satisfies*

$$u(x, \kappa) = -\kappa^2 \mathscr{G}[qu](x) + \mathscr{G}[g](x) + \mathscr{G}[\sigma \dot{W}](x), \quad P\text{-}a.s., \tag{2.5}$$

where

$$\mathscr{G}[\sigma \dot{W}](x) = \int_{B_R} G(x, y, \kappa) \sigma(y) \mathrm{d}W_y.$$

Proof.　First we show that there exists a continuous modification of the random field

$$v(x, \kappa) = \mathscr{G}[\sigma \dot{W}] = \int_{B_R} G(x, y, \kappa) \sigma(y) \mathrm{d}W_y, \quad x \in \Omega.$$

For any $x, z \in \Omega$, we have from the Itô isometry and the Hölder inequality that

$$\mathbf{E}(|v(x, \kappa) - v(z, \kappa)|^2) = \int_{B_R} |G(x, y, \kappa) - G(z, y, \kappa)|^2 \sigma^2(y) \mathrm{d}y$$

$$\leqslant \left(\int_{B_R} |G(x, y, \kappa) - G(z, y, \kappa)|^{\frac{2p}{p-2}} \mathrm{d}y \right)^{\frac{p-2}{p}} \left(\int_{B_R} |\sigma(y)|^p \mathrm{d}y \right)^{\frac{2}{p}}.$$

For $p > 2$, it follows from (2.3) that

$$\int_{B_R} |G(x, y, \kappa) - G(z, y, \kappa)|^{\frac{2p}{p-2}} \mathrm{d}y \lesssim |x-z|^{\frac{3}{2}},$$

which gives

$$\mathbf{E}(|v(x, \kappa) - v(z, \kappa)|^2) \lesssim \|\sigma\|_{L^p(B_R)}^2 |x-z|^{\frac{3p-6}{2p}}.$$

Using the fact that $v(x, \kappa) - v(z, \kappa)$ is a random Gaussian variable, we have (see [20, proposition 3.14]) for any integer m that

$$\mathbf{E}(|v(x, \kappa) - v(z, \kappa)|^{2m}) \lesssim \left(\mathbf{E}(|v(x, \kappa) - v(z, \kappa)|^2) \right)^m \lesssim \|\sigma\|_{L^p(B_R)}^{2m} |x-z|^{\frac{m(3p-6)}{2p}}.$$

Taking $m > \frac{2p}{3p-6}$, we obtain from Kolmogorov's continuity theorem that there exists a P-a.s. continuous modification of the random field v.

Next we present a constructive proof to show the existence. We shall construct a sequence of processes $\dot{W}_x^n$ satisfying $\sigma \dot{W}^n \in L^2(B_R)$ and a sequence

$$v^n(x, \kappa) = \mathscr{G}[\sigma \dot{W}^n](x) = \int_{B_R} G(x, y, \kappa) \sigma(y) \mathrm{d}W_y^n, \quad x \in B_R,$$

which satisfies $v^n \to v$ in $L^2(B_R)$ a.s. as $n \to \infty$.

Recall that σ has a compact support in $\bar{D} \subset B_R$. Let $\mathcal{T}_n = \cup_{j=1}^n K_j$ be a regular triangulation of D, where K_j are triangles. Denote

$$\xi_j = |K_j|^{-\frac{1}{2}} \int_{K_j} \mathrm{d}W_x, \quad 1 \leqslant j \leqslant n,$$

where $|K_j|$ is the area of K_j. It is known from [30] that $\{\xi_j\}_{j=1}^n$ is a family of independent identically distributed normal random variables with mean zero and variance one. We obtain a piecewise constant approximation sequence:

$$\dot{W}_x^n = \sum_{j=1}^n |K_j|^{-\frac{1}{2}} \xi_j \chi_j(x),$$

where χ_j is the characteristic function of K_j. Clearly we have for any $p \geqslant 1$ that

$$\mathbf{E}\left(\|\dot{W}^n\|_{L^p(D)}^p\right) = \mathbf{E}\left(\int_D \left|\sum_{j=1}^n |K_j|^{-\frac{1}{2}} \xi_j \chi_j(x)\right|^p \mathrm{d}x\right) \lesssim \mathbf{E}\left(\int_D \sum_{j=1}^n |K_j|^{-\frac{p}{2}} |\xi_j|^p \chi_j(x) \mathrm{d}x\right)$$

$$= \sum_{j=1}^n \mathbf{E}(|\xi_j|^p) |K_j|^{1-\frac{p}{2}} < \infty,$$

which shows that $\dot{W}^n \in L^p(D), p \geqslant 1$. It follows from the Hölder inequality that $\sigma \dot{W}^n \in L^2(D)$.

Using the Itô isometry, we have

$$\mathbf{E}\left(\int_{B_R} \left|\mathscr{G}[\sigma \dot{W}](x) - \mathscr{G}[\sigma \dot{W}^n](x)\right|^2 \mathrm{d}x\right)$$

$$= \mathbf{E}\left(\int_{B_R} \left|\int_D G(x, y, \kappa) \sigma(y) \mathrm{d}W_y - \int_D G(x, y, \kappa) \sigma(y) \mathrm{d}W_y^n\right|^2 \mathrm{d}x\right)$$

$$= \mathbf{E}\left(\int_{B_R} \left|\sum_{j=1}^n \int_{K_j} G(x, y, \kappa) \sigma(y) \mathrm{d}W_y - \sum_{j=1}^n |K_j|^{-1} \int_{K_j} G(x, z, \kappa) \sigma(z) \mathrm{d}z \int_{K_j} \mathrm{d}W_y\right|^2 \mathrm{d}x\right)$$

$$= \mathbf{E}\left(\int_{B_R} \left|\sum_{j=1}^n \int_{K_j} \int_{K_j} |K_j|^{-1} (G(x, y, \kappa) \sigma(y) - G(x, z, \kappa) \sigma(z)) \mathrm{d}z \mathrm{d}W_y\right|^2 \mathrm{d}x\right)$$

$$= \int_{B_R} \left(\sum_{j=1}^n \int_{K_j} \left||K_j|^{-1} \int_{K_j} (G(x, y, \kappa) \sigma(y) - G(x, z, \kappa) \sigma(z)) \mathrm{d}z\right|^2 \mathrm{d}y\right) \mathrm{d}x$$

$$\leqslant \int_{B_R} \left(\sum_{j=1}^n |K_j|^{-1} \int_{K_j} \int_{K_j} |G(x, y, \kappa) \sigma(y) - G(x, z, \kappa) \sigma(z)|^2 \mathrm{d}z \mathrm{d}y\right) \mathrm{d}x$$

$$= \sum_{j=1}^n |K_j|^{-1} \int_{K_j} \int_{K_j} \int_{B_R} |G(x, y, \kappa) \sigma(y) - G(x, z, \kappa) \sigma(z)|^2 \mathrm{d}x \mathrm{d}z \mathrm{d}y.$$

