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On an inverse source problem for the Biot equations in electro-seismic imaging*

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Abstract

Electro-seismic imaging is a novel hybrid imaging modality in geophysical exploration where electromagnetic wave and seismic wave are coupled in porous medium. This paper concerns with an inverse source problem for the Biot's equations arising in electro-seismic imaging. Using the time reversal method, we derive an explicit reconstruction formula, which immediately gives the uniqueness and stability of the reconstructed source.

Keywords: electro-seismic, hybrid modality, inverse source problem, Biot's equations, time reversal, reconstruction

1. Introduction

Electro-seismic (ES) imaging and *seismo-electric (SE)* imaging are emerging modalities in geophysical exploration. They provide images of high accuracy with low cost in comparison with traditional modalities, and have been applied in locating groundwater acquifers and petroleum hydrocarbon reservoir. The goal of this article is to study an inverse problem arising in the mathematical model of the ES imaging.

The underlying physical phenomena of ES and SE imaging are known as *electro-seismic conversion* and *seismo-electric conversion*, respectively. These conversions usually occur in a porous medium, that is a solid skeletal material saturated with fluid electrolyte. When the solid is positively charged and the liquid negatively charged, or vice versa, the charges tend to

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move and gather at the interface, forming an *electrical double layer*, typically of 10 nm scale, with the two sides oppositely charged. This double layer then leads to the aforementioned two types of conversion according to electrodynamics. More specifically, as electromagnetic wave propagates through the medium, the electric field acts on the charges and generates pressure difference and consequently seismic disturbance. This is the ES conversion. On the other hand, external seismic disturbance could cause rearrangement of the charge distribution, producing an electric field. This is the SE conversion. These conversions have been used as bore-hole logging and cross-hole logging tools [13, 14, 19, 36–38] and measurements has been recorded and studied in [6, 12, 16, 22–24, 29, 33, 34].

Both ES and SE imaging involve conversions of electromagnetic energy and seismic energy. The governing equations were derived by Pride [27]:

$$\nabla \times E = -\mu \partial_t H, \tag{1}$$

$$\nabla \times H = (\epsilon \partial_t + \sigma)E + L(-\nabla p - \rho_{12}\partial_t^2 \mathbf{u}^{\mathbf{s}}), \tag{2}$$

$$\rho_{11}\partial_t^2 \mathbf{u}^{\mathbf{s}} + \rho_{12}\partial_t^2 \mathbf{u}^{\mathbf{f}} = \operatorname{div} \tau, \tag{3}$$

$$\rho_{12}\partial_t^2 \mathbf{u}^{\mathbf{s}} + \rho_{22}\partial_t^2 \mathbf{u}^{\mathbf{f}} + \frac{\eta}{\kappa}\partial_t \mathbf{u}^{\mathbf{f}} + \nabla p = \frac{\eta}{\kappa} LE, \tag{4}$$

$$\tau = (\lambda \operatorname{div} \mathbf{u}^{\mathbf{s}} + q \operatorname{div} \mathbf{u}^{\mathbf{f}}) I_3 + \mu(\nabla \mathbf{u}^{\mathbf{s}} + (\nabla \mathbf{u}^{\mathbf{s}})^T), \tag{5}$$

$$-p = q \operatorname{div} \mathbf{u}^{\mathbf{s}} + r \operatorname{div} \mathbf{u}^{\mathbf{f}}, \tag{6}$$

where E is the electric field, H is the magnetic field, $\mathbf{u}^s$ is the solid displacement, $\mathbf{u}^f$ is the fluid displacement, μ is the magnetic permeability, ϵ is the dielectric constant, σ is the conductivity, L is the electro-kinetic mobility parameter, ρ_{11} is a linearly combined density of the solid and the fluid, ρ_{12} is the density of the pore fluid, ρ_{22} is the mass coupling coefficient, κ is the fluid flow permeability, η is the viscosity of pore fluid, λ and μ are the Lamé parameters, q and r are the Biot moduli parameters, τ is the bulk stress tensor, p is the pore pressure, I_3 is the 3×3 identity matrix. Equations (1) and (2) are Maxwell's equations, modeling the electromagnetic wave propagation. Equations (3)–(6) are Biot's equations, describing the seismic wave propagation in the porous medium [4, 5].

In this work, we focus on the ES imaging. The inverse problem in ES imaging concerns recovery of the physical parameters in (1)–(6) from boundary measurement of the displacements $\mathbf{u}^{\mathbf{s}}$ and $\mathbf{u}^{\mathbf{f}}$. This problem can be studied in two mutually relevant steps. The first step concerns an inverse source problem for the Biot equations (3)–(6) to recover the coupling term $\frac{\eta}{\kappa}LE$. The second step utilizes this term as internal measurement to retrieve physical parameters in the Maxwell equations (1) and (2). The second step has been considered in [8, 9].

The first step, however, has not been well understood mathematically. In an unpublished work of Chen and de Hoop [7], they suggested using Gassmann's approximation [17] to reduce Biot's equations to the elastic equation and then applying the result in [35]. This approach has the limitation that Gassmann's approximation is valid only when the fluid permeability is small and the wave frequency is sufficiently low. An inverse source problem for the Biot's equations was investigated in [2] in a different context by assuming access to internal measurement rather than boundary measurement. Recently, the work [3] studied the electro-seismic model (1)–(6) and derived a Hölder stability estimate for recovery of the electric parameters and the coupling coefficient from internal measurement near the boundary.

Our goal is to demonstrate a general approach to the first step and complete the two-step approach towards the coupled physics inverse problem in ES imaging. We study the inverse source problem from boundary measurement of $\mathbf{u}^{\mathbf{s}}$ and $\mathbf{u}^{\mathbf{f}}$ directly for Biot's equations, without resorting to Gassmann's approximation or any internal measurement. We derive an explicit reconstruction formula in terms of a Neumann series. Uniqueness and stability of the reconstructed solution are immediate consequences of the explicit formula.

2. Problem formulation and main result

We make two simplifications to the Biot equations (3)–(6). First, in this paper we only study the corresponding non-attenuated model, that is, we ignore the attenuation term $\frac{\eta}{\kappa}\partial_t\mathbf{u}^{\mathbf{f}}$ in (4). Second, we assume that the source term $\frac{\eta}{\kappa}LE$ can be separated into a known temporal component $h(t)$ and an unknown spatial component $f(x)$, that is

$$\frac{\eta(x)}{\kappa(x)}L(x)E(t,x) = h(t)f(x).$$

We deal with the delta pulse $h(t) = \delta(t)$ in theorem 1. The more general case where $h(t)$ is a continuous function can be readily derived, see corollary 2.

Let us rewrite Biot's equations (without attenuation) in different forms for subsequent analysis. It follows from (5) to (6) that we have

$$\operatorname{div}\tau = \Delta_{\mu,\lambda}\mathbf{u}^{\mathbf{s}} + \nabla(q\operatorname{div}\mathbf{u}^{\mathbf{f}}), \quad \nabla p = -\nabla(q\operatorname{div}\mathbf{u}^{\mathbf{s}}) - \nabla(r\operatorname{div}\mathbf{u}^{\mathbf{f}}),$$

where the elastic operator $\Delta_{\mu,\lambda}$ is defined by

$$\begin{aligned}\Delta_{\mu,\lambda}\mathbf{u}^{\mathbf{s}} &:= \operatorname{div}(\mu(\nabla\mathbf{u}^{\mathbf{s}} + (\nabla\mathbf{u}^{\mathbf{s}})^T)) + \nabla(\lambda\operatorname{div}\mathbf{u}^{\mathbf{s}}),\\ &= \mu\Delta\mathbf{u}^{\mathbf{s}} + (\mu+\lambda)\nabla(\operatorname{div}\mathbf{u}^{\mathbf{s}}) + (\operatorname{div}\mathbf{u}^{\mathbf{s}})\nabla\lambda + (\nabla\mathbf{u}^{\mathbf{s}} + (\nabla\mathbf{u}^{\mathbf{s}})^T)\nabla\mu.\end{aligned} \tag{7}$$

Then the Biot equations (without attenuation) can be written as

$$\begin{cases}\rho_{11}\partial_t^2\mathbf{u}^{\mathbf{s}} + \rho_{12}\partial_t^2\mathbf{u}^{\mathbf{f}} - \Delta_{\mu,\lambda}\mathbf{u}^{\mathbf{s}} - \nabla(q\operatorname{div}\mathbf{u}^{\mathbf{f}}) = 0\\ \rho_{12}\partial_t^2\mathbf{u}^{\mathbf{s}} + \rho_{22}\partial_t^2\mathbf{u}^{\mathbf{f}} - \nabla(q\operatorname{div}\mathbf{u}^{\mathbf{s}}) - \nabla(r\operatorname{div}\mathbf{u}^{\mathbf{f}}) = \delta(t)f(x).\end{cases} \tag{8}$$

One can further write the system (8) into a matrix equation. Let

$$M = \begin{pmatrix}\rho_{11}I_3 & \rho_{12}I_3\\ \rho_{12}I_3 & \rho_{22}I_3\end{pmatrix}, \qquad P(D) = \begin{pmatrix}-\Delta_{\mu,\lambda}\cdot & -\nabla(q\operatorname{div}\cdot)\\ -\nabla(q\operatorname{div}\cdot) & -\nabla(r\operatorname{div}\cdot)\end{pmatrix}. \tag{9}$$