Using the triangle and Cauchy–Schwartz inequalities yields

$$\int_{B_R} |G(x,y,\kappa)\sigma(y) - G(x,z,\kappa)\sigma(z)|^2 \mathrm{d}x \lesssim \int_{B_R} |G(x,y,\kappa) - G(x,z,\kappa)|^2 |\sigma(y)|^2 \mathrm{d}x$$
$$+ \int_{B_R} |G(x,z,\kappa)|^2 |\sigma(y) - \sigma(z)|^2 \mathrm{d}x.$$

It follows from (2.3), lemma 2.1, and the η-Hölder continuity of σ that

$$\int_{B_R} |G(x,y,\kappa)\sigma(y) - G(x,z,\kappa)\sigma(z)|^2 \mathrm{d}x \lesssim \sigma^2(y)|y-z|^{\frac{3}{2}} + |y-z|^{2\eta},$$

which gives

$$\mathbf{E}\left(\int_{B_R} \left| \mathscr{G}[\sigma \dot{W}](x) - \mathscr{G}[\sigma \dot{W}^n](x) \right|^2 \mathrm{d}x \right)$$
$$\lesssim \sum_{j=1}^n |K_j|^{-1} \int_{K_j}\int_{K_j} \sigma^2(z)|y-z|^{\frac{3}{2}} \mathrm{d}z\mathrm{d}y + \sum_{j=1}^n |K_j|^{-1} \int_{K_j}\int_{K_j} |y-z|^{2\eta} \mathrm{d}z\mathrm{d}y$$
$$\leqslant \|\sigma\|_{L^2(D)}^2 \max_{1\leqslant j\leqslant n} (\mathrm{diam}K_j)^{\frac{3}{2}} + |D| \max_{1\leqslant j\leqslant n} (\mathrm{diam}K_j)^{2\eta} \to 0$$

as $n \to \infty$ since the diameter of $K_j \to 0$ as $n \to \infty$.

For each $n \in \mathbb{N}$, we consider the scattering problem

$$\begin{cases} \Delta u^n(x,\kappa) + \kappa^2(1 + q(x))u^n(x,\kappa) = g(x) + \sigma(x)\dot{W}_x^n, & x \in \mathbb{R}^2, \\ \partial_r u^n - \mathrm{i}\kappa u^n = o(r^{-1/2}), & r \to \infty. \end{cases} \tag{2.6}$$

It follows from $\sigma \dot{W}_x^n \in L^2(B_R)$ and theorem 2.3 that the scattering problem (2.6) has a unique solution $u^n \in H^1(B_R)$ which satisfies the Lippmann–Schwinger integral equation

$$u^n(x,\kappa) = -\kappa^2 \mathscr{G}[qu^n](x) + \mathscr{G}[g](x) + \mathscr{G}[\sigma \dot{W}^n](x). \tag{2.7}$$

Consider the following sequence:

$$u^n(x,\kappa) - u^m(x,\kappa) = -\kappa^2 \mathscr{G}[q(u^n - u^m)](x) + \mathscr{G}[\sigma \dot{W}^n - \sigma \dot{W}^m](x). \tag{2.8}$$

Given $q \in L^\infty(B_R)$, define an integral operator $\mathscr{G}_q : L^2(B_R) \to L^2(B_R)$ by

$$\mathscr{G}_q(\phi)(x) = \mathscr{G}[q\phi](x) = \int_{B_R} G(\kappa,x,y)q(y)\phi(y)\mathrm{d}y, \quad \forall \phi \in L^2(B_R).$$

It is clear to note that the operator $\mathscr{G}_q$ is compact from $L^2(B_R)$ to $L^2(B_R)$ and the equation (2.8) can be written as

$$(\mathscr{I} + \kappa^2 \mathscr{G}_q)(u^n - u^m)(x,\kappa) = \mathscr{G}[\sigma \dot{W}^n - \sigma \dot{W}^m](x), \tag{2.9}$$

where $\mathscr{I}$ is the identity operator. Using the Fredholm alternative theorem and the uniqueness result in theorem 2.3, we obtain that the operator equation (2.9) has a unique solution for any $\kappa > 0$ and the solution satisfies

$$\|u^n - u^m\|_{L^2(B_R)} \lesssim \|\mathscr{G}[\sigma \dot{W}^n - \sigma \dot{W}^m]\|_{L^2(B_R)}.$$

Since $\mathbf{E}(\|\mathscr{G}[\sigma \dot{W}^n - \sigma \dot{W}^m]\|_{L^2(B_R)}^2) \to 0$ as $n, m \to \infty$, the sequence $\{u^n\}$ is a Cauchy sequence, i.e.

$$\mathbf{E}(\|u^n - u^m\|_{L^2(B_R)}^2) \to 0 \quad \text{as } n, m \to \infty,$$

which shows that the sequence $\{u^n\}$ is convergent. Denoting by u the limit of the sequence in (2.7), we obtain the mild solution (2.5).

To show the uniqueness, let u_1, u_2 be two solutions of (2.4) and denote $u = u_1 - u_2$. It is easy to verify that u satisfies

$$u(x, \kappa) = -\kappa^2 \mathscr{G}[qu](x),$$

which is equivalent to the homogeneous operator equation

$$(\mathscr{I} + \kappa^2 \mathscr{G}_q)u(x, \kappa) = 0.$$

It follows from the uniqueness result again in theorem 2.3 that $u = 0$, which completes the proof. □

3. Inverse scattering problem

In this section, we shall derive integral equations connecting the unknown source functions to the data on ∂B_R, and present a regularized block Kaczmarz method to solve the stochastic inverse problem by using multi-frequency data.

3.1. Reconstruction formulas

Consider the inhomogeneous stochastic Helmholtz equation

$$\Delta u(x, \kappa) + \kappa^2(1 + q(x))u(x, \kappa) = g(x) + \sigma(x)\dot{W}_x \quad \text{in } B_R. \tag{3.1}$$

Multiplying the complex conjugate of a smooth test function v on both sides of (3.1) and integrating by parts, we obtain

$$\int_{B_R} u(x)\Delta \bar{v}(x)\mathrm{d}x + \kappa^2 \int_{B_R} (1 + q(x))u(x)\bar{v}(x)\mathrm{d}x + \int_{\partial B_R} (\partial_\nu u(x)\bar{v}(x) - \partial_\nu \bar{v}(x)u(x))\mathrm{d}S$$
$$= \int_{B_R} g(x)\bar{v}(x)\mathrm{d}x + \int_{B_R} \sigma(x)\bar{v}(x)\mathrm{d}W(x), \tag{3.2}$$

where ν is the unit outward normal vector on ∂B_R.