The matrix form of (8) is

$$M\partial_t^2\mathbf{u} + P(D)\mathbf{u} = \delta(t)\mathbf{f}$$

which, by Duhamel's principle, is equivalent to the initial value problem:

$$\begin{cases}M\partial_t^2\mathbf{u} + P(D)\mathbf{u} = \mathbf{0} & \text{in } \mathbb{R}\times\mathbb{R}^3,\\ \mathbf{u}|_{t=0} = \mathbf{f}(\mathbf{x}),\\ \partial_t\mathbf{u}|_{t=0} = \mathbf{0}.\end{cases} \tag{10}$$

Hereafter, a vector with 6 components is written in bold face such as $\mathbf{u}$. It is often split into two 3-vectors such as $\mathbf{u} = (\mathbf{u}^{\mathbf{s}}, \mathbf{u}^{\mathbf{f}}) \in \mathbb{R}^3\times\mathbb{R}^3$. The names of the two 3-vectors are chosen to be consistent with the splitting for solutions of the Biot equations. Following this notation, the right hand side is defined as $\mathbf{f} = (\mathbf{f}^{\mathbf{s}}, \mathbf{f}^{\mathbf{f}}) := (0, f)$. Hence we will not refer to the specific form any longer but instead consider a general $\mathbf{f}$.

Well-posedness of the initial value problem (10) can be proved in a similar way as [2, theorem 1.1]. More precisely, for $\mathbf{f} \in H^2(\Omega) \cap H_0^1(\Omega)$, there is a solution $\mathbf{u} = (\mathbf{u}^s, \mathbf{u}^f)$ to (10) such that

$$\mathbf{u}^s \in \mathcal{C}([-T,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([-T,T]; H^1(\Omega)) \cap C^2([-T,T]; L^2(\Omega))$$
$$\mathbf{u}^f \in C^2([-T,T]; L^2(\Omega)), \qquad \operatorname{div} \mathbf{u}^f \in C([-T,T]; H^1(\Omega)) \cap C^1([-T,T]; L^2(\Omega))$$

and the following estimate holds for some constant $C > 0$:

$$\|\mathbf{u}\|_{C^2([-T,T];L^2(\Omega))} \leqslant C\|\mathbf{f}\|_{H^2(\Omega)}.$$

Higher regularity on $\mathbf{f}$ leads to smoother $\mathbf{u}$ as usual.

The following hypothesis are crucial and assumed throughout the paper.

(**H1**): The functions ρ_{11}, ρ_{12}, ρ_{22}, μ, λ, q, r are bounded from below by a constant, say $c > 0$, and have bounded derivatives.
(**H2**): $\rho_{11}\rho_{22} - \rho_{12}^2 > 0$ in $\mathbb{R}^3$.
(**H3**): $\lambda r - q^2 > 0$ in $\mathbb{R}^3$.

Here (**H1**) states that all the physical parameters are positive and sufficiently smooth. (**H2**) ensures that the matrix M is positive definite. (**H3**) guarantees that the energy functional defined in section 5 is non-negative.

In ES imaging, the measurement is the solid displacement $\mathbf{u}^{\mathbf{s}}$ and the fluid displacement $\mathbf{u}^{\mathbf{f}}$ on the boundary of a domain-of-interest Ω which is a smooth bounded open subset in $\mathbb{R}^3$. Introduce the source-to-measurement operator $\mathbf{\Lambda}$ as follows:

$$\mathbf{\Lambda f} := \mathbf{u}|_{[0,T]\times\partial\Omega}$$

where $\mathbf{u}$ is the solution of the initial value problem (10) and $T > 0$ represents a duration of the measurement. Note that the initial displacement occurs inside the domain-of-interest, which means that $\mathbf{f}$ is supported in the interior of Ω. We are interested in recovering information on $\mathbf{f}$ from the boundary measurement $\mathbf{\Lambda f}$.

A quick look at the equation (8) suggests that there is a certain gauge transform for the recovery of $\mathbf{u}^{\mathbf{f}}$. In fact, adding to $\mathbf{u}^{\mathbf{f}}$ by any vector field g that is divergence free (i.e, $\operatorname{div} g = 0$) and vanishes on the boundary (i.e. $g|_{\partial\Omega} = 0$) does not affect (8) and $\mathbf{\Lambda f}$. This is a reflection of the simple fact that $\mathbf{u}^{\mathbf{f}}$ appears in the equations only in the form of $\operatorname{div} \mathbf{u}^{\mathbf{f}}$. Taking such gauge into account, we raise the following question which is the central topic of this paper.

2.1. Inverse source problem

Suppose that M and $P(D)$ are known and satisfy the hypotheses (**H1**)–(**H3**), can one reconstruct the initial source $\mathbf{f} = (\mathbf{f}^{\mathbf{s}}, \mathbf{f}^{\mathbf{f}})$, compactly supported in Ω, from the boundary measurement $\mathbf{\Lambda f}$ up to a pair of vector fields $(0, g)$ with g divergence free and $g|_{\partial\Omega} = 0$?

We give an affirmative answer to the question, including a reconstruction formula. Our main result can be summarized as follows. A rigorous restatement of this theorem is given in theorem 11.

Theorem 1. *Under appropriate assumptions, the initial source* $\mathbf{f} \in H^2(\Omega) \cap H_0^1(\Omega)$ *can be uniquely and stably reconstructed from* $\mathbf{\Lambda f}$ *by a convergent Neumann series, up to a pair of vector fields* $(0, g)$ *with* $\operatorname{div} g = 0$ *and* $g|_{\partial\Omega} = 0$.

Our proof is based on the modified time reversal method proposed by Stefanov and Uhlmann [30] for thermo-acoustic tomography (TAT). TAT is a hybrid modality in medical imaging where optical or electromagnetic waves are exerted to trigger ultrasound wave in tissue through thermo-elastic conversion. Conventional time reversal method is known to provide an approximate reconstruction of the source [20]. Stefanov and Uhlmann improved it and obtained an accurate reconstruction of the source by a Neumann series [30] which was numerically implemented in [10, 26]. This improved time reversal method has since been adapted and generalized to many other models [11, 18, 21, 25, 31, 35]. We refer to [1] for a survey on more hybrid modalities in the context of medical imaging.

Remark. A slight variation of our approach can be used to reconstruct $\mathbf{f}(\mathbf{x})$ when the source term takes the form $L_0\frac{\eta}{\kappa}E = \delta'(t)\mathbf{f}(\mathbf{x})$. Indeed, it follows from Duhamel's principle that we have

$$\begin{cases} M\partial_t^2\mathbf{u} + P(D)\mathbf{u} &= \mathbf{0} \qquad \text{in } \mathbb{R}\times\mathbb{R}^3,\\ \mathbf{u}|_{t=0} &= \mathbf{0},\\ \partial_t\mathbf{u}|_{t=0} &= \mathbf{f}(\mathbf{x}). \end{cases}$$

If $\mathbf{u}$ solves this problem, the $\partial_t\mathbf{u}$ solves (10) by [32]. Hence one can first reconstruct $\partial_t\mathbf{u}$ using our approach and then integrate to obtain $\mathbf{u}$.

Theorem 1 can be extended to a general source of the form

$$\frac{\eta(x)}{\kappa(x)}L(x)E(t,x) = h(t)f(x)$$

with $h(t)\in C([0,T])$ a known continuous function and $f(x)$ an unknown function to be recovered. In this case, the Biot's equations are reduced as before to the following system

$$\begin{cases} M\partial_t^2\mathbf{u} + P(D)\mathbf{u} &= h(t)\mathbf{f}(\mathbf{x}) \qquad \text{in } \mathbb{R}\times\mathbb{R}^3,\\ \mathbf{u}|_{t=0} &= \mathbf{0},\\ \partial_t\mathbf{u}|_{t=0} &= \mathbf{0} \end{cases} \tag{11}$$

with $\mathbf{f}(\mathbf{x}) = (0, f(x))$. The measurement is $\mathbf{u}|_{[0,T]\times\partial\Omega}$.

Corollary 2. *Let $h(t)\in C([0,T])$ be a known continuous function. Under the same assumptions of theorem 1, the spatial function $\mathbf{f}\in H^2(\Omega)\cap H^1_0(\Omega)$ can be determined from $\mathbf{u}|_{[0,T]\times\partial\Omega}$ where $\mathbf{u}$ is the solution of (11), up to a pair of vector fields $(0,g)$ with $\operatorname{div} g = 0$ and $g|_{\partial\Omega} = 0$.*

Our proof of the corollary leads to an explicit reconstruction approach, see Section 6. We would like to remark that the sources considered in (11), which can be written as the product of a spatial function and a temporal function, have been considered in the inverse problems literature [39–42].

The rest of the paper is organized as follows. In section 3 we convert Biot's equations (without attenuation) to two hyperbolic systems. These systems are used in section 4 to prove finite speed of propagation and unique continuation results. Section 5 is devoted to discussion of function spaces. Theorem 1 and corollary 2 are proved in section 6.

3. Biot's equations

In this section we transform Biot's equations into two hyperbolic systems: a principally scalar system and a symmetric hyperbolic system.