We choose $v \in C^2(B_R)$ to be the eigenfunction for the following problem:

$$\begin{cases} \Delta v(x, \kappa) + \kappa^2(1 + q(x))v(x, \kappa) = 0 & \text{in } B_R, \\ v(x, \kappa) = 0 & \text{on } \partial B_R. \end{cases} \tag{3.3}$$

Substituting such a choice of v into (3.2), we get

$$-\int_{\partial B_R} \partial_\nu \bar{v}(x,\kappa)u(x,\kappa)\mathrm{d}S = \int_{B_R} g(x)\bar{v}(x,\kappa)\mathrm{d}x + \int_{B_R} \sigma(x)\bar{v}(x,\kappa)\mathrm{d}W(x). \quad (3.4)$$

Remark 3.1. We point out that the problem (3.3) falls into the Sturm–Liouville problem [29, chapter 8.6]. Hence there exits a countable set of eigenpairs $(\kappa_j, v_j(x,\kappa_j))_{j=1}^\infty$ with $\kappa_j > 0$ such that the eigenfunctions $\{v_j(x,\kappa_j)\}_{j=1}^\infty$ form an orthonormal basis of the weighted Hilbert space $L_q^2(B_R)$ with respect to the inner product

$$(f,g)_{L_q^2(B_R)} = ((1+q)f, g)_{L^2(B_R)}.$$

If we have a knowledge of u on ∂B_R, or its statistic characters, e.g. the mean and covariance functions, the equation (3.4) clearly provides a connection between the known boundary measurement of the data and the characters of the unknown source functions. Next we present the formulas to reconstruct the mean g and the variance σ^2 of the random source function.

We split the equation (3.4) into the real and imaginary parts:

$$-\mathrm{Re}\int_{\partial B_R} \partial_\nu \bar{v}(x,\kappa)u(x,\kappa)\mathrm{d}S = \mathrm{Re}\int_{B_R} g(x)\bar{v}(x,\kappa)\mathrm{d}x + \mathrm{Re}\int_{B_R} \sigma(x)\bar{v}(x,\kappa)\mathrm{d}W(x) \quad (3.5)$$

and

$$-\mathrm{Im}\int_{\partial B_R} \partial_\nu \bar{v}(x,\kappa)u(x,\kappa)\mathrm{d}S = \mathrm{Im}\int_{B_R} g(x)\bar{v}(x,\kappa)\mathrm{d}x + \mathrm{Im}\int_{B_R} \sigma(x)\bar{v}(x,\kappa)\mathrm{d}W(x). \quad (3.6)$$

Noting that g is a real function and

$$\mathbf{E}\Big(\mathrm{Re}\int_{B_R} \sigma(x)\bar{v}(x,\kappa)\mathrm{d}W(x)\Big) = 0, \quad \mathbf{E}\Big(\mathrm{Im}\int_{B_R} \sigma(x)\bar{v}(x,\kappa)\mathrm{d}W(x)\Big) = 0,$$

we take the expectation on both sides of (3.5) and (3.6) and obtain the Fredholm integral equations to reconstruct g:

$$-\mathbf{E}\Big(\mathrm{Re}\int_{\partial B_R} \partial_\nu \bar{v}(x,\kappa)u(x,\kappa)\mathrm{d}S\Big) = \int_{B_R} g(x)\mathrm{Re}\bar{v}(x,\kappa)\mathrm{d}x, \quad (3.7)$$

$$-\mathbf{E}\Big(\mathrm{Im}\int_{\partial B_R} \partial_\nu \bar{v}(x,\kappa)u(x,\kappa)\mathrm{d}S\Big) = \int_{B_R} g(x)\mathrm{Im}\bar{v}(x,\kappa)\mathrm{d}x. \quad (3.8)$$

It follows from (3.7) and (3.8) and [2] that the mean g can be uniquely determined from the set of eigenfunctions $\{v_j(x,\kappa_j)\}_{j=1}^\infty$ of the eigenvalue problem (3.3). In fact, for any $g \in L^2(B_R)$, we have $g(1+q)^{-1} \in L_q^2(B_R)$ and the generalized Fourier expansion

$$g(1+q)^{-1}(x) = \sum_{j=1}^\infty g_j v_j(x,\kappa_j), \quad (3.9)$$

where the coefficient

$$\begin{aligned} g_j &= (g(1+q)^{-1}, v_j(\cdot,\kappa_j))_{L_q^2(B_R)} \\ &= (g, v_j(\cdot,\kappa_j))_{L^2(B_R)} = -\mathbf{E}\int_{\partial B_R} \partial_\nu \bar{v}_j(\cdot,\kappa_j)u(x,\kappa)\mathrm{d}S. \end{aligned}$$

9

At this point, we have the uniqueness result for the reconstruction of the mean g for the stochastic inverse source problem in an inhomogeneous medium by using multi-frequency scattering data.

Lemma 3.2. *Let $\mathcal{K} = \{\kappa_j^{1/2}\}_{j=1}^{\infty}$ with $\kappa_j > 0$ being the eigenvalues of the problem (3.3). Given the standard deviation function σ, suppose that we have two mean functions g_1 and g_2 such that the expectation of their radiated waves coincide on the boundary ∂B_R for all wavenumbers in $\mathcal{K}$, then $g_1 = g_2$.*

Proof. The proof of this lemma is readily seen from the Fourier expansion (3.9). Let u_1 and u_2 be the random wave fields radiated by the sources (g_1, σ) and (g_2, σ), respectively. Denote $u = \mathbf{E}(u_1 - u_2)$ and $g = g_1 - g_2$. It is easy to verify that u satisfies

$$\begin{cases} \Delta u(x, \kappa) + \kappa^2 (1 + q(x)) u(x, \kappa) = g(x) & \text{in } B_R, \\ u(x, \kappa) = 0 & \text{on } \partial B_R, \end{cases}$$

where we have used the linearity of the Helmholtz equation and the coincidence of the radiated wave fields on the boundary ∂B_R. By the Fourier expansion (3.9), we obtain that all the Fourier coefficients are

$$g_j = -\int_{\partial B_R} \partial_\nu \bar{v}_j(\cdot, \kappa_j) u(x, \kappa_j) \mathrm{d}S = 0,$$

which gives $g = 0$, i.e. $g_1 = g_2$. $\square$

In numerical experiments, the left-hand sides of (3.7) and (3.8) could be approximated by a numerical quadrature, e.g.

$$\mathbf{E}\left(\mathrm{Re} \int_{\partial B_R} \partial_\nu \bar{v}(x, \kappa) u(x, \kappa) \mathrm{d}S\right) = \mathbf{E}\left(\mathrm{Re} \int_0^{2\pi} \partial_r \bar{v}(x, \kappa) u(x, \kappa) R \mathrm{d}\theta\right)$$

$$\approx \mathbf{E}\left[\mathrm{Re} \sum_{m=1}^M \partial_r \bar{v}(x_m, \kappa) u(x_m, \kappa) R \Delta\theta\right] = \mathrm{Re} \sum_{m=1}^M \partial_r \bar{v}(x_m, \kappa) \mathbf{E}(u(x_m, \kappa)) R \Delta\theta$$

and

$$\mathbf{E}\left(\mathrm{Im} \int_{\partial B_R} \partial_r \bar{v}(x, \kappa) u(x, \kappa) \mathrm{d}S\right) = \mathbf{E}\left(\mathrm{Im} \int_0^{2\pi} \partial_r \bar{v}(x, \kappa) u(x, \kappa) R \mathrm{d}\theta\right)$$

$$\approx \mathbf{E}\left[\mathrm{Im} \sum_{m=1}^M \partial_r \bar{v}(x_m, \kappa) u(x_m, \kappa) R \Delta\theta\right] = \mathrm{Im} \sum_{m=1}^M \partial_r \bar{v}(x_m, \kappa) \mathbf{E}(u(x_m, \kappa)) R \Delta\theta,$$

where $\Delta\theta = 2\pi/M$ and $x_m = (R\cos\theta_m, R\sin\theta_m)$ with $\theta_m = m\Delta\theta$.

Therefore, we can reconstruct the mean g from the known data $\mathbf{E}u(x)$, $x \in \partial B_R$ via (3.7) or (3.8) for a set of wavenumbers $\kappa \in \{\kappa_1, \kappa_2, \cdots, \kappa_N\}$.

Recalling that σ is also a real function and using the Itô isometry for stochastic integrals:

$$\mathbf{E}\left|\mathrm{Re} \int_{B_R} \sigma(x) \bar{v}(x, \kappa) \mathrm{d}W(x)\right|^2 = \int_{B_R} \sigma^2(x) |\mathrm{Re}\bar{v}(x, \kappa)|^2 \mathrm{d}x$$

and

$$\mathbf{E}\left|\mathrm{Im}\int_{B_R}\sigma(x)\bar{v}(x,\kappa)\mathrm{d}W(x)\right|^2 = \int_{B_R}\sigma^2(x)|\mathrm{Im}\bar{v}(x,\kappa)|^2\mathrm{d}x,$$

we take the variance on both sides of (3.5) and (3.6) and obtain the Fredholm integral equations to reconstruct the variance σ^2:

$$\mathbf{V}\left(\mathrm{Re}\int_{\partial B_R}\partial_\nu\bar{v}(x,\kappa)u(x,\kappa)\mathrm{d}S\right) = \int_{B_R}\sigma^2(x)|\mathrm{Re}v(x,\kappa)|^2\mathrm{d}x \tag{3.10}$$

and

$$\mathbf{V}\left(\mathrm{Im}\int_{\partial B_R}\partial_\nu\bar{v}(x,\kappa)u(x,\kappa)\mathrm{d}S\right) = \int_{B_R}\sigma^2(x)|\mathrm{Im}v(x,\kappa)|^2\mathrm{d}x, \tag{3.11}$$

where $\mathbf{V}(u) = \mathbf{E}(u^2) - (\mathbf{E}u)^2$ is the variance of the random variable u.

It is severely ill-posed to reconstruct the variance σ^2 as little properties are known for $|v|^2$ where v satisfies (3.3). The uniqueness of the reconstruction of the mean g follows from the fact that the set of eigenfunctions $\{v_n(x,\kappa_n)\}_{n=1}^\infty$ form an orthonormal basis in $L_q^2(B_R)$. However, we do not have such a nice property for the set of functions $\{|v_n(x,\kappa_n)|^2\}_{n=1}^\infty$.

Similarly, the left-hand sides of (3.10) and (3.11) can be approximated by a numerical quadrature:

$$\mathbf{V}\left(\mathrm{Re}\int_{\partial B_R}\partial_\nu\bar{v}(x,\kappa)u(x,\kappa)\mathrm{d}S\right) = \mathbf{V}\left(\mathrm{Re}\int_0^{2\pi}\partial_r\bar{v}(x,\kappa)u(x,\kappa)R\mathrm{d}\theta\right)$$

$$\approx\mathbf{V}\left(\mathrm{Re}\sum_{m=1}^M\partial_r\bar{v}(x_m,\kappa)u(x_m,\kappa)R\Delta\theta\right)$$

$$=\mathbf{V}\left[\sum_{m=1}^M\left(\mathrm{Re}\partial_r\bar{v}(x_m,\kappa)\mathrm{Re}u(x_m,\kappa) - \mathrm{Im}\partial_r\bar{v}(x_m,\kappa)\mathrm{Im}u(x_m,\kappa)\right)R\Delta\theta\right]$$

$$=(R\Delta\theta)^2\sum_{m_1=1}^M\sum_{m_2=1}^M\left\{\mathrm{Re}\partial_r\bar{v}(x_{m_1},\kappa)\mathrm{Re}\partial_r\bar{v}(x_{m_2},\kappa)\mathbf{C}(\mathrm{Re}u(x_{m_1},\kappa),\ \mathrm{Re}u(x_{m_2},\kappa))\right.$$

$$-\,2\mathrm{Re}\partial_r\bar{v}(x_{m_1},\kappa)\mathrm{Im}\partial_r\bar{v}(x_{m_2},\kappa)\mathbf{C}(\mathrm{Re}u(x_{m_1},\kappa),\ \mathrm{Im}u(x_{m_2},\kappa))$$

$$+\left.\mathrm{Im}\partial_r\bar{v}(x_{m_1},\kappa)\mathrm{Im}\partial_r\bar{v}(x_{m_2},\kappa)\mathbf{C}(\mathrm{Im}u(x_{m_1},\kappa),\ \mathrm{Im}u(x_{m_2},\kappa))\right\}$$

and

$$\mathbf{V}\left(\mathrm{Im}\int_{\partial B_R}\partial_r\bar{v}(x,\kappa)u(x,\kappa)\mathrm{d}S\right) = \mathbf{V}\left(\mathrm{Im}\int_0^{2\pi}\partial_r\bar{v}(x,\kappa)u(x,\kappa)R\mathrm{d}\theta\right)$$

$$\approx\mathbf{V}\left(\mathrm{Im}\sum_{m=1}^M\partial_r\bar{v}(x_m,\kappa)u(x_m,\kappa)R\Delta\theta\right)$$

$$=\mathbf{V}\left[\sum_{m=1}^M\left(\mathrm{Re}\partial_r\bar{v}(x_m,\kappa)\mathrm{Im}u(x_m,\kappa) + \mathrm{Im}\partial_r\bar{v}(x_m,\kappa)\mathrm{Re}u(x_m,\kappa)\right)R\Delta\theta\right]$$

$$=(R\Delta\theta)^2\sum_{m_1=1}^M\sum_{m_2=1}^M\left\{\mathrm{Re}\partial_r\bar{v}(x_{m_1},\kappa)\mathrm{Re}\partial_r\bar{v}(x_{m_2},\kappa)\mathbf{C}(\mathrm{Im}u(x_{m_1},\kappa),\ \mathrm{Im}u(x_{m_2},\kappa))\right.$$

$$+\,2\mathrm{Re}\partial_r\bar{v}(x_{m_1},\kappa)\mathrm{Im}\partial_r\bar{v}(x_{m_2},\kappa)\mathbf{C}(\mathrm{Im}u(x_{m_1},\kappa),\ \mathrm{Re}u(x_{m_2},\kappa))$$

$$+\left.\mathrm{Im}\partial_r\bar{v}(x_{m_1},\kappa)\mathrm{Im}\partial_r\bar{v}(x_{m_2},\kappa)\mathbf{C}(\mathrm{Re}u(x_{m_1},\kappa),\ \mathrm{Re}u(x_{m_2},\kappa))\right\}$$

where $\mathbf{C}(u,v) = \mathbf{E}[(u - \mathbf{E}u)(v - \mathbf{E}v)]$ is the covariance of the random wave fields u and v.