3.1. The principally scalar system

Let us start with the principally scalar system. Recall the definition of a principally scalar system [15]. For a function $a = a(x)$, define the scalar wave operator $\Box_a := a\partial_t^2 - \Delta$.

Definition 3. A principally scalar system refers to

$$\square_{a_j} u_j + b_j(t,x,\nabla u) + c_j(t,x,u) = f_j, \qquad j = 1,\ldots,m. \tag{12}$$

Here $u = (u_1,\ldots,u_m)$, $a_j = a_j(x) \in C^1$ are real-valued functions, b_j and c_j are linear functions with L^∞_{loc}-coefficients of ∇u and u.

The system is called principally scalar as the principal part of each equation is a scalar wave operator. We shall see in section 4 that a principally scalar systems can be uniquely continued towards inside from proper boundary data. This unique continuation property is crucial to obtain proposition 8.

We can write the equation in (10) as a principally scalar system by following the procedures in [2]. Let $\rho(x) = \rho_{11}(x)\rho_{22}(x) - \rho_{12}^2(x)$. We have $\rho > 0$ by (**H2**) and

$$M^{-1} = \frac{1}{\rho}\begin{pmatrix} \rho_{22} I_3 & -\rho_{12} I_3 \\ -\rho_{12} I_3 & \rho_{11} I_3 \end{pmatrix}.$$

Multiplying the equation in (10) by M^{-1} to get

$$\partial_t^2 \begin{pmatrix} \mathbf{u}^{\mathbf{s}} \\ \mathbf{u}^{\mathbf{f}} \end{pmatrix} + M^{-1}P(D)\begin{pmatrix} \mathbf{u}^{\mathbf{s}} \\ \mathbf{u}^{\mathbf{f}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where

$$M^{-1}P(D) = \frac{1}{\rho}\begin{pmatrix} -\rho_{22}\Delta_{\mu,\lambda}\cdot + \rho_{12}\nabla(q\,\mathrm{div}\,\cdot) & -\rho_{22}\nabla(q\,\mathrm{div}\,\cdot) + \rho_{12}\nabla(r\,\mathrm{div}\,\cdot) \\ \rho_{12}\Delta_{\mu,\lambda}\cdot - \rho_{11}\nabla(q\,\mathrm{div}\,\cdot) & \rho_{12}\nabla(q\,\mathrm{div}\,\cdot) - \rho_{11}\nabla(r\,\mathrm{div}\,\cdot). \end{pmatrix}.$$

Now, we write the matrix equation as a system of equations and move all the first and zeroth order derivatives to the right hand side to get

$$\begin{cases} \partial_t^2 \mathbf{u}^{\mathbf{s}} - \mu_1 \Delta \mathbf{u}^{\mathbf{s}} - (\mu_1 + \lambda_1)\nabla \mathrm{div}\,\mathbf{u}^{\mathbf{s}} - q_1 \nabla \mathrm{div}\,\mathbf{u}^{\mathbf{f}} &= \mathscr{P}(\mathbf{u}^{\mathbf{s}}, \mathbf{u}^{\mathbf{f}}), \\ \partial_t^2 \mathbf{u}^{\mathbf{f}} + \mu_2 \Delta \mathbf{u}^{\mathbf{s}} - r_2 \nabla(\mathrm{div}\,\mathbf{u}^{\mathbf{f}}) - q_2 \nabla(\mathrm{div}\,\mathbf{u}^{\mathbf{s}}) &= \mathscr{Q}(\mathbf{u}^{\mathbf{s}}, \mathbf{u}^{\mathbf{f}}), \end{cases} \tag{13}$$

where the new coefficients are

$$\mu_1 = \rho^{-1}\rho_{22}\mu, \quad \lambda_1 = \rho^{-1}(\rho_{22}\lambda - \rho_{12}q), \quad q_1 = \rho^{-1}(\rho_{22}q - \rho_{12}r),$$

$$\mu_2 = \rho^{-1}\rho_{12}\mu, \quad r_2 = \rho^{-1}(\rho_{11}r - \rho_{12}q), \quad q_2 = \rho^{-1}(\rho_{11}q - \rho_{12}(\mu + \lambda)).$$

Here and below, the script letters $\mathscr{P}, \mathscr{Q}, \mathscr{R}, \ldots$ denote various first-order linear differential operators.

A straightforward calculation shows that first order derivatives in $\mathscr{P}(\mathbf{u}^{\mathbf{s}}, \mathbf{u}^{\mathbf{f}})$ and $\mathscr{Q}(\mathbf{u}^{\mathbf{s}}, \mathbf{u}^{\mathbf{f}})$ appear in terms of $\nabla \mathbf{u}^{\mathbf{s}}$, $(\nabla \mathbf{u}^{\mathbf{s}})^T$, $\mathrm{div}\,\mathbf{u}^{\mathbf{s}}$, and $\mathrm{div}\,\mathbf{u}^{\mathbf{f}}$. The left hand side can be simplified by the substitutions as indicated below.

Introduce the substitutions:

$$v^s = \mathrm{div}\,\mathbf{u}^{\mathbf{s}}, \quad v^f = \mathrm{div}\,\mathbf{u}^{\mathbf{f}}, \quad \mathbf{v}^{\mathbf{s}} = \mathrm{curl}\,\mathbf{u}^{\mathbf{s}}. \tag{14}$$

Applying div to (13) and utilizing the relations

$$\mathrm{div}\,\nabla \mathbf{u}^{\mathbf{s}} = \Delta \mathbf{u}^{\mathbf{s}} = \nabla(\mathrm{div}\,\mathbf{u}^{\mathbf{s}}) - \mathrm{curl}\ \mathrm{curl}\ \mathbf{u}^{\mathbf{s}}, \quad \mathrm{div}\,(\nabla \mathbf{u}^{\mathbf{s}})^T = \nabla(\mathrm{div}\,\mathbf{u}^{\mathbf{s}}),$$

we get

$$\begin{cases} \partial_t^2 v^s - a_{11}\Delta v^s - a_{12}\Delta v^f &= \mathscr{R}(v^s, v^f, \mathbf{u}^{\mathbf{s}}, \mathbf{v}^{\mathbf{s}}) \\ \partial_t^2 v^f - a_{21}\Delta v^s - a_{22}\Delta v^f &= \mathscr{S}(v^s, v^f, \mathbf{u}^{\mathbf{s}}, \mathbf{v}^{\mathbf{s}}); \end{cases}$$

or equivalently

$$\partial_t^2 \begin{pmatrix} v^s \\ v^f \end{pmatrix} - A(x)\Delta \begin{pmatrix} v^s \\ v^f \end{pmatrix} = \mathscr{L}(v^s, v^f, \mathbf{u^s}, \mathbf{v^s}),$$

where the coefficient $A(x)$ is

$$A(x) := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{1}{\rho} \begin{pmatrix} \rho_{22} & -\rho_{12} \\ -\rho_{12} & \rho_{11} \end{pmatrix} \cdot \begin{pmatrix} 2\mu + \lambda & q \\ q & r \end{pmatrix}.$$

The hypotheses (**H1**)–(**H3**) ensures that A is a symmetric positive definite matrix. Let a_1, a_2 be its eigenvalues, then there exists a non-singular matrix $Q(x)$ such that

$$(Q^{-1}AQ)(x) = \mathrm{Diag}\,(a_1, a_2)(x).$$

Making the change of variable

$$(\tilde{v}^s, \tilde{v}^f) = Q^{-1}(v^s, v^f)$$

yields that $(\tilde{v}^s, \tilde{v}^f)$ solves

$$\begin{cases} \partial_t^2 \tilde{v}^s - a_1 \Delta \tilde{v}^s &= \mathscr{M}(\tilde{v}^s, \tilde{v}^f, \mathbf{u^s}, \mathbf{v^s}), \\ \partial_t^2 \tilde{v}^f - a_2 \Delta \tilde{v}^f &= \mathscr{N}(\tilde{v}^s, \tilde{v}^f, \mathbf{u^s}, \mathbf{v^s}). \end{cases}$$

Applying curl to the first equation of (13) and using curl $\nabla \mathbf{u} = 0$ gives

$$\partial_t^2 \mathbf{v^s} - \mu_1 \Delta \mathbf{v^s} = \mathscr{T}(\tilde{v}^s, \tilde{v}^f, \mathbf{u^s}, \mathbf{v^s}).$$

The first equation of (13) can be written as

$$\partial_t^2 \mathbf{u^s} - \mu_1 \Delta \mathbf{u^s} = \mathscr{U}(\tilde{v}^s, \tilde{v}^f, \mathbf{u^s}, \mathbf{v^s}).$$

Thus we obtain the following principally scalar system in the variables $(\tilde{v}^s, \tilde{v}^f, \mathbf{u^s}, \mathbf{v^s})$:

$$\begin{cases} \square_{\frac{1}{a_1}} \tilde{v}^s - \frac{1}{a_1}\mathscr{M}(\tilde{v}^s, \tilde{v}^f, \mathbf{u^s}, \mathbf{v^s}) &= 0, \\ \square_{\frac{1}{a_2}} \tilde{v}^f - \frac{1}{a_2}\mathscr{N}(\tilde{v}^s, \tilde{v}^f, \mathbf{u^s}, \mathbf{v^s}) &= 0, \\ \square_{\frac{1}{\mu_1}} \mathbf{u^s} - \frac{1}{\mu_1}\mathscr{T}(\tilde{v}^s, \tilde{v}^f, \mathbf{u^s}, \mathbf{v^s}) &= 0, \\ \square_{\frac{1}{\mu_1}} \mathbf{u^f} - \frac{1}{\mu_1}\mathscr{U}(\tilde{v}^s, \tilde{v}^f, \mathbf{u^s}, \mathbf{v^s}) &= 0. \end{cases} \tag{15}$$

This is the desired principally scalar system. Note that a_1, a_2, μ_1 are smooth and strictly positive by (**H1**), hence their reciprocals exist and are smooth as well. Another observation is that $(\tilde{v}^s, \tilde{v}^f) = 0$ if and only if $(v^s, v^f) = 0$.

3.2. The symmetric hyperbolic system

We proceed to write the principally scalar system (15) as a first order symmetric hyperbolic system [28], which will be exploited to show that the Biot equations have finite speed of propagation. We restrict to the case where the coefficients are time-independent matrices.

For $x \in \mathbb{R}^3$, let $A_0(x), \ldots, A_4(x)$ be matrix-valued functions. Denote by $\mathcal{L}$ the partial differential operator

$$\mathcal{L}(x, \partial_t, \partial_x) = A_0(x)\partial_t + \sum_{j=1}^{3} A_j(x)\partial_{x_j} + A_4(x), \tag{16}$$

where the coefficient matrices $A_0, \ldots, A_4$ are assumed to have uniformly bounded derivatives, i.e.

$$\sup_{x\in\mathbb{R}^3} \|\partial_x^\alpha (A_0(x), \ldots, A_4(x))\| < \infty \quad \text{for any multi} - \text{index}\alpha.$$

Definition 4. $\mathcal{L}$ is called symmetric hyperbolic if the following conditions hold:

(i) $A_0(x), \ldots, A_3(x)$ are symmetric;
(ii) A_0 is strictly positive, i.e. there is a constant $C > 0$ such that for all $x \in \mathbb{R}^3$,

$$A_0(x) \geqslant CI,$$

where I is the identity matrix.

Our goal is to convert the principally scalar system (15) to a symmetric hyperbolic system. We begin with a scalar wave equation to demonstrate the procedures. The principally scalar system is treated afterwards.

Let $v(x)$ be a scalar function defined in $\mathbb{R}^3$ and satisfy the equation

$$\partial_t^2 v - a_1(x)\Delta v = \mathscr{L}(\partial_{x_1} v, \partial_{x_2} v, \partial_{x_3} v, v), \tag{17}$$

where $\mathscr{L}(\partial_{x_1} v, \partial_{x_2} v, \partial_{x_3} v, v)$ is linear in each derivative. We define a vector $V(v)$:

$$V(v) = (V_0, V_1, V_2, V_3, V_4) := (\partial_t v, \partial_{x_1} v, \partial_{x_2} v, \partial_{x_3} v, v).$$

The scalar equation (17) can be written in terms of $V_0, \ldots, V_4$ as follows:

$$\begin{cases} \partial_t V_0 - a_1\partial_{x_1} V_1 - a_1\partial_{x_2} V_2 - a_1\partial_{x_3} V_3 - \mathscr{L}(V_1, \ldots, V_4) &= 0, \\ a_1\partial_t V_1 - a_1\partial_{x_1} V_0 &= 0, \\ a_1\partial_t V_2 - a_1\partial_{x_2} V_0 &= 0, \\ a_1\partial_t V_3 - a_1\partial_{x_3} V_0 &= 0, \\ a_1\partial_t V_4 - a_1 V_0 &= 0. \end{cases}$$

In the matrix form, this system reads

$$B_0(x)\partial_t V + B_1(x)\partial_{x_1} V + B_2(x)\partial_{x_2} V + B_3(x)\partial_{x_3} V + B_4(x)V = 0,$$

where $B_0, \ldots, B_4$ are 5×5 matrices. More explicitly, $B_0 = \mathrm{Diag}(1, a_1, a_1, a_1, a_1)$, B_4 is determined by the concrete form of $\mathscr{L}$, and the other matrices are

$$B_i(x) := \begin{cases} -a_1 & \text{for } (1, i+1) \text{ and } (i+1, 1) \text{ entry} \\ 0 & \text{for other entries} \end{cases}, \quad i = 1, 2, 3.$$

Note this is a symmetric hyperbolic system since $B_0, \ldots, B_3$ are symmetric matrices and B_0 is strictly positive.

Now we turn to the principally scalar system (15). As each equation in the system takes the form (17), we define a vector U, which has 40 components and is obtained by juxtaposing $V(v)$ with v replaced successively by the components of $(\tilde{v}^s, \tilde{v}^f, \mathbf{u}^s, \mathbf{v}^s)$, i.e.

$$U = (U_1, \ldots, U_{40}) := (V(\tilde{v}^s), V(\tilde{v}^f), V(\mathbf{u}_1^s), V(\mathbf{u}_2^s), V(\mathbf{u}_3^s), V(\mathbf{v}_1^s), V(\mathbf{v}_2^s), V(\mathbf{v}_3^s)).$$

Since the principal part of each equation in (15) is uncoupled, one can write (15) as a first order system in a similar manner:

$$A_0(x)\partial_t U + A_1(x)\partial_{x_1} U + A_2(x)\partial_{x_2} U + A_3(x)\partial_{x_3} U + A_4(x)U = 0. \tag{18}$$

Here each A_i is a 40×40 matrix, A_0 represents the diagonal matrix

$$\mathrm{Diag}\,((1, a_1, a_1, a_1, a_1), (1, a_2, a_2, a_2, a_2), (1, \mu_1, \mu_1, \mu_1, \mu_1), \ldots, (1, \mu_1, \mu_1, \mu_1, \mu_1))$$

which is strictly positive, and A_1, A_2, A_3 are symmetric. This is the desired symmetric hyperbolic system that is equivalent to the principally scalar system (15) and the Biot equations in (10).

4. Finite speed of propagation and unique continuation

We derive some results for the Biot equations regarding finite speed of propagation and unique continuation. It is crucial to have the symmetric hyperbolic system (18) and the principally scalar system (15).

4.1. Finite speed of propagation

Let $\mathcal{L}$ be the symmetric hyperbolic operator defined as in (16). For any $\xi \in \mathbb{R}^3 \backslash \{0\}$, let

$$\Lambda(\xi) = \inf\{\ell : A_0(x)^{-\frac{1}{2}} \left(\sum_{j=1}^{3} A_j(x)\xi_j \right) A_0(x)^{-\frac{1}{2}} \leqslant \ell I\},$$

which is the smallest upper bound of the eigenvalues for $A_0^{-\frac{1}{2}} \left(\sum_{j=1}^{3} A_j \xi_j \right) A_0^{-\frac{1}{2}}$. We state the following result, which shows that the solution of a symmetric hyperbolic system has finite speed of propagation.

Proposition 5. *[28, theorem 2.3.2] Suppose $s \in \mathbb{R}$ and $u \in C([0, \infty); H^s(\mathbb{R}^3))$ satisfies a symmetric hyperbolic system $\mathcal{L}u = 0$. If*

$$\mathrm{supp}\, u(0, x) \subset \{x \in \mathbb{R}^3 : x \cdot \xi \leqslant 0\},$$

then for $t \geqslant 0$,

$$\mathrm{supp}\, u(t, x) \subset \{x \in \mathbb{R}^3 : x \cdot \xi \leqslant \Lambda(\xi) t\}.$$

In particular, set $c_{\max} := \max_{|\xi|=1} \Lambda(\xi)$, if

$$\mathrm{supp}\, u(0, x) \subset \{x \in \mathbb{R}^3 : |x| \leqslant R\},$$

then

$$\mathrm{supp}\, u(t, x) \subset \{x \in \mathbb{R}^3 : |x| \leqslant R + c_{\max}|t|\}.$$

Here $|\xi|$ denotes the Euclidean norm of ξ. If u is a vector, $\mathrm{supp}\, u$ stands for the union of the supports of its components.

Using this proposition, one can deduce finite speed of propagation for the solutions of Biot's equation. Let $c_{\max}$ be the number in proposition 5 with $\mathcal{L}$ defined on the left hand side of (18).

Corollary 6. *Let $\mathbf{u} = (\mathbf{u}^{\mathrm{s}}, \mathbf{u}^{\mathbf{f}})$ solve $M\partial_t^2 \mathbf{u} + P(D)\mathbf{u} = \mathbf{0}$. If*

$$\mathrm{supp}\,(\mathbf{u}^{\mathrm{s}}(0, \cdot), \mathrm{div}\, \mathbf{u}^{\mathbf{f}}(0, \cdot)) \subset \{x \in \mathbb{R}^3 : |x| \leqslant r\},$$

then

$$\operatorname{supp}(\mathbf{u}^{\mathbf{s}}(t,\cdot), \operatorname{div}\mathbf{u}^{\mathbf{f}}(t,\cdot)) \subset \{x \in \mathbb{R}^3 : |x| \leqslant r + c_{\max}|t|\}.$$

Here the support of a vector field is defined to the the union of the supports of its components.