Therefore, we can reconstruct the variance σ^2 from the covariance of data $\mathbf{C}(u(x), u(y))$, $x, y \in \partial B_R$ via the linear combination of equations (3.10) and (3.11) for a set of wavenumbers $\kappa \in \{\kappa_1, \kappa_2, \cdots, \kappa_N\}$.

3.2. Numerical method

In this section, we present a regularized block Kaczmarz method to solve the ill-posed integral equations. As an iterative method, the classical Kaczmarz method is used to solve linear systems of algebraic equations [28]. The regularized block Kaczmarz method has also been adopted in [7, 8] for solving the random source scattering problems for acoustic and elastic waves in homogeneous media.

The Fredholm integral equations can be formulated as the following operator equations

$$\mathscr{A}_j y = d_j, \quad j = 1, \ldots, m, \tag{3.12}$$

where the index j is for different wavenumber, y represents the unknown g or σ^2, d_j is the given data, and $\mathscr{A}_j$ is the integral operators. Given an arbitrary initial guess y^0, the classical Kaczmarz method for solving (3.12) reads: For $k = 0, 1, \ldots,$

$$\begin{cases} y_0 = y^k, \\ y_j = y_{j-1} + \mathscr{A}_j^*(\mathscr{A}_j \mathscr{A}_j^*)^{-1}(d_j - \mathscr{A}_j y_{j-1}), \quad j = 1, \ldots, m, \\ y^{k+1} = y_m, \end{cases} \tag{3.13}$$

where $\mathscr{A}_j^*$ is the adjoint operator of A_j. In (3.13), there are two loops: the outer loop is carried for iterative index k and the inner loop is done for the different wavenumber κ_j. Since the operator $\mathscr{A}_j$ is compact, the operator $\mathscr{A}_j \mathscr{A}_j^*$ is ill-pose. A regularization technique is needed.

We propose a regularized Kaczmarz method: Given an arbitrary initial guess y^0,

$$\begin{cases} y_0 = y^k, \\ y_j = y_{j-1} + \mathscr{A}_j^*(\mu \mathscr{I} + \mathscr{A}_j \mathscr{A}_j^*)^{-1}(d_j - \mathscr{A}_j y_{j-1}), \quad j = 1, \ldots, m, \\ y^{k+1} = y_m, \end{cases} \tag{3.14}$$

for $k = 0, 1, \ldots,$ where $\mu > 0$ is the regularization parameter and $\mathscr{I}$ is the identity operator. Although there are two loops in (3.14), the operator $\mu \mathscr{I} + \mathscr{A}_j \mathscr{A}_j^*$ leads to small scale linear system of equations with size equal to the number of measurements after discretization. Moreover, they essentially need to be solved only m times by a direct solver such as the LU decomposition since $\mathscr{A}_j$ does not change during the outer loop. In practice, the initial guess y^0 is usually set to be zero and the Kacmarz method converges to the minimum norm solution. We refer to [8] for the convergence analysis of the regularized block Kaczmarz method.

4. Numerical experiments

In this section, we present two representative numerical examples, one has a homogeneous background medium and another has an inhomogeneous background medium, to demonstrate the validity and effectiveness of the proposed method.

The scattering data is obtained by the numerical solution of the stochastic Helmholtz equation in order to avoid the so-called inverse crime. Although the stochastic Helmholtz equation may be efficiently solved by using the Wiener Chaos expansions to obtain statistical moments such as the mean and variance [6], we choose a different numerical procedure, i.e. Monte Carlo method to simulate the actual process of measuring data. In each realization, the
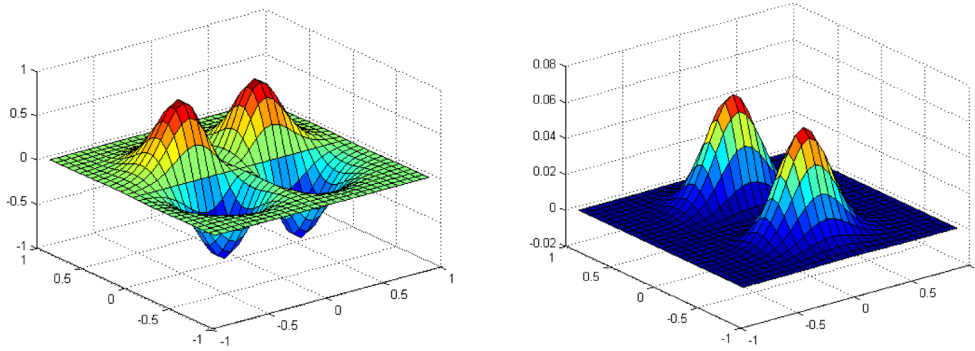
Figure 1. Example 1 (homogeneous background medium): (left) surface plot of the exact mean g; (right) surface plot of the exact variance σ^2.

stochastic Helmholtz equation is solved by using the finite element method with the perfectly matched layer (PML) technique. After all the realizations are done, we take an average of the solutions and use it as an approximated scattering data to either the mean or the covariance. It is clear to note that the data is more accurate as more number of realizations is taken.

Example 1. First we present an example with a homogeneous background medium where the analytic solutions of the eigenfunctions are available. Consider the homogeneous stochastic Helmholtz equation:

$$\Delta u(x, \kappa) + \kappa^2 u(x, \kappa) = g(x) + \sigma(x)\dot{W}_x, \quad x = (x_1, x_2)^\top \in \mathbb{R}^2,$$

where the mean $g(x_1, x_2) = g_1(3x_1, 3x_2)$ and the standard variation $\sigma(x_1, x_2) = \sigma_1(2x_1, 2x_2)$. Here

$$g_1(x_1, x_2) = 5x_1^2 x_2 e^{-x_1^2 - x_2^2}, \quad \sigma_1(x_1, x_2) = x_2^2 e^{-1.5(x_1^2 + x_2^2)}.$$

See figure 1 for the surface plots of the exact g (left) and σ^2 (right) inside the rectangular domain $D = [-1, 1] \times [-1, 1]$. The computational domain is set to be $[-3, 3] \times [-3, 3]$ with the PML thickness 0.5. After the direct problem is solved and the values of the wave field u are obtained at the grid points, the linear interpolation is used to generate the synthetic data at 40 uniformly distributed points on the circle with radius 2, i.e. $x_1 = 2\cos\theta_j, x_2 = 2\sin\theta_j, \theta_j = j\pi/20, j = 0, 1, \ldots, 39$. The corresponding eigenvalue problem is

$$\begin{cases} \Delta v_{mn} + \kappa_{mn}^2 v_{mn} = 0 & \text{in } B_2, \\ v_{mn} = 0 & \text{on } \partial B_2, \end{cases}$$

where B_2 is the disc with center 0 and radius 2. Under the polar coordinates

$$x = (x_1, x_2)^\top = (r\cos\theta, r\sin\theta)^\top, \quad 0 \leqslant r \leqslant R := 2, 0 \leqslant \theta \leqslant 2\pi,$$