Proof. Given $(\mathbf{u}^{\mathbf{s}}, \mathbf{u}^{\mathbf{f}})$, one can construct the functions $(\tilde{v}^s, \tilde{v}^f, \mathbf{u}^{\mathbf{s}}, \mathbf{v}^{\mathbf{s}})$ as in (14), and further the solution U of (18). The proof is completed by applying proposition 5 to the symmetric hyperbolic system (18). □

4.2. Unique continuation

A principally scalar system satisfies certain unique continuation property [15], which will be used as an intermediate step towards the main theorem. Let $T' > 0$ be a positive number, and let $D \subset \mathbb{R}^3$ be a C^2-domain containing the origin. Denote by $B(0;R) = \{x \in \mathbb{R}^3 : |x| < R\}$ the ball of radius R centered at the origin. We state the following unique continuation result.

Proposition 7. *[15, corollary 3.5] Let $u = (u_1, \dots, u_m)$ be a solution of a general principally scalar system (12). Suppose that there exists $\theta > 0$ such that $D \subset B(0;\theta T')$ and the coefficients a_j in (12) satisfy the constraints:*

$$\begin{aligned} \theta^2 a_j(a_j + a_j^{-\frac{1}{2}}|t\nabla a_j|) &< a_j + \tfrac{1}{2}x \cdot \nabla a_j && \text{in } (-T',T') \times \overline{D} \\ \theta^2 a_j &\leqslant 1 && \text{in } \overline{D}. \end{aligned}$$

Then $u = \partial_\nu u = 0$ on $(-T',T') \times \partial D$ implies $u = 0$ on $\{(t,x) \in (-T',T') \times D : |x| > \theta t\}$.

The inequality constraints in the theorem justify the pseudo-convexity of a certain phase function with respect to the wave operators $\Box_{a_j}$ [15]. Observe that if a_j are positive constants, the above constraints reduce to $\theta^2 a_j < 1$ in $\overline{D}$. In this case, any $\theta > 0$ that is less than $\min\{\frac{1}{\sqrt{a_j}} : j = 1, \cdots, m\}$ fulfills the inequalities.

Next proposition is the main result of this section. Choose $R > 0$ sufficiently large such that $\Omega \subset B(0;R)$. Recall that $c_{\max}$ is defined in corollary 5.

Proposition 8. *Let $\mathbf{u} = (\mathbf{u}^{\mathbf{s}}, \mathbf{u}^{\mathbf{f}})$ be the solution of the forward problem (10) with $\mathbf{f}$ compactly supported in Ω. Suppose there exists $\theta > 0$ such that $B(0;R + c_{\max}\frac{3T}{2}) \subset B(0;\theta T)$ and the following inequality constraints hold when a is replaced by $a_1(x)$, $a_2(x)$, $\mu_1(x)$ respectively:*

$$\begin{aligned} \theta^2 a(a + a^{-\frac{1}{2}}|t\nabla a|) &< a + \tfrac{1}{2}x \cdot \nabla a, && \text{in } (-\tfrac{3T}{2}, \tfrac{3T}{2}) \times \overline{B(0;R + c_{\max}\tfrac{3T}{2})} \\ \theta^2 a &\leqslant 1 && \text{in } \overline{B(0;R + c_{\max}\tfrac{3T}{2})}. \end{aligned}$$

If $\mathbf{u}^{\mathbf{s}}(T,x) = \operatorname{div}\mathbf{u}^{\mathbf{f}}(T,x) = 0$ for $x \in \mathbb{R}^3 \backslash \Omega$, then

$$\mathbf{u}^{\mathbf{s}}(0,x) = \operatorname{div}\mathbf{u}^{\mathbf{f}}(0,x) = 0 \quad \text{for } x \in \Omega, x \neq 0.$$

Proof. In view of the initial conditions in (10), we can extend the solution $\mathbf{u}$ as an even function of t to $(-T,T) \times \Omega$. This extension is denoted by $\mathbf{u}$ again.

Given $\mathbf{u}^{\mathbf{s}}(T,x) = \operatorname{div}\mathbf{u}^{\mathbf{f}}(T,x) = 0$ for $x \in \mathbb{R}^3 \backslash \Omega$, it follows from corollary 6 that

$$\mathbf{u}^{\mathbf{s}}(t,x) = \operatorname{div}\mathbf{u}^{\mathbf{f}}(t,x) = 0 \quad \text{in } \{(t,x) : |x| \geqslant R + c_{\max}|t - T|\}.$$

As $\mathbf{u}$ is an even function of t, we also have

$$\mathbf{u}^{\mathbf{s}}(t,x) = \operatorname{div}\mathbf{u}^{\mathbf{f}}(t,x) = 0 \quad \text{in } \{(t,x) : |x| \geqslant R + c_{\max}|t+T|\}.$$

On the other hand, the fact that $\mathbf{f}$ has compact support in Ω implies $\mathbf{u}^{\mathbf{s}}(0,x) = \operatorname{div}\mathbf{u}^{\mathbf{f}}(0,x) = 0$ for $x \in \mathbb{R}^3\backslash\Omega$. Hence by corollary 6, we get

$$\mathbf{u}^{\mathbf{s}}(t,x) = \operatorname{div}\mathbf{u}^{\mathbf{f}}(t,x) = 0 \quad \text{in } \{(t,x) : |x| \geqslant R + c_{\max}|t|\}.$$

Combing the above equations gives

$$\mathbf{u}^{\mathbf{s}}(t,x) = \operatorname{div}\mathbf{u}^{\mathbf{f}}(t,x) = 0 \quad \text{on } \{(t,x) : |x| = R + c_{\max}\frac{T}{2}, -\frac{3T}{2} < t < \frac{3T}{2}\}.$$

Using proposition 7 with the choice $T' := \frac{3T}{2}$, $D := B(0, R + c_{\max}\frac{T}{2})$ yields

$$\mathbf{u}^{\mathbf{s}}(0,x) = \operatorname{div}\mathbf{u}^{\mathbf{f}}(0,x) = 0 \quad \text{for } x \in \Omega, x \neq 0,$$

which completes the proof. □

5. Energy conservation

Define a Sobolev space

$$H(\operatorname{div};\Omega) := \{\mathbf{v}^{\mathbf{f}} \in (L^2(\Omega))^3 : \operatorname{div}\mathbf{v}^{\mathbf{f}} \in L^2(\Omega)\},$$

which is equipped with the norm

$$\|\mathbf{v}^{\mathbf{f}}\|^2_{H(\operatorname{div};\Omega)} := \|\mathbf{v}^{\mathbf{f}}\|^2_{L^2(\Omega)} + \|\operatorname{div}\mathbf{v}^{\mathbf{f}}\|^2_{L^2(\Omega)}.$$

Consider the space

$$V := \{\mathbf{v} = (\mathbf{v}^{\mathbf{s}}, \mathbf{v}^{\mathbf{f}}) \in (H^1(\Omega))^3 \times H(\operatorname{div};\Omega)\},$$

which has the norm

$$\|(\mathbf{v}^{\mathbf{s}}, \mathbf{v}^{\mathbf{f}})\|^2_V := \|\mathbf{v}^{\mathbf{s}}\|^2_{H^1(\Omega)} + \|\mathbf{v}^{\mathbf{f}}\|^2_{H(\operatorname{div};\Omega)}.$$

Introduce a symmetric bilinear form on V:

$$\begin{aligned} B(\mathbf{v},\mathbf{w}) = &\int_\Omega \left[\lambda \operatorname{div}\mathbf{v}^{\mathbf{s}} \cdot \operatorname{div}\mathbf{w}^{\mathbf{s}} + 2\mu\left(\epsilon(\mathbf{v}^{\mathbf{s}}) : \epsilon(\mathbf{w}^{s})\right) + r \operatorname{div}\mathbf{v}^{\mathbf{f}} \cdot \operatorname{div}\mathbf{w}^{\mathbf{f}}\right] \mathrm{d}x \\ &+ \int_\Omega q\left[\operatorname{div}\mathbf{v}^{\mathbf{f}} \cdot \operatorname{div}\mathbf{w}^{\mathbf{s}} + \operatorname{div}\mathbf{v}^{\mathbf{s}} \cdot \operatorname{div}\mathbf{w}^{\mathbf{f}}\right] \mathrm{d}x, \end{aligned}$$

where $\epsilon(\mathbf{v}^{\mathbf{s}}) = \frac{1}{2}(\nabla\mathbf{v}^{\mathbf{s}} + (\nabla\mathbf{v}^{\mathbf{s}})^T)$, $A : B = \operatorname{tr}(AB^\top)$ is the Frobenius inner product of matrices A and B.

It is clear to note that $B(\mathbf{v},\mathbf{v}) \geqslant 0$ for all $\mathbf{v} \in V$ in view of (**H3**); moreover, $B(\mathbf{v},\mathbf{v}) = 0$ if and only if $\mathbf{v}^{\mathbf{s}} = 0$ and $\operatorname{div}\mathbf{v}^{\mathbf{f}} = 0$. Thus B induces a semi-norm $\|\mathbf{v}\|_B := (\mathbf{v},\mathbf{v})_B$. We relate the bilinear form B to the differential operator $P(D)$.