the normalized eigenfunctions are given explicitly as

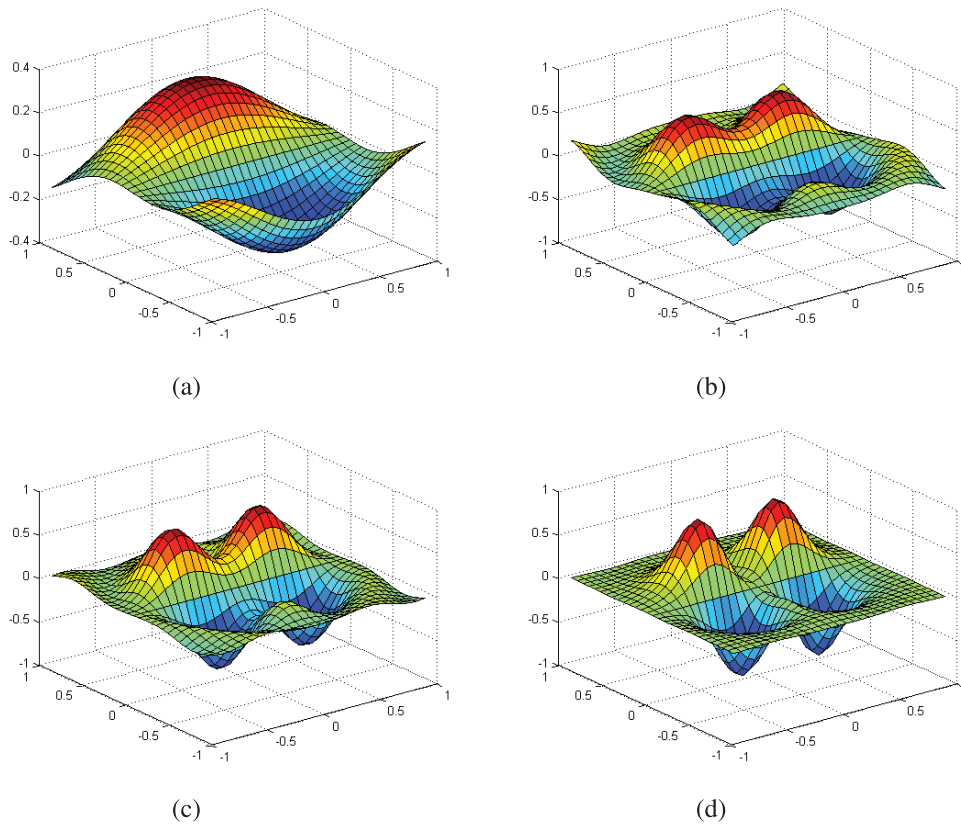$$v_{mn}(r, \theta) = \frac{1}{RJ_{n+1}(z_{mn})} J_n\left(\frac{z_{mn}r}{R}\right) e^{in\theta}$$

Figure 2. Example 1 (homogeneous background medium): surface plots of the reconstructed mean g by using different number of eigenfunctions. (a) $N = 2$. (b) $N = 4$. (c) $N = 5$. (d) $N = 8$.

for $n = 0, 1, 2, \cdots$ and $m = 1, 2, 3, \cdots$. Here J_n denotes the Bessel function of the first kind with order n and z_{mn} the mth zero of J_n. It is clear to note that

$$\kappa_{mn} = \frac{z_{mn}}{R}.$$

A simple calculation yields that

$$\partial_r v_{mn}(R, \theta) = -\frac{z_{mn}}{R^2} e^{in\theta}.$$

We use a total number of N^2 ($n = 0, 1, \cdots, N - 1$; $m = 1, 2, \cdots, N$) wavenumbers to reconstruct the mean g and the variance σ^2. The regularization parameter is $\mu = 1.0 \times 10^{-7}$ and the total number of the outer loop for the Kaczmarz method is 5. Figures 2 and 3 show the reconstructions of the mean g and the variance σ^2 by using different N. As is expected, more accurate results can be obtained as more eigenfunctions are used for the reconstruction.

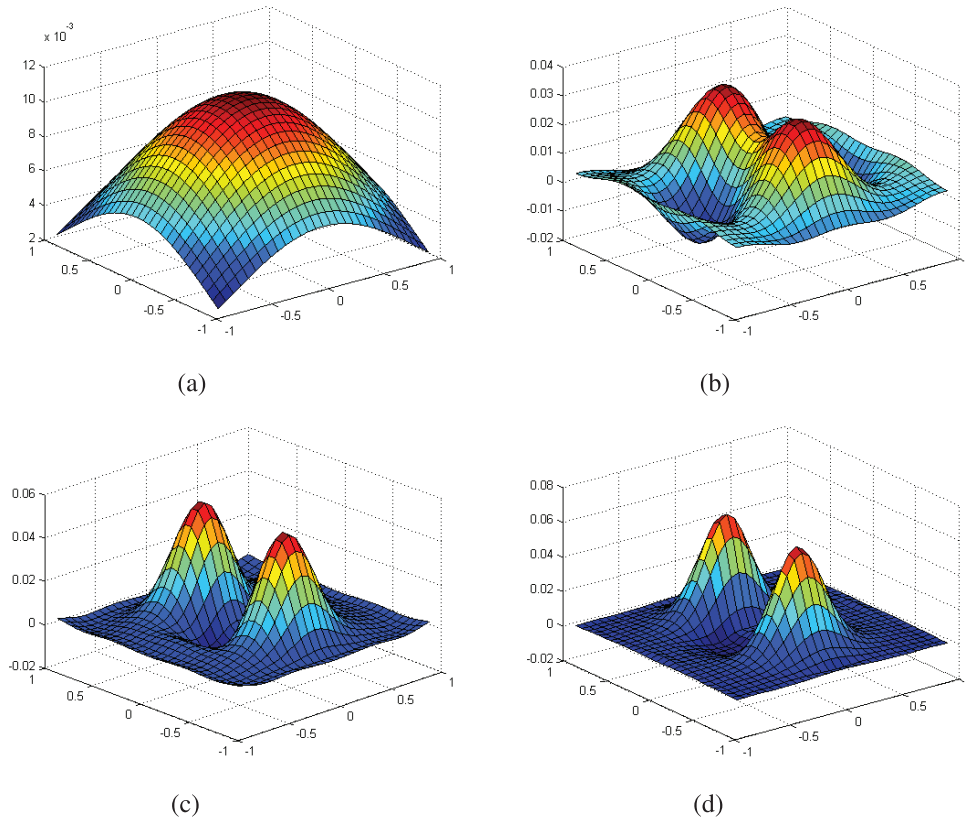**Figure 3.** Example 1 (homogeneous background medium): surface plots of the reconstructed variance σ^2 by using different number of eigenfunctions. (a) $N = 1$. (b) $N = 2$. (c) $N = 3$. (d) $N = 10$.