Lemma 9. *Suppose* $\mathbf{v}, \mathbf{w} \in V$ *with* $\mathbf{v}|_{\partial\Omega} = \mathbf{0}$ *or* $\mathbf{w}|_{\partial\Omega} = \mathbf{0}$*, then*

$$B(\mathbf{v}, \mathbf{w}) = (P(D)\mathbf{v}, \mathbf{w})_{L^2(\Omega)} = (\mathbf{v}, P(D)\mathbf{w})_{L^2(\Omega)}.$$

Proof. By symmetry, we only need to prove the first equality with the assumption that $\mathbf{w}|_{\partial\Omega} = \mathbf{0}$. Recalling $\mathbf{w} = (\mathbf{w}^{\mathbf{s}}, \mathbf{w}^{\mathbf{f}})^T$ and

$$P(D)\mathbf{v} = (-\Delta_{\mu,\lambda}\mathbf{v}^{\mathbf{s}} - \nabla(q\mathrm{div}\,\mathbf{v}^{\mathbf{f}}), -\nabla(q\mathrm{div}\,\mathbf{v}^{\mathbf{s}}) - \nabla(r\mathrm{div}\,\mathbf{v}^{\mathbf{f}}))^T,$$

we have

$$\begin{aligned}(P\mathbf{v}, \mathbf{w})_{L^2(\Omega)} = \int_\Omega \big[&- \Delta_{\mu,\lambda}\mathbf{v}^{\mathbf{s}} \cdot \mathbf{w}^{\mathbf{s}} - \nabla(q\mathrm{div}\,\mathbf{v}^{\mathbf{f}}) \cdot \mathbf{w}^{\mathbf{s}} \\ &- \nabla(q\mathrm{div}\,\mathbf{v}^{\mathbf{s}}) \cdot \mathbf{w}^{\mathbf{f}} - \nabla(r\mathrm{div}\,\mathbf{v}^{\mathbf{f}}) \cdot \mathbf{w}^{\mathbf{f}}\big]\,\mathrm{d}x. \end{aligned} \tag{19}$$

To deal with the first integrand, we expand $\Delta_{\mu,\lambda}$ using (7) to have

$$\int_\Omega [-\Delta_{\mu,\lambda}\mathbf{v}^{\mathbf{s}} \cdot \mathbf{w}^{\mathbf{s}}]\,\mathrm{d}x = -2\int_\Omega \mathrm{div}\,(\mu\epsilon(\mathbf{v}^{\mathbf{s}})) \cdot \mathbf{w}^{\mathbf{s}}\,\mathrm{d}x - \int_\Omega \nabla(\lambda\mathrm{div}\,\mathbf{v}^{\mathbf{s}}) \cdot \mathbf{w}^{\mathbf{s}}\,\mathrm{d}x.$$

We claim that

$$\mathrm{div}\,(\mu\epsilon(\mathbf{v}^{\mathbf{s}})) \cdot \mathbf{w}^{\mathbf{s}} = \mathrm{div}\,(\mu\epsilon(\mathbf{v}^{\mathbf{s}}))\,\mathbf{w}^{\mathbf{s}} - \mu\epsilon(\mathbf{v}^{\mathbf{s}}) : \epsilon(\mathbf{w}^{\mathbf{s}}).$$

To justify this, we write $\epsilon(\mathbf{v}^{\mathbf{s}}) := (\epsilon_1, \epsilon_2, \epsilon_3)^T$, where $\epsilon_1, \epsilon_2, \epsilon_3$ are the three rows; write $\mathbf{w}^{\mathbf{s}} = (\mathbf{w}^{\mathbf{s}}_{\mathbf{1}}, \mathbf{w}^{\mathbf{s}}_{\mathbf{2}}, \mathbf{w}^{\mathbf{s}}_{\mathbf{3}})^T$. Then

$$\begin{aligned}\mathrm{div}\,(\mu\epsilon(\mathbf{v}^{\mathbf{s}})) \cdot \mathbf{w}^{\mathbf{s}} &= \sum_{j=1}^{3} \mathrm{div}\,(\mu\epsilon_j)\mathbf{w}^{\mathbf{s}}_{\mathbf{j}} = \sum_{j=1}^{3} \left[\mathrm{div}\,\left(\mu\mathbf{w}^{\mathbf{s}}_{\mathbf{j}}\epsilon_j\right) - \mu\epsilon_j \cdot \nabla\mathbf{w}^{\mathbf{s}}_{\mathbf{j}}\right] \\ &= \mathrm{div}\,(\mu\epsilon(\mathbf{v}^{\mathbf{s}}))\,\mathbf{w}^{\mathbf{s}} - \mu\epsilon(\mathbf{v}^{\mathbf{s}}) : \epsilon(\mathbf{w}^{\mathbf{s}}).\end{aligned}$$

Using the integration by parts yields

$$\begin{aligned}\int_\Omega [-\Delta_{\mu,\lambda}\mathbf{v}^{\mathbf{s}} \cdot \mathbf{w}^{\mathbf{s}}]\,\mathrm{d}x = &\int_\Omega [2\mu\epsilon(\mathbf{v}^{\mathbf{s}}) : \epsilon(\mathbf{w}^{\mathbf{s}}) + \lambda\mathrm{div}\,\mathbf{v}^{\mathbf{s}} \cdot \mathrm{div}\,\mathbf{w}^{\mathbf{s}}]\,\mathrm{d}x \\ &+ \int_{\partial\Omega} [-2\mu\,(\epsilon(\mathbf{v}^{\mathbf{s}})\mathbf{w}^{\mathbf{s}}) - (\lambda\mathrm{div}\,\mathbf{v}^{\mathbf{s}})\mathbf{w}^{\mathbf{s}}] \cdot \nu\,\mathrm{d}x.\end{aligned} \tag{20}$$

The boundary term vanishes owing to the compact support of $\mathbf{w}$.

The remaining three integrands in (19) can be treated using the standard integration by parts:

$$\begin{aligned}&\int_\Omega \left[-\nabla(q\mathrm{div}\,\mathbf{v}^{\mathbf{f}}) \cdot \mathbf{w}^{\mathbf{s}} - \nabla(q\mathrm{div}\,\mathbf{v}^{\mathbf{s}}) \cdot \mathbf{w}^{\mathbf{f}} - \nabla(r\mathrm{div}\,\mathbf{v}^{\mathbf{f}}) \cdot \mathbf{w}^{\mathbf{f}}\right]\,\mathrm{d}x \\ = &\int_\Omega \left[q\,\mathrm{div}\,\mathbf{v}^{\mathbf{f}} \cdot \mathrm{div}\,\mathbf{w}^{\mathbf{s}} + q\,\mathrm{div}\,\mathbf{v}^{\mathbf{s}} \cdot \mathrm{div}\,\mathbf{w}^{\mathbf{f}} + r\,\mathrm{div}\,\mathbf{v}^{\mathbf{f}} \cdot \mathrm{div}\,\mathbf{w}^{\mathbf{f}}\right]\,\mathrm{d}x \\ &+ \int_{\partial\Omega} \left[-(q\mathrm{div}\,\mathbf{v}^{\mathbf{f}})\mathbf{w}^{\mathbf{s}} - (q\mathrm{div}\,\mathbf{v}^{\mathbf{s}})\mathbf{w}^{\mathbf{f}} - (r\mathrm{div}\,\mathbf{v}^{\mathbf{f}})\mathbf{w}^{\mathbf{f}}\right] \cdot \nu\,\mathrm{d}x.\end{aligned} \tag{21}$$

The boundary term again vanishes. This completes the proof. $\square$

Let $L^2(\Omega; M)$ be a weighted L^2-space with measure $M(x)\mathrm{d}x$ where $M(x)$ is the positive definite matrix defined in (9); in other words, for any $\mathbf{v} \in (L^2(\Omega))^6$, $\|\mathbf{v}\|_{L^2(\Omega;M)} := (\mathbf{v}, M\mathbf{v})_{L^2}$. The space $L^2(\mathbb{R}^3; M)$ and the norm $\|\cdot\|_{L^2(\mathbb{R}^3;M)}$ is defined similarly with the domain of integration replaced by $\mathbb{R}^3$.

Given a time-dependent function $\mathbf{u}(t, x)$, define its total energy over the domain Ω at time t:

$$\begin{aligned} E_\Omega(t, \mathbf{u}) = \|\partial_t \mathbf{u}(t, \cdot)\|^2_{L^2(\Omega;M)} + \|\mathbf{u}(t, \cdot)\|^2_B = \int_\Omega \big(M(x)\partial_t \mathbf{u} \cdot \partial_t \mathbf{u} \\ + \lambda|\mathrm{div}\, \mathbf{u}^\mathbf{s}|^2 + 2\mu|\epsilon(\mathbf{u}^\mathbf{s})|^2 + r|\mathrm{div}\, \mathbf{u}^\mathbf{f}|^2 + 2q(\mathrm{div}\, \mathbf{u}^\mathbf{f})(\mathrm{div}\, \mathbf{u}^\mathbf{s})\big)\, \mathrm{d}x. \end{aligned}$$

This quantity is conservative on a bounded domain Ω for the solution of Biot's equations when imposed with appropriate boundary conditions.

Lemma 10. *Let* $\mathbf{u}$ *satisfy Biot's equations with zero Dirichlet boundary condition:*

$$\begin{cases} M\partial_t^2 \mathbf{u} + P(D)\mathbf{u} = \mathbf{0} & \text{in } (0, T) \times \Omega, \\ \mathbf{u}|_{[0,T]\times\partial\Omega} = \mathbf{0}, & \end{cases}$$

then

$$E_\Omega(t, \mathbf{u}) = E_\Omega(0, \mathbf{u}) \qquad 0 \leqslant t \leqslant T.$$

Proof. We briefly sketch the proof since it is similar to that of lemma 9. Taking the inner product of the equation with $\partial_t \mathbf{u} = (\partial_t \mathbf{u}^\mathbf{s}, \partial_t \mathbf{u}^\mathbf{f})^T$, we obtain

$$\begin{aligned} 0 &= (\partial_t^2 \mathbf{u}, \partial_t \mathbf{u})_{L^2(\Omega;M)} + (P(D)\mathbf{u}, \partial_t \mathbf{u})_{L^2(\Omega)} \\ &= (\partial_t^2 \mathbf{u}, \partial_t \mathbf{u})_{L^2(\Omega;M)} + B(\mathbf{u}, \partial_t \mathbf{u}) + b.t. \\ &= 2\frac{\mathrm{d}}{\mathrm{d}t} E_\Omega(t, \mathbf{u}) + b.t.. \end{aligned}$$

Here the second equality is justified by lemma 9; '$b.t.$' represents the arising boundary terms, which are the boundary integrals in (20) and (21) with $\mathbf{v}$ and $\mathbf{w}$ replaced by $\mathbf{u}$ and $\partial_t \mathbf{u}$ respectively. The zero Dirichlet boundary condition annihilates $b.t.$, which completes the proof of conservation of energy. $\square$

The proof verifies the well known fact that zero Dirichlet boundary condition preserves energy. In fact each of the following boundary conditions

(i) $\mathbf{u}^\mathbf{s}(t, x) = 0$ and $\mathbf{u}^\mathbf{f} \cdot \nu(t, x) = 0$ on $(0, T) \times \partial\Omega$;
(ii) $\mathbf{u}^\mathbf{s} \cdot \nu(t, x) = 0$ and $\mathbf{u}^\mathbf{f}(t, x) = 0$ on $(0, T) \times \partial\Omega$;
(iii) $\mathbf{u}^\mathbf{s} \cdot \nu(t, x) = 0$ and $\mathbf{u}^\mathbf{f} \cdot \nu(t, x) = 0$ on $(0, T) \times \partial\Omega$.

is energy preserving as well, since each of them is sufficient to annihilate the boundary term '$b.t.$' in the proof.

Let $\mathbf{u}$ be the solution of the direct problem (10). We can also consider the energy over the entire space $E_{\mathbb{R}^3}(t, \mathbf{u})$. This global energy is conservative as well, i.e.

$$E_{\mathbb{R}^3}(t, \mathbf{u}) = E_{\mathbb{R}^3}(0, \mathbf{u}) = \|\mathbf{f}\|^2_B \qquad 0 \leqslant t \leqslant T.$$

To show this, one can take a large ball $B(0;R)$ so that the solution $\mathbf{u}(t,\cdot)$ is supported inside $B(0;R)$ for any $t \in [0,T]$. Then lemma 10 is applicable since $\mathbf{u}|_{[0,T]\times\partial B(0;R)} = \mathbf{0}$ and $E_{\mathbb{R}^3}(t,\mathbf{u}) = E_\Omega(t,\mathbf{u})$ for any $t \in [0,T]$.

6. Main theorem

Let $\mathbf{v}$ be the solution of

$$\begin{cases} M\partial_t^2\mathbf{v} + P(D)\mathbf{v} &= 0 \quad \text{in } (0,T)\times\Omega, \\ \mathbf{v}|_{[0,T]\times\partial\Omega} &= h, \\ \mathbf{v}(T,\cdot) &= \phi, \\ \partial_t\mathbf{v}(T,\cdot) &= \mathbf{0}, \end{cases} \tag{22}$$

where ϕ is the function satisfying

$$P(D)\phi = 0, \quad \phi|_{\partial\Omega} = h(T,\cdot).$$

The solution ϕ exists since the analysis in section 3 manifests that $P(D)\phi = 0$ can be transformed into an elliptic system, which is the time-independent counterpart of (15). Define the time reversal operator

$$Ah := \mathbf{v}(0,\cdot).$$

Now we take $h = \mathbf{\Lambda f}$ as the boundary measurement and expect $A\mathbf{\Lambda f}$ to be a reasonable approximation of $\mathbf{f}$. The rationale, from a microlocal viewpoint, is that the hyperbolic operator in the forward problem propagates microlocal singularities of $\mathbf{f}$ to $\partial\Omega$, while the time-reversal process tends to send back these singularities. This suggests a possible reconstruction of $\mathbf{f}$, as least all the microlocal singularities will be restored.

The microlocal viewpoint also suggests the necessity of an additional assumption to make sure that all the microlocal singularities of $\mathbf{u}$, the solution of (10), are not trapped, i.e. all of them are able to reach $\partial\Omega$ in a finite time. This indicates the necessity of the following *non-trapping* condition.

For the wave operator $\square_a = a\partial_t^2 - \Delta$ with $a = a(x) > 0$, the associated Hamiltonian is $H = \frac{1}{2}\frac{1}{a}|\xi|^2$ and the Hamiltonian system is

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial H}{\partial \xi} = \frac{1}{a}\xi, & x|_{t=0} = x_0 \\ \frac{\mathrm{d}\xi}{\mathrm{d}t} = -\frac{\partial H}{\partial x} = -\frac{1}{\sqrt{a}}\nabla_x\left(\frac{1}{\sqrt{a}}\right)|\xi|^2, & \xi|_{t=0} = \xi_0. \end{cases}$$

We say the wave speed $\frac{1}{\sqrt{a}}$ is *non-trapping* if for any $\xi_0 \neq 0$, the spatial component $x(t) \to \infty$ in $\mathbb{R}^n$ as $t \to \infty$. As the Biot system is equivalent to the principal scalar system (15) where the highest order derivatives are decoupled, we need to ensure each wave speed $\sqrt{a}_1$, $\sqrt{a}_2$, $\sqrt{\mu}_1$ in this system is non-trapping. Then there exists a maximal escaping time $T(M,P(D),\Omega) > 0$, depending on M, $P(D)$ and the geometry of Ω, such that all the (microlocal) singularities of $\mathbf{u}$ are out of Ω whenever $t > T(M,P(D),\Omega)$; in other words, $\mathbf{u}(t,x)$ is smooth for $x \in \Omega$ whenever $t > T(M,P(D),\Omega)$.

Now we are in the position to state and prove the main theorem. We show that A is the inverse of $\mathbf{\Lambda}$ up to a compact operator and the compact operator becomes a contraction on a suitable function space. Recall that $\|\cdot\|_B$ is merely a semi-norm on V: it is not positive definite since $\|\mathbf{w}\|_B = 0$ implies only $\mathbf{w^s} = 0$ and $\operatorname{div}\mathbf{w^f} = 0$. We can take V modulo the closed subspace $\{\mathbf{w} \in V : \mathbf{w^s} = 0,\ \operatorname{div}\mathbf{w^f} = 0\}$ to make it a genuine norm. Denote by B the quotient space, then $(B,\|\cdot\|_B)$ is a Banach space and $\|\mathbf{w}\|_B = 0$ implies $\mathbf{w} = 0$ in B. We also project

$Ah = \mathbf{v}(0, \cdot)$ to this quotient space and abuse the notation to call A composed with the canonical projection as A. So A maps into $(B, \|\cdot\|_B)$.

Theorem 11. *Let Ω be non-trapping and $T > T(M, P(D), \Omega)$. Suppose that the hypotheses ($\mathbf{H1}$)–($\mathbf{H3}$) and the assumption of proposition 8 are satisfied. Then $K := I - A\mathbf{\Lambda}$ is compact and contractive on B in the sense that $\|K\|_{B\to B} < 1$. As a consequence, $I - K$ is invertible on B and*

$$\mathbf{f} = \sum_{j=0}^{\infty} K^j A\mathbf{\Lambda}\mathbf{f} = A\mathbf{\Lambda}\mathbf{f} + KA\mathbf{\Lambda}\mathbf{f} + K^2A\mathbf{\Lambda}\mathbf{f} + \ldots . \qquad \text{in } B.$$

Proof. The proof is divided into two claims. We first show the inequality $\|K\|_{B\to B} \leqslant 1$, and then prove by a contra-positive argument that the inequality is strict. Given a time-dependent function $g(t, x)$, we abbreviate $g(t)$ for the spatial function $g(t, \cdot)$.

Claim 1. $\|K\mathbf{f}\|_B < \|\mathbf{f}\|_B$ unless $\mathbf{f} = \mathbf{0}$ in B.