Example 2.   Now we present an example with an inhomogeneous background medium. Consider the inhomogeneous Helmholtz equation:

$$\Delta u(x, \kappa) + \kappa^2(1 + q(x))u(x, \kappa) = g(x) + \sigma(x)\dot{W}_x, \quad x = (x_1, x_2)^\top \in \mathbb{R}^2,$$

where the scatterer $q(x) = e^{-(3x_1)^2 - (3x_2)^2}$. Let

$$g_2(x_1, x_2) = 0.3(1 - x_1)^2 e^{-x_1^2 - (x_2+1)^2} - (0.2x_1 - x_1^3 - x_2^5)e^{-x_1^2 - x_2^2} - 0.03e^{-(x_1+1)^2 - x_2^2}$$

and

$$\sigma(x_1, x_2) = 0.6e^{-8(r^3 - 0.75r^2)}, \quad r = (x_1^2 + x_2^2)^{1/2},$$

and reconstruct the mean $g(x_1, x_2) = g(3x_1, 3x_2)$ and the variance σ^2 inside the rectangular domain $D = [-1, 1] \times [-1, 1]$. See figure 4 for the surface plots of the exact mean g (left) and variance σ^2 (right). Again, the computational domain is set to be $[-3, 3] \times [-3, 3]$ with the PML thickness 0.5. After the direct problem is solved and the value of u is obtained at the grid points, the linear interpolation is used to generate the synthetic data at 40 uniformly distributed
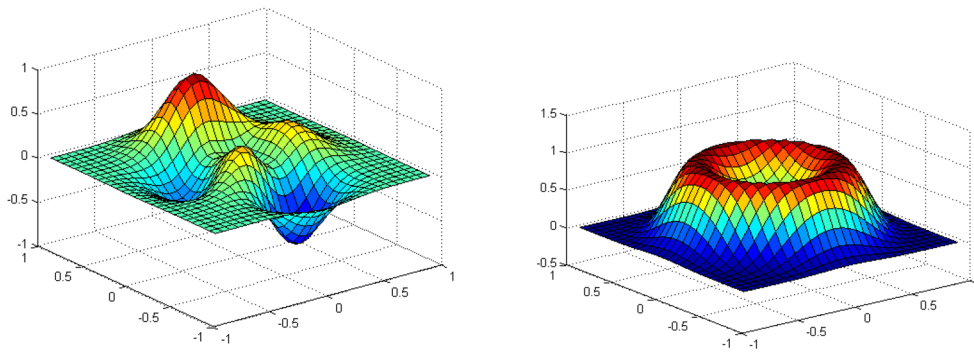
Figure 4. Example 2 (inhomogeneous background medium): (left) surface plot of the exact mean g; (right) surface plot of the exact variance σ^2.
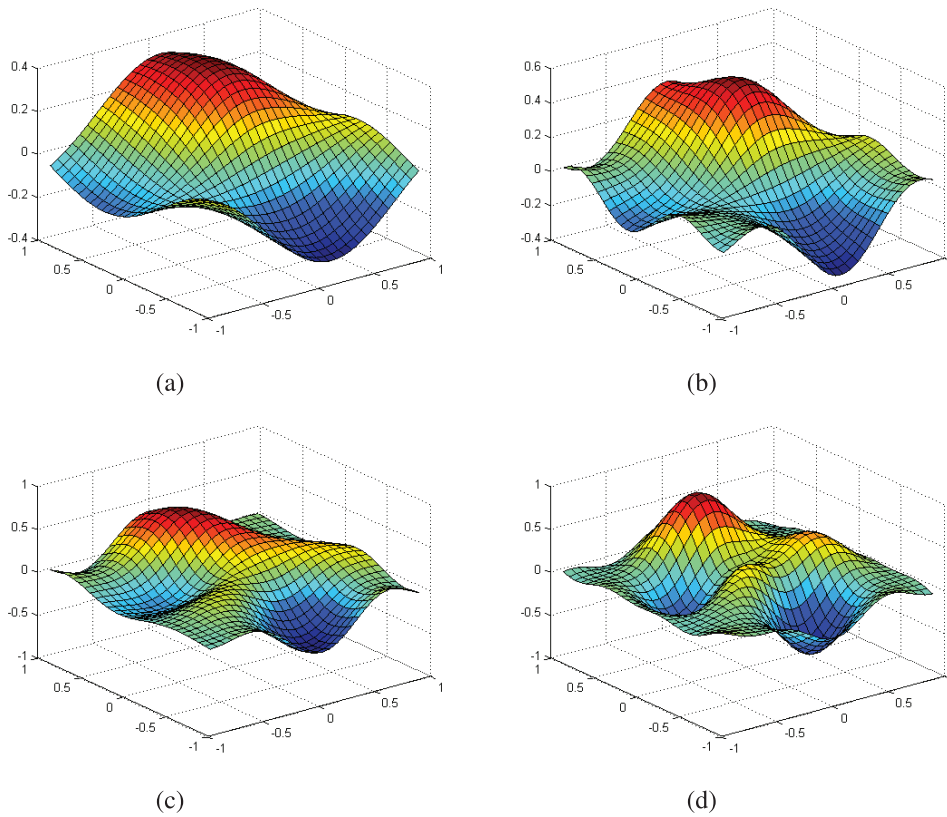

(a)

(b)

(c)

(d)

Figure 5. Example 2 (inhomogeneous background medium): surface plots of the reconstructed mean g by using different number of eigenfunctions. (a) $N = 15$. (b) $N = 30$. (c) $N = 45$. (d) $N = 60$.
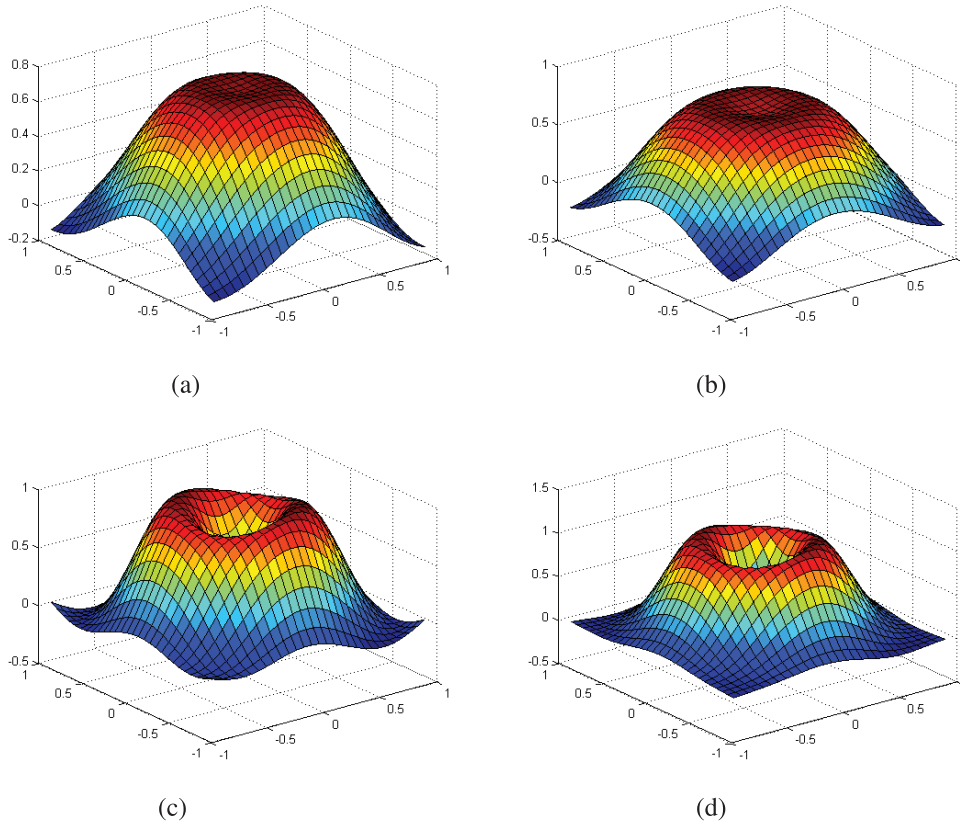
Figure 6. Example 2 (inhomogeneous background medium): surface plots of the reconstructed variance σ^2 by using different number of eigenfunctions. (a) $N = 5$. (b) $N = 8$. (c) $N = 10$. (d) $N = 15$.