Let us give another representation of $K\mathbf{f}$. Let $\mathbf{u}$ be the solution of (10); let $\mathbf{v}$ be the solution of (22) with h replaced by $\mathbf{\Lambda}\mathbf{f}$. Denote $\mathbf{w} := \mathbf{u} - \mathbf{v}$, then $\mathbf{w}$ satisfies

$$\begin{cases} M\partial_t^2\mathbf{w} + P(D)\mathbf{w} &= \mathbf{0} \quad \text{in } (0, T) \times \Omega, \\ \mathbf{w}|_{[0,T]\times\partial\Omega} &= \mathbf{0}, \\ \mathbf{w}(T) &= \mathbf{u}(T) - \phi, \\ \partial_t\mathbf{w}(T) &= \partial_t\mathbf{u}(T). \end{cases} \tag{23}$$

Moreover, we have

$$K\mathbf{f} = \mathbf{f} - A\mathbf{\Lambda}\mathbf{f} = \mathbf{u}(0) - \mathbf{v}(0) = \mathbf{w}(0). \tag{24}$$

On the other hand, it is clear to note that $(\mathbf{u}(T) - \phi)|_{\partial\Omega} = \mathbf{0}$ by the construction of ϕ. It follows from lemma 9 that

$$(\mathbf{u}(T) - \phi, \phi)_B = (\mathbf{u}(T) - \phi, P(D)\phi)_{L^2(\Omega)} = 0,$$

which gives $\|\mathbf{u}(T) - \phi\|_B^2 = \|\mathbf{u}(T)\|_B^2 - \|\phi\|_B^2$. It is easy to verify that

$$\begin{aligned} E_\Omega(T, \mathbf{w}) &= \|\partial_t\mathbf{w}(T)\|^2_{L^2(\Omega;M)} + \|\mathbf{w}(T)\|_B^2 \\ &= \|\partial_t\mathbf{u}(T)\|^2_{L^2(\Omega;M)} + \|\mathbf{u}(T) - \phi\|_B^2 \\ &= \|\partial_t\mathbf{u}(T)\|^2_{L^2(\Omega;M)} + \|\mathbf{u}(T)\|_B^2 - \|\phi\|_B^2 \\ &= E_\Omega(T, \mathbf{u}) - \|\phi\|_B^2 \\ &\leqslant E_\Omega(T, \mathbf{u}). \end{aligned}$$

Combining lemma 10 and conservation of energy in $\mathbb{R}^3$ yields

$$E_\Omega(0, \mathbf{w}) = E_\Omega(T, \mathbf{w}) \leqslant E_\Omega(T, \mathbf{u}) \leqslant E_{\mathbb{R}^3}(T, \mathbf{u}) = E_{\mathbb{R}^3}(0, \mathbf{u}) = \|\mathbf{f}\|_B^2.$$

Thus we have from (24) that

$$\|K\mathbf{f}\|_B^2 = \|\mathbf{w}(0)\|_B^2 \leqslant E_\Omega(0, \mathbf{w}) \leqslant \|\mathbf{f}\|_B^2.$$

Suppose the equality holds for some $\mathbf{f} \in B$, then all the above inequalities become equalities. In particular $E_\Omega(T, \mathbf{u}) = E_{\mathbb{R}^3}(T, \mathbf{u})$, which implies

$$\mathbf{u}^{\mathbf{s}}(T, x) = \operatorname{div} \mathbf{u}^{\mathbf{f}}(T, x) = 0 \quad \text{for } x \in \mathbb{R}^3 \backslash \Omega.$$

By proposition 8, we obtain $\mathbf{f}^{\mathbf{s}}(x) = 0$ and $\operatorname{div} \mathbf{f}^{\mathbf{f}}(x) = 0$ for $x \neq 0$. Changing the value at the single point $x = 0$ does not affect a function in B. This completes the proof of claim 1.

Claim 2. $K : B \to B$ is compact and $\|K\|_{B \to B} < 1$.

Claim 1 alone implies $\|K\|_{B \to B} \leqslant 1$. To prove the strict inequality, we show K is a compact operator on B. The spectrum of a compact operator consists of countably many eigenvalues which may accumulate only at 0. Since claim 1 excludes eigenvalues of modulus 1, the spectral radius of K must be strictly less than 1, proving that $\|K\|_{B \to B} < 1$.

Next we prove the compactness of K. Using the representation (24), we decompose K into composition of bounded operators:

$$\begin{aligned} \mathbf{f} &\xmapsto{\pi_1^*} (\mathbf{f}, \mathbf{0}) \xmapsto{U_1} (\mathbf{u}|_{t=T} - \phi, \partial_t \mathbf{u}|_{t=T}) \\ &= (\mathbf{w}(T), \partial_t \mathbf{w}(T)) \xmapsto{U_2} (\mathbf{w}(0), \partial_t \mathbf{w}(0)) \xmapsto{\pi_1} \mathbf{w}(0). \end{aligned}$$

Here $\pi_1 : (\mathbf{f}, \mathbf{g}) \mapsto \mathbf{f}$ is the natural projection onto the first component; π_1^* is its adjoint; U_1 is the solution operator of the forward problem (10), mapping the state $t = 0$ to the state $t = T$; and U_2 is the solution operator of (23) sending $t = T$ to $t = 0$. These are all bounded operators.

Consider

$$U_1 : (\mathbf{f}, \mathbf{0}) \longmapsto (\mathbf{u}|_{t=T} - \phi, \partial_t \mathbf{u}|_{t=T}).$$

In view of the assumption that $T > T(P(D), \Omega)$, all the microlocal singularities of $\mathbf{f}$ have escaped from Ω at the moment T, hence $(\mathbf{u}|_{t=T}, \partial_t \mathbf{u}|_{t=T})$ is a pair of smooth functions. On the other hand, the function ϕ, as a solution of the elliptic equations $P(D)\phi = 0$, is smooth by elliptic regularity. We conclude that U_1 is a smoothing operator, hence compact. This means K is compact as well, since it is the composition of U_1 with other bounded operators.

We know that K is a contraction on B, $(I - K)^{-1}$ exists as a bounded operator. Applying $(I - K)^{-1}$ to the identity $(I - K)\mathbf{f} = A\mathbf{\Lambda f}$ and expanding it in terms of Neumann series, we obtain the reconstruction formula in the statement of the theorem. □

The following stability estimate shows that the faster the energy escapes from Ω, the faster the convergence of the Neumann series is.

Corollary 12. *Under the assumption of theorem 11, the following stability estimate holds*

$$\|K\mathbf{f}\|_B \leqslant \left(\frac{E_\Omega(T, \mathbf{u})}{E_\Omega(0, \mathbf{u})} \right)^{\frac{1}{2}} \|\mathbf{f}\|_B \quad \text{for } \mathbf{f} \neq 0 \text{ in } B.$$

Proof. A simple calculation yields that

$$\frac{\|K\mathbf{f}\|_B^2}{\|\mathbf{f}\|_B^2} = \frac{\|\mathbf{w}(0)\|_B^2}{E_\Omega(0, \mathbf{u})} \leqslant \frac{E_\Omega(0, \mathbf{w})}{E_\Omega(0, \mathbf{u})} \leqslant \frac{E_\Omega(T, \mathbf{u})}{E_\Omega(0, \mathbf{u})}.$$

□

Finally, we prove corollary 2. We will deal with a general source of the form

$$\frac{\eta(x)}{\kappa(x)}L(x)E(t,x) = h(t)f(x),$$

where $h(t) \in C([0,T])$ is a known continuous function and $f(x)$ is to be recovered. The Biot's system in this case is reduced to (11) with $\mathbf{f}(\mathbf{x}) = (0,f(x))$.

Proof of corollary 2. Let $\mathbf{u}$ be the solution of (11). By Duhamel's principle $\mathbf{u}$ can be written as

$$\mathbf{u}(t,x) = \int_0^t \mathbf{v}(t-s,x;s)\,\mathrm{d}s \tag{25}$$

where $\mathbf{v}(t,x;s)$ is the solution of the initial value problem

$$\begin{cases} M\partial_t^2\mathbf{v} + P(D)\mathbf{v} = \mathbf{0} & \text{in } \mathbb{R}\times\mathbb{R}^3, \\ \mathbf{v}|_{t=0} = \mathbf{0}, & \\ \partial_t\mathbf{v}|_{t=0} = h(s)\mathbf{f}. & \end{cases}$$

Set $\mathbf{w}(t,x;s) = \partial_t\mathbf{v}(t,x;s)$; then $\mathbf{w}$ solves

$$\begin{cases} M\partial_t^2\mathbf{w} + P(D)\mathbf{w} = \mathbf{0} & \text{in } \mathbb{R}\times\mathbb{R}^3, \\ \mathbf{w}|_{t=0} = h(s)\mathbf{f}, & \\ \partial_t\mathbf{w}|_{t=0} = \mathbf{0}. & \end{cases}$$

In (25), we fix $x \in \partial\Omega$ and differentiate in t to have

$$\partial_t\mathbf{u}(t,x) = \int_0^t \partial_t\mathbf{v}(t-s,x,s)\,\mathrm{d}s = \int_0^t \mathbf{w}(t-s,x,s)\,\mathrm{d}s = \int_0^t h(s)\mathbf{\Lambda f}(t-s,x)\,\mathrm{d}s.$$

This is the convolution of $h(\cdot)$ and $\mathbf{\Lambda f}(\cdot,x)$. As the left hand side is known for $x \in \partial\Omega$, one can take the Laplace transform to recover $\mathbf{\Lambda f}(t,x)$. Theorem 11 then recovers $\mathbf{f}$ up to a pair of vector fields $(0,g)$ with $\operatorname{div} g = 0$ and $g|_{\partial\Omega} = 0$. $\square$

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