points on the circle with radius 2, i.e. $x_1 = 2\cos\theta_j, x_2 = 2\sin\theta_j, \theta_j = j\pi/20, i = 0, 1, \ldots, 39$. The corresponding eigenvalue problem is

$$\begin{cases} \Delta v_n + \kappa_n^2(1 + q)v_n = 0 & \text{in } B_2, \\ v_n = 0 & \text{on } \partial B_2. \end{cases}$$

where $n = 1, 2, \ldots, N$. Due to the inhomogeneous background medium, we solve the above eigenvalue problem numerically since the analytic solution is not available. A total number of N wavenumbers are used to reconstruct the mean g and the variance σ^2. The regularization parameter is $\mu = 1.0 \times 10^{-7}$ and the total number of the outer loop for the Kaczmarz method is 5. Figures 5 and 6 plot the reconstructions of the mean g and the variance σ^2, respectively, at a different number of N.

5. Conclusion

We have studied the inverse random source scattering problem in an inhomogeneous background medium, where the wave propagation is governed by stochastic Helmholtz equation.

As a source, the electric current density is assumed to be a random function driven by additive white noise. Under a suitable regularity assumptions of the relative electric permittivity q, the mean, g and the standard deviation σ of the source, the direct scattering problem is shown constructively to have a unique mild solution. Using the corresponding eigenvalue problem for the Helmholtz equation, we deduce Fredholm integral equations which connect the unknown source with the known measurement of the radiated wave field for the inverse scattering problem to reconstruct the mean and the variance of the random source. We present a regularized block Kaczmarz method to solve the ill-posed integral equations by using multiple frequency data. Numerical examples, one homogeneous background medium and one inhomogeneous background medium, are shown to demonstrate the validity and effectiveness of the proposed method.

ORCID iDs

Peijun Li ⬤ https://orcid.org/0000-0001-5119-6435

References

[1] Eller M and Valdivia N P 2009 Acoustic source identification using multiple frequency information *Inverse Problems* **25** 115005
[2] Acosta S, Chow S, Taylor J and Villamizar V 2012 On the multi-frequency inverse source problem in heterogeneous media *Inverse Problems* **28** 075013
[3] Albanese R and Monk P 2006 The inverse source problem for Maxwell's equations *Inverse Problems* **22** 1023–35
[4] Ammari H, Bao G and Fleming J 2002 An inverse source problem for Maxwell's equations in magnetoencephalography *SIAM J. Appl. Math.* **62** 1369–82
[5] Badia A and Nara T 2011 An inverse source problem for Helmholtz's equation from the Cauchy data with a single wave number *Inverse Problems* **27** 105001
[6] Badieirostami M, Adibi A, Zhou H-M and Chow S-N 2010 Wiener chaos expansion and simulation of electromagnetic wave propagation excited by a spatially incoherent source *Multiscale Model. Simul.* **8** 591–604
[7] Bao G, Chen C and Li P 2016 Inverse random source scattering problems in several dimensions *SIAM/ASA J. Uncertain. Quantification* **4** 1263–87
[8] Bao G, Chen C and Li P Inverse random source scattering for elastic waves *SIAM J. Numer. Anal.* **55** 2616–43
[9] Bao G, Chow S-N, Li P and Zhou H-M 2014 An inverse random source problem for the Helmholtz equation *Math. Comput.* **83** 215–33
[10] Bao G, Li P, Lin J and Triki F 2015 Inverse scattering problems with multi-frequencies *Inverse Problems* **31** 093001
[11] Bao G, Lin J and Triki F 2010 A multi-frequency inverse source problem *J. Diff. Equa.* **249** 3443–65
[12] Bao G, Lu S, Rundell W and Xu B 2015 A recursive algorithm for multifrequency acoustic inverse source problems *SIAM J. Numer. Anal.* **53** 1608–28
[13] Bao G and Xu X 2013 An inverse random source problem in quantifying the elastic modulus of nano-materials *Inverse Problems* **29** 015006
[14] Devaney A 1979 The inverse problem for random sources *J. Math. Phys.* **20** 1687–91
[15] Devaney A, Marengo E and Li M 2007 Inverse source problem in nonhomogeneous background media *SIAM J. Appl. Math.* **67** 1353–78
[16] Devaney A and Sherman G 1982 Nonuniqueness in inverse source and scattering problems *IEEE Trans. Antennas Propag.* **30** 1034–7
[17] Fokas A, Kurylev Y and Marinakis V 2004 The unique determination of neuronal currents in the brain via magnetoencephalogrphy *Inverse Problems* **20** 1067–82
[18] Friedman A 2006 *Stochastic Differential Equations and Applications* (New York: Academic)

[19] Hauer K-H, Kühn L and Potthast R 2005 On uniqueness and non-uniqueness for current reconstruction from magnetic fields *Inverse Problems* **21** 955–67

[20] Hairer M 2009 *Introduction to Stochastic PDEs (Lecture Notes)*

[21] Isakov V 1989 *Inverse Source Problems* (Providence, RI: American Mathematical Society)

[22] Kaipio J and Somersalo E 2005 *Statistical and Computational Inverse Problems* (New York: Springer)

[23] Li P 2011 An inverse random source scattering problem in inhomogeneous media *Inverse Problems* **27** 035004

[24] Li P and Yuan G 2017 Increasing stability for the inverse source scattering problem with multi-frequencies *Inverse Problems Imaging* **11** 745–59

[25] Li P and Yuan G 2017 Stability on the inverse random source scattering problem for the one-dimensional Helmholtz equation *J. Math. Anal. Appl.* **450** 872–87

[26] Marengo E and Devaney A 1999 The inverse source problem of electromagnetics: linear inversion formulation and minimum energy solution *IEEE Trans. Antennas Propag.* **47** 410–2

[27] Marengo E, Khodja M and Boucherif A 2008 Inverse source problem in nonhomogeneous backgroud media. Part II: vector formulation and antenna substrate performance characterization *SIAM J. Appl. Math.* **69** 81–110

[28] Natterer F 1986 *The Mathematics of Computerized Tomography* (Stuttgart: Teubner)

[29] Renardy M and Rogers R 2004 *An Introduction to Partial Differential Equations* (*Texts in Applied Mathematics* vol 13) 2nd edn (New York: Springer)

[30] Walsh J 1986 *An Introduction to Stochastic Partial Differential Equations* (New York: Springer)

[31] Zhang D and Guo Y 2015 Fourier method for solving the multi-frequency inverse acoustic source problem for the Helmholtz equation *Inverse Problems* **31** 035